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EINSTEIN WARPED PRODUCT MANIFOLDS WITH 3− DIMENSIONAL FIBER MANIFOLDS

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ABSTRACT. In this paper, we consider the existence of nonconstant warping functions on a warped product manifold $M = B \times_{f^2} F$, where B is a $q(> 2)-$dimensional base manifold with a nonconstant scalar curvature $S_B(x)$ and F is a $3-$ dimensional fiber Einstein manifold and discuss that the resulting warped product manifold is an Einstein manifold, using the existence of the solution of some partial differential equation.

1. Introduction

In [2], A.L. Besse studied a new compact Einstein manifold using the warped product. Then A.L. Besse asked the following: "Does there exist an Einstein warped product manifold with a nonconstant warping function?"

In [9],[10], and [11], the authors proved that there does not exist a compact Einstein warped product space with a nonconstant warping function, if the scalar curvature on M is nonpositive or the base is a compact $2-$dimensional manifold. Hence here we assume that the base manifold B is a compact $q(> 2)-$ dimensional manifold with the positive scalar curvature somewhere.

DEFINITION 1.1. Let (B, g_B) and (F, g_F) be two manifolds. Let g_B be a metric tensor of B and g_F be a metric tensor of F. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. For a positive smooth function f on B the warped product manifold $M = B \times_{f^2} F$ is the product manifold $M = B \times F$ furnished with the metric tensor g

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defined by $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$. We denote by π^* and σ^* the pullback π and σ, respectively. Here B is called the base of M and F the fiber([1,3,4,5,12]).

Now we recall the formula for the Ricci curvature tensor Ric of the warped product manifold $M = B \times_{f^2} F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

PROPOSITION 1.2. *On a warped product manifold $M = B \times_{f^2} F$ with $p = dimF > 1$, let X, Y be horizontal and V, W vertical. Then*
(i) $Ric(X,Y) = Ric^B(X,Y) - \frac{p}{f} H^f(X,Y)$,
(ii) $Ric(X,V) = 0$,
(iii) $Ric(V,W) = Ric^F(V,W) - g(V,W)(\frac{\Delta f}{f} + (p-1)\frac{g(df,df)}{f^2})$,
where H^f and Δf denote by the Hessian of f and the Laplacian of f for g_B.

Proof. See Proposition 9.106 in ([2, p.266]). □

COROLLARY 1.3. *Let F be a 3 - dimensional manifold. The warped product $M = B \times_{f^2} F$ is an Einstein manifold (with $Ric = \lambda g$) if and only if g_F, g_B and f satisfy*
(i) $(F,\ g_F)$ *is Einstein (with* $Ric_F = \lambda_0 g_F$*),*
(ii) $\frac{\Delta f}{f} - 2\frac{\|df\|^2}{f^2} + \frac{\lambda_0}{f^2} = \lambda$,
(iii) $Ric_B - \frac{3}{f} H^f = \lambda g_B$.

Proof. See Corollary 9.107 in [2, p.267] □

Obviously, (i) gives a condition on (F, g_F) alone, whereas (ii) and (iii) are two differential equations for f on (B, g_B).

In this paper, we consider the following question:

Question A : If the base manifold B is a compact $q(>2)-$ dimensional manifold and the fiber manifold F is a 3 - dimensional Einstein manifold with $Ric_F = \lambda_0 g_F$, then do there exist a constant λ and a nonconstant warping function f such that the resulting warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold with $Ric = \lambda g$?

In [6], the author proved that if B is a compact $q(>2)-$ dimensional manifold with a nonconstant scalar curvature and F is a $p(>3)$ - dimensional Einstein manifold, then there exist a constant λ and a nonconstant warping function f such that the resulting warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold with $Ric = \lambda g$.

In this paper, the similar results are proved in case of 3 - dimensional fiber manifolds, using a partial differential equation.

REMARK 1.4. We denote by $\dim B = q(> 2)$ and $\dim F = 3$. Then, using Corollary 1.3 (ii) and (iii), we may replace the unique equation

$$(1.1) \quad Ric_B - \frac{3}{f}H^f = \frac{1}{2}[\ S_B + 6\frac{\Delta f}{f} - 6\frac{\|df\|^2}{f^2} + 3\frac{\lambda_0}{f^2} - (1+q)\lambda\ ]g_B,$$

where S_B is a scalar curvature of B (See also (9.108) in [2, p.267]).

In order to solve Question A, we study equation (1.1) on M with $\dim B = q(> 2)$ and $\dim F = 3$. Recalling that $Ric_B - \frac{3}{f}H^f = \lambda g_B$ and that S_B is a scalar curvature on B, equation (1.1) implies that we have equation

$$(1.2) \qquad 0 = \Delta f - \frac{\|df\|^2}{f} + \frac{\lambda_0}{2f} + \frac{S_B - (3+q)\lambda}{6}f.$$

From now on, we study the nonconstant solution of equation (1.2). If $S_B = C$ is constant, then the solution f is maybe also constant. In case that S_B is not a constant, then the solution also is not a constant. So we assume that S_B is not a constant.

Using the change of variable $f = e^{-u}$, equation (1.2) is changed into

$$(1.3) \qquad 0 = \Delta u - \frac{S_B(x) - (3+q)\lambda}{6} - \frac{\lambda_0}{2}e^{2u}.$$

We put $\frac{\lambda_0}{2} = C_{\lambda_0}$ and $\frac{S_B(x) - (3+q)\lambda}{6} = h_\lambda(x)$ (a function depending on $S_B(x)$ and λ). Then equation (1.3) is changed into

$$(1.4) \qquad 0 = \Delta u - h_\lambda(x) - C_{\lambda_0}e^{2u},$$

where λ is a constant.

In order to solve equation (1.4), we consider the following functional J_λ for a fixed constant λ, i.e.,

$$J_\lambda(u) = \frac{\int_B |\nabla u|^2 dB - 2\int_B h_\lambda(x) u dB}{\int_B e^{2u} dB}$$

2. Main results

Let B be a compact connected manifold, which is not necessarily orientable and possesses a given Riemannian structure g. We denote the volume element of this metric by dB, the gradient by ∇, and the associated Laplacian by Δ (we use the sign convention which gives $\triangle u = -u_{xx} - u_{yy}$ for the standard metric on R^2). We let $H_{s,r}(B)$ denote the Sobolev space of functions on B whose derivatives through order s are in $L_r(B)$. The norm on $H_{s,r}(B)$ will be denoted by $|| \ ||_{s,r}$. In the special case $s = 0, H_{s,r}(B)$ is just $L_r(B)$, and we denote the norm by $|| \ ||_r$. We have the following elementary inequality.

LEMMA 2.1. *For all $v \in H_{1,2}(B)$, if $v \not\equiv 0$ and $\int_B v dB = 0$, then*

$$\int_B e^v dB \ > \ vol(B), \tag{2.1}$$

where $vol(B)$ is the volume of B.

Proof. See [7, Theorem 2.5 and Corollary 1, p.3228]. □

By Lemma 2.1, if we choose a function $v \in H_{1,2}(B)$ and v is not a constant, then $\int_B e^{v-\bar{v}} dB \ > \ vol(B)$, where $\bar{v} = \int_B v dB$. Hence we can consider the functional J_λ on $V_\sigma = \{v \in H_{1,2}(B) \ |v \not\equiv 0, \int_B v dB = 0, \int_B e^{2v} \ dB = \sigma\}$ for some constant $\sigma(> vol(B))$,

$$\begin{aligned} J_\lambda(v) &= \frac{\int_B |\nabla v|^2 \ dB - 2\int_B h_\lambda(x) v \ dB}{\int_B e^{2v} \ dB} \\ &= \frac{1}{\sigma}[\int_B |\nabla v|^2 \ dB - 2\int_B h_\lambda(x) v \ dB]. \end{aligned}$$

THEOREM 2.2. *For a fixed constant λ, let $\{v_i\}$ be a minimizing sequence in V_σ such that $J_\lambda(v_i) \to C$ for some constant C. If $v_i \to v_0$ in V_σ and $J_\lambda(v_0) = C$, then equation (1.4) has a solution v_0 for some constant C.*

Proof. For a fixed constant λ, let v_0 satisfy

$$J_\lambda(v_0) = \frac{\int_B |\nabla v_0|^2 \ dB - 2\int_B h_\lambda(x) v_0 \ dB}{\int_B e^{2v_0} \ dB} = C.$$

For all $\psi \in H_{1,2}(B)$,

$$\frac{dJ_\lambda(v_0+t\psi)}{dt}\mid_{t=0}$$
$$=\frac{d}{dt}[\frac{\int_B(|\nabla v_0+t\nabla\psi|^2-2h_\lambda(x)(v_0+t\psi))\ dB}{\int e^{2(v_0+t\psi)}\ dB}]\mid_{t=0}$$
$$=\frac{1}{(\int_B e^{2(v_0+t\psi)}\ dB)^2}[\{\int_B 2\nabla v_0\nabla\psi\ dB-2\int_B h_\lambda(x)\psi\ dB$$
$$+2t\int_B|\nabla\psi|^2\ dB\ \}\{\int_B e^{2(v_0+t\psi)}\ dB\}$$
$$-\{\int_B|\nabla v_0+t\nabla\psi|^2\ dB-2\int_B h_\lambda(x)(v_0+t\psi)\ dB\}$$
$$\times\ \{2\int_B e^{2(v_0+t\psi)}\psi\ dB\}]\mid_{t=0}$$
$$=\frac{1}{(\int_B e^{2(v_0)}\ dB)^2}[\{2\int_B\nabla v_0\nabla\psi\ dB-2\int_B h_\lambda(x)\psi\ dB\}$$
$$\times\ \{\int_B e^{2(v_0)}\ dB\}\ -\{\int_B|\nabla v_0|^2\ dB-2\int_B h_\lambda(x)v_0\ dB\}$$
$$\times\ \{2\int_B e^{2(v_0)}\psi\ dB\}]=0.$$

Therefore

$$\int_B\nabla v_0\nabla\psi\ dB-\int_B h_\lambda(x)\psi\ dB-C\int_B e^{2(v_0)}\psi\ dB=0,$$

for all $\psi \in H_{1,2}(B)$. Since Δ is the negative Laplacian, we have

$$\Delta v_0-h_\lambda(x)-Ce^{2v_0}=0, \qquad (2.2)$$

where C is a constant. □

For $v \in H_{1,2}(B)$, let $v^+(x)=max\{0,v(x)\}$ and $v^-(x)=min\{0,v(x)\}$. Then we know easily that $2v^+(x)\le e^{2v^+(x)+2v^-(x)}=e^{2v(x)}$ for each x, hence we have the following key lemma.

LEMMA 2.3. *For a fixed constant λ, if $v \in V_\sigma$, then $|\int_B h_\lambda(x)v\ dB|\le N_0\sigma$, where $N_0=\ max_{x\in B}|h_\lambda(x)|$.*

Proof. If $v\in V_\sigma$ and $\int_B v dB=0$, then $\int_B|v|dB=2\int_B v^+dB$. Hence $\int_B|v|dB\le\int_B e^{2v}dB$. Thus

$$|\int_B h_\lambda(x)v\ dB|\le N_0\sigma,$$

where $N_0 = \ max_{x\in B}|h_\lambda(x)|$. □

THEOREM 2.4. *On* $V_\sigma \ = \ \{v \in H_{1,2}(B)|v \not\equiv 0, \int_B v dB = 0, \int_B e^{2v} dB = \sigma\}$ *for some constant* $\sigma(> vol(B))$, *the functional* $J_\lambda(v)$ *is bounded below for a fixed constant* λ.

Proof. Since B is compact, $\max_{x\in B}|h_\lambda(x)| \leq N_0$ for some positive constant N_0. If $v \in V_\sigma$, then by Lemma 2.3

$$\begin{aligned} J_\lambda(v) &\geq \frac{1}{\sigma}[\int_B |\nabla v|^2 \ dB - 2N_0 \int_M |v| \ dB] \\ &\geq -2N_0. \end{aligned}$$

This means that $J_\lambda(v)$ is bounded below on V_σ. □

We consider the following functional J_λ for a fixed constant λ on $V_\sigma = \{v \in H_{1,2}(B)|v \not\equiv 0, \int_B v dB = 0, \int_B e^{2v} \ dB = \sigma\}$ for some constant $\sigma(> vol(B))$,

$$J_\lambda(v) = \frac{\int_B |\nabla v|^2 dB - 2\int_B h_\lambda(x) v dB}{\int_B e^{2v} dB} = \frac{1}{\sigma}[\int_B |\nabla v|^2 dB - 2\int_B h_\lambda(x) v dB].$$

THEOREM 2.5. *Let* $C = \inf_{v\in V_\sigma} J_\lambda(v)$ *for a fixed constant* λ *and for some constant* $\sigma(> \ vol(B))$. *If* $C_{\lambda_0} \ = \ \frac{\lambda_0}{2} \ = \ C$, *then there exists a nonconstant solution of equation (1.4) for* $C_{\lambda_0} = \frac{\lambda_0}{2} = C$.

Proof. Since $h_\lambda(x)$ is smooth on B, Theorem 2.4 implies that J_λ is bounded below on V_σ. Hence there exists a minimizing sequence $\{v_i\}$ in V_σ such that $J_\lambda(v_i) \to C$. Because V_σ is not empty, there is some $v_1 \in V_\sigma$. Hence there is a $b > 0$ such that $J_\lambda(v_1) < b$ and $J_\lambda(v_n) \leq b$ for all n.

For $v_n \in V_\sigma$,

$$\sigma J_\lambda(v_n) = \int_B |\nabla v_n|^2 \ dB - 2\int_B h_\lambda(x) v_n \ dB \geq \int_B |\nabla v_n|^2 \ dB - 2N_0\sigma.$$

Hence $\int_B |\nabla v_n|^2 \ dB \leq (b + 2N_0)\sigma$. It follows that $||v_n||^2_{1,2} \leq$ constant for all n. Since the unit ball in any Hilbert space is weakly compact ([1,p.74]), there exist a subsequence $\{v_i\}$ of $\{v_n\}$ and a function $v_0 \in H_{1,2}(B)$ such that :

i) $v_i \to v_0$ strongly in $L_2(B)$

ii) $v_i \to v_0$ weakly in $H_{1,2}(B)$
iii) $v_i \to v_0$ pointwise almost everywhere.

This implies that $\int_B e^{2v_0}\, dB = \sigma$, $\int_B v_0 dB = 0$ and $\int_B h_\lambda(x) v_i\, dB \to \int_B h_\lambda(x) v_0\, dB$. Therefore $v_0 \in V_\sigma$. Hence $J_\lambda(v_0) \geq C$.

To conclude that v_0 minimizes J_λ for all $v \in V_\sigma$, we use the general result that whenever v_n converges to v_0 weakly in a Hilbert space, then $||\nabla v_0||_2 \leq \lim\inf ||\nabla v_n||_2$. Thus $J_\lambda(v_0) \leq J_\lambda(v_n)$ for all n and $J_\lambda(v_0) \leq C$. Therefore v_0 minimizes J_λ in V_σ. □

THEOREM 2.6. *Let* $V_\sigma = \{v \in H_{1,2}(B) | v \not\equiv 0, \int_B v dB = 0, \int_B e^{2v} dB = \sigma\}$ *for some constant* $\sigma(> vol(B))$. *For each* λ_0, *there exists a constant* λ *such that*

$$inf_{v \in V_\sigma} J_\lambda = \frac{\lambda_0}{2},$$

which implies that Question A holds.

Proof. Since B is compact, the scalar curvature $S_B(x)$ is bounded. Hence $h_\lambda(x) \to -\infty$ as $\lambda \to \infty$. And $0 < \delta \leq inf_{v \in V_\sigma} \frac{\int_B |v| dB}{\int_B e^{2v} dB} \leq 1$, where δ is a positive constant (If $inf_{v \in V_\sigma} \frac{\int_B |v| dB}{\int_B e^{2v} dB} = 0$, then $\lim \int_B |v_n| dB = 0$, which means a contradiction to the fact that $\sigma = \int_B e^{2v_n} dB \to vol(B)$.). Therefore $inf_{v \in V_\sigma} J_\lambda \to -\infty$ as $\lambda \to \infty$. Similarly $inf_{v \in V_\sigma} J_\lambda \to +\infty$ as $\lambda \to -\infty$. Since J_λ is linear with respect to λ, for each λ_0 there exists a constant λ such that

$$inf_{v \in V_\sigma} J_\lambda = \frac{\lambda_0}{2}.$$

Therefore Theorem 2.5 implies that there exists a nonconstant warping function v_0 such that v_0 is a solution of equation (1.4), which implies that the warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold. □

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