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GENERALIZED FUZZY CLOSED SETS ON INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce three different concepts of closed sets on the intuitionistic fuzzy topological spaces, i.e., the generalized fuzzy (r, s)-closed, semi-generalized fuzzy (r, s)-closed, and generalized fuzzy (r, s)-semiclosed sets on intuitionistic fuzzy topological spaces in Šostak's sense. Also we investigate their properties and the relationships among these generalized fuzzy closed sets.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [13]. Chang [3] defined fuzzy topological spaces with the fuzzy sets. These spaces and its generalizations are later studied by several authors [12, 4, 11]. The concept of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of fuzzy sets [1]. Later, many researchers studied the topological spaces on the intuitionistic fuzzy sets [5, 11, 6].

Since closed sets play an important role in the study of continuity of spaces, many researchers studied generalized closed sets. G. Balasubramanian and P. Sundaram [2] introduced the notion of generalized fuzzy closed sets to study the generalization of fuzzy continuity. M. E. El-Shafei and A. Zakari [7] introduced the concept of semi-generalized fuzzy closed sets on Chang's fuzzy topological spaces, and investigated some of their properties.

In this paper, as a prior work to studying the various continuity on intuitionistic fuzzy topological spaces, we introduce three different concepts of closed sets on the intuitionistic fuzzy topological spaces. Specifically, the generalized fuzzy (r, s)-closed sets, semi-generalized fuzzy

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(r,s)-closed sets, and generalized fuzzy (r,s)-semiclosed sets on intuitionistic fuzzy topological spaces in Šostak's sense are discussed, and the relationships among these generalized fuzzy closed sets are discussed.

2. Preliminaries

For the nonstandard definitions and notations we refer to [8, 9, 10].

Let $I(X)$ be a family of all intuitionistic fuzzy sets in X and let $I \otimes I$ be the set of the pair (r,s) such that $r,s \in I$ and $r+s \leq 1$.

DEFINITION 2.1. ([6]) Let X be a nonempty set. An *intuitionistic fuzzy topology in Šostak's sense*(SoIFT for short) $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ on X is a mapping $\mathcal{T} : I(X) \to I \otimes I$ which satisfies the following properties:

(1) $\mathcal{T}_1(\underline{0}) = \mathcal{T}_1(\underline{1}) = 1$ and $\mathcal{T}_2(\underline{0}) = \mathcal{T}_2(\underline{1}) = 0$.
(2) $\mathcal{T}_1(A \cap B) \geq \mathcal{T}_1(A) \wedge \mathcal{T}_1(B)$ and $\mathcal{T}_2(A \cap B) \leq \mathcal{T}_2(A) \vee \mathcal{T}_2(B)$.
(3) $\mathcal{T}_1(\bigcup A_i) \geq \bigwedge \mathcal{T}_1(A_i)$ and $\mathcal{T}_2(\bigcup A_i) \leq \bigvee \mathcal{T}_2(A_i)$.

The space $(X, \mathcal{T}) = (X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be an *intuitionistic fuzzy topological space in Šostak's sense*(SoIFTS for short). Also, we call $\mathcal{T}_1(A)$ a *gradation of openness* of A and $\mathcal{T}_2(A)$ a *gradation of nonopenness* of A.

DEFINITION 2.2. ([8]) Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r,s) \in I \otimes I$. Then A is said to be

(1) *fuzzy (r,s)-semiopen* if there is a fuzzy (r,s)-open set B in X such that $B \subseteq A \subseteq \mathrm{cl}(B,r,s)$,
(2) *fuzzy (r,s)-semiclosed* if there is a fuzzy (r,s)-closed set B in X such that $\mathrm{int}(B,r,s) \subseteq A \subseteq B$.

THEOREM 2.3. ([8]) Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r,s) \in I \otimes I$. Then the following statements are equivalent:

(1) A is a fuzzy (r,s)-semiopen set.
(2) A^c is a fuzzy (r,s)-semiclosed set.
(3) $\mathrm{cl}(\mathrm{int}(A,r,s),r,s) \supseteq A$.
(4) $\mathrm{int}(\mathrm{cl}(A^c,r,s),r,s) \subseteq A^c$.

DEFINITION 2.4. ([8]) Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a SoIFTS. For each $(r,s) \in I \otimes I$ and for each $A \in I(X)$, the *fuzzy (r,s)-semiclosure* is defined as

$$\mathrm{scl}(A,r,s) = \bigcap \{B \in I(X) \mid A \subseteq B, \\ B \text{ is fuzzy } (r,s)\text{-semiclosed}\}.$$

Definition 2.5. ([2]) A fuzzy set μ of a fuzzy topological space (X, τ) is said to be a *generalized closed fuzzy set* if and only if $\mathrm{cl}(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is an open fuzzy set.

3. Generalized fuzzy (r,s)-closed sets

We define the notion of generalized fuzzy (r,s)-closed sets on intuitionistic fuzzy topological spaces in Šostak's sense, and then we investigate some of their properties.

Definition 3.1. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r,s) \in I \otimes I$. Then A is said to be *generalized fuzzy* (r,s)*-closed* if $\mathrm{cl}(A, r, s) \subseteq B$ whenever $A \subseteq B$ and B is fuzzy (r,s)-open. The complement of a generalized fuzzy (r,s)-closed set is called *generalized fuzzy* (r,s)*-open.*

Theorem 3.2. Let A_1 and A_2 be intuitionistic fuzzy sets in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r,s) \in I \otimes I$. If A_1 and A_2 are generalized fuzzy (r,s)-closed sets, then $A_1 \cup A_2$ is a generalized fuzzy (r,s)-closed set.

Proof. Suppose that A_1 and A_2 are generalized fuzzy (r,s)-closed sets in X. Let B be a fuzzy (r,s)-open set in X such that $A_1 \cup A_2 \subseteq B$. Then $A_1 \subseteq B$ and $A_2 \subseteq B$. Since A_1 and A_2 are generalized fuzzy (r,s)-closed sets, we have $\mathrm{cl}(A_1, r, s) \subseteq B$, $\mathrm{cl}(A_2, r, s) \subseteq B$. Thus $\mathrm{cl}(A_1 \cup A_2, r, s) = \mathrm{cl}(A_1, r, s) \cup \mathrm{cl}(A_2, r, s) \subseteq B$. Hence $A_1 \cup A_2$ is generalized fuzzy (r,s)-closed. □

The following example shows that the intersection of two generalized fuzzy (r,s)-closed sets need not be a generalized fuzzy (r,s)-closed set.

Example 3.3. Let $X = \{x, y, z\}$ and let A_1, A_2, and A_3 be intuitionistic fuzzy sets in X defined as

$$A_1(x) = (0.7, 0.1),\ A_1(y) = (0.1, 0.7),\ A_1(z) = (0.1, 0.7);$$

$$A_2(x) = (0.7, 0.1),\ A_2(y) = (0.7, 0.1),\ A_2(z) = (0.1, 0.7);$$

and

$$A_3(x) = (0.7, 0.1),\ A_3(y) = (0.1, 0.7),\ A_3(z) = (0.7, 0.1).$$

Define $\mathcal{T} : I(X) \to I \otimes I$ by

$$\mathcal{T}(A) = (\mathcal{T}_1(A), \mathcal{T}_2(A)) = \begin{cases} (1,0) & \text{if } A = \underline{0}, \underline{1}, \\ (\frac{1}{2}, \frac{1}{3}) & \text{if } A = A_1, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a SoIFT on X. It is easy to see that A_2 and A_3 are generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed sets. But $A_2 \cap A_3$ is not generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed. For $A_2 \cap A_3 \subseteq A_1$ and A_1 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-open but $\mathrm{cl}(A_2 \cap A_3, \frac{1}{2}, \frac{1}{3}) = \underline{1} \nsubseteq A_1$.

THEOREM 3.4. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If A is a generalized fuzzy (r, s)-closed set and $A \subseteq B \subseteq \mathrm{cl}(A, r, s)$, then B is generalized fuzzy (r, s)-closed.

Proof. Let C be a fuzzy (r, s)-open set such that $B \subseteq C$. Since $A \subseteq B \subseteq C$ and A is generalized fuzzy (r, s)-closed, we have $\mathrm{cl}(A, r, s) \subseteq C$. Thus $\mathrm{cl}(B, r, s) \subseteq \mathrm{cl}(A, r, s) \subseteq C$. Hence B is generalized fuzzy (r, s)-closed. □

THEOREM 3.5. Let A be intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then A is generalized fuzzy (r, s)-open if and only if $B \subseteq \mathrm{int}(A, r, s)$ whenever $B \subseteq A$ and B is fuzzy (r, s)-closed.

Proof. Suppose that A is generalized fuzzy (r, s)-open. Let B be a fuzzy (r, s)-closed set such that $B \subseteq A$. Then $A^c \subseteq B^c$ and B^c is fuzzy (r, s)-open. Since A^c is generalized fuzzy (r, s)-closed, we have $\mathrm{cl}(A^c, r, s) \subseteq B^c$. Thus $B \subseteq \mathrm{int}(A, r, s)$.

Conversely, let B be a fuzzy (r, s)-open set such that $A^c \subseteq B$. Then $B^c \subseteq A$ and B^c is fuzzy (r, s)-closed. By the assumption, we have $B^c \subseteq \mathrm{int}(A, r, s)$. Thus $\mathrm{cl}(A^c, r, s) \subseteq B$, and hence A^c is generalized fuzzy (r, s)-closed. Therefore A is generalized fuzzy (r, s)-open. □

REMARK 3.6. (1) The union of two generalized fuzzy (r, s)-open sets need not be generalized fuzzy (r, s)-open. Example 3.3 serves the purpose.
(2) The intersection of any two generalized fuzzy (r, s)-open sets is generalized fuzzy (r, s)-open. Theorem 3.2 serves the purpose.

THEOREM 3.7. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If $\mathrm{int}(A, r, s) \subseteq B \subseteq A$ and A is generalized fuzzy (r, s)-open, then B is generalized fuzzy (r, s)-open.

Proof. Since $\mathrm{int}(A, r, s) \subseteq B \subseteq A$, we have $A^c \subseteq B^c \subseteq \mathrm{int}(A, r, s)^c = \mathrm{cl}(A^c, r, s)$. Since A is generalized fuzzy (r, s)-open, A^c is generalized fuzzy (r, s)-closed. So it follows by Theorem 3.4 that B^c is generalized fuzzy (r, s)-closed. Thus B is generalized fuzzy (r, s)-open. □

THEOREM 3.8. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be SoIFTSs and $(r, s) \in I \otimes I$. Let A be a generalized fuzzy (r, s)-closed set in X. If $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$

is fuzzy (r,s)-continuous and closed, then $f(A)$ is generalized fuzzy (r,s)-closed in Y.

Proof. If $f(A) \subseteq B$ where B is fuzzy (r,s)-open in Y, then $A \subseteq f^{-1}(B)$. Since A is generalized fuzzy (r,s)-closed and $f^{-1}(B)$ is fuzzy (r,s)-open, we have $\mathrm{cl}(A,r,s) \subseteq f^{-1}(B)$. Thus $f(\mathrm{cl}(A,r,s)) \subseteq B$. By the assumption, $f(\mathrm{cl}(A,r,s))$ is fuzzy (r,s)-closed, and hence

$$\begin{aligned} \mathrm{cl}(f(A),r,s) &\subseteq \mathrm{cl}(f(\mathrm{cl}(A,r,s)),r,s) \\ &= f(\mathrm{cl}(A,r,s)) \subseteq B. \end{aligned}$$

This means $f(A)$ is generalized fuzzy (r,s)-closed. □

DEFINITION 3.9. Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a SoIFTS. For each $(r,s) \in I \otimes I$ and for each $A \in I(X)$, the *generalized fuzzy (r,s)-closure* is defined by

$$\mathrm{gcl}(A,r,s) = \bigcap\{B \in I(X) \mid A \subseteq B,\ B \text{ is generalized fuzzy } (r,s)\text{-closed}\}.$$

If A is generalized fuzzy (r,s)-closed, then $\mathrm{gcl}(A,r,s) = A$. The converse is not true, because the intersection of generalized fuzzy (r,s)-closed sets need not be generalized fuzzy (r,s)-closed.

THEOREM 3.10. Let A be an intuitionistic fuzzy set in a SoIFTS $(X,\mathcal{T}_1,\mathcal{T}_2)$ and $(r,s) \in I \otimes I$. Then $A \subseteq \mathrm{gcl}(A,r,s) \subseteq \mathrm{cl}(A,r,s)$.

Proof. It is obvious. □

4. Semi-generalized fuzzy (r,s)-closed sets

DEFINITION 4.1. Let A be an intuitionistic fuzzy set in a SoIFTS $(X,\mathcal{T}_1,\mathcal{T}_2)$ and $(r,s) \in I \otimes I$. Then A is said to be *semi-generalized fuzzy (r,s)-closed* if $\mathrm{scl}(A,r,s) \subseteq B$ whenever $A \subseteq B$ and B is fuzzy (r,s)-semiopen. The complement of a semi-generalized fuzzy (r,s)-closed set is called *semi-generalized fuzzy (r,s)-open.*

The following example shows that the concepts of generalized fuzzy (r,s)-closed and semi-generalized fuzzy (r,s)-closed are independent.

EXAMPLE 4.2. Let $X = \{x,y\}$ and let A_1, A_2, A_3 and A_4 be intuitionistic fuzzy sets in X defined as

$$A_1(x) = (0,1),\ A_1(y) = (0.6,0.3);$$
$$A_2(x) = (0,1),\ A_2(y) = (0.3,0.6);$$
$$A_3(x) = (0.5,0.5),\ A_3(y) = (0.5,0.5);$$

and

$$A_4(x) = (0.5, 0.5),\ A_4(y) = (0.7, 0.2).$$

Define $\mathcal{T} : I(X) \to I \otimes I$ by

$$\mathcal{T}(A) = (\mathcal{T}_1(A), \mathcal{T}_2(A)) = \begin{cases} (1,0) & \text{if } \ A = \underline{0}, \underline{1}, \\ (\frac{1}{2}, \frac{1}{3}) & \text{if } \ A = A_1, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a SoIFT on X. It is easy to show that A_2 is a semi-generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed set. But A_2 is not generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed set. For $A_2 \subseteq A_1$ and A_1 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-open but $\mathrm{cl}(A_2, \frac{1}{2}, \frac{1}{3}) = A_1^c \nsubseteq A_1$. Furthermore, A_3 is generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed. However, it is not semi-generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed, because $A_3 \subseteq A_4$ and A_4 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiopen but $\mathrm{scl}(A_3, \frac{1}{2}, \frac{1}{3}) = \underline{1} \nsubseteq A_4$.

Remark 4.3. It is clear that every fuzzy (r, s)-semiclosed set is semi-generalized fuzzy (r, s)-closed. However, the following example shows that the converse need not be true.

Example 4.4. Let $X = \{x, y\}$ and let A_1 and A_2 be intuitionistic fuzzy sets in X defined as

$$A_1(x) = (0.3, 0.4),\ A_1(y) = (0, 1);$$

and

$$A_2(x) = (0, 1),\ A_2(y) = (0.3, 0.6).$$

Define $\mathcal{T} : I(X) \to I \otimes I$ by

$$\mathcal{T}(A) = (\mathcal{T}_1(A), \mathcal{T}_2(A)) = \begin{cases} (1,0) & \text{if } \ A = \underline{0}, \underline{1}, \\ (\frac{1}{2}, \frac{1}{3}) & \text{if } \ A = A_1, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a SoIFT on X. It is easy to see that A_2 is semi-generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed but not fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed.

The following example shows that the concepts of generalized fuzzy (r, s)-closed and fuzzy (r, s)-semiclosed are independent.

Example 4.5. Let $(X, \mathcal{T})$ be the SoIFTS as described in Example 4.2. Then A_2 is a fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed set. But A_2 is not a generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed set. Furthermore, A_3 is a generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed set but not fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed set.

THEOREM 4.6. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If A is semi-generalized fuzzy (r, s)-closed and $A \subseteq B \subseteq \mathrm{scl}(A, r, s)$, then B is semi-generalized fuzzy (r, s)-closed.

Proof. Let C be a fuzzy (r, s)-semiopen set such that $B \subseteq C$. Since $A \subseteq B \subseteq C$ and A is semi-generalized fuzzy (r, s)-closed, we have $\mathrm{scl}(A, r, s) \subseteq C$. Thus $\mathrm{scl}(B, r, s) \subseteq \mathrm{scl}(A, r, s) \subseteq C$. Hence B is semi-generalized fuzzy (r, s)-closed. □

THEOREM 4.7. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If $\mathrm{sint}(A, r, s) \subseteq B \subseteq A$ and A is semi-generalized fuzzy (r, s)-open, then B is semi-generalized fuzzy (r, s)-open.

Proof. It is obvious. □

THEOREM 4.8. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then A is semi-generalized fuzzy (r, s)-open if and only if $B \subseteq \mathrm{sint}(A, r, s)$ whenever $B \subseteq A$ and B is fuzzy (r, s)-semiclosed.

Proof. Let A be a semi-generalized fuzzy (r, s)-open set and B a fuzzy (r, s)-semiclosed set such that $B \subseteq A$. Then $A^c \subseteq B^c$. Since A^c is semi-generalized fuzzy (r, s)-closed, we have $\mathrm{scl}(A^c, r, s) \subseteq B^c$. Thus $B \subseteq \mathrm{sint}(A, r, s)$.

Conversely, let B be a fuzzy (r, s)-semiopen set such that $A^c \subseteq B$. Then $B^c \subseteq A$ and B^c is fuzzy (r, s)-semiclosed. By the assumption, we have $B^c \subseteq \mathrm{sint}(A, r, s)$, and hence $\mathrm{scl}(A^c, r, s) \subseteq B$. Thus A^c is semi-generalized fuzzy (r, s)-closed. Therefore A is semi-generalized fuzzy (r, s)-open. □

THEOREM 4.9. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a SoIFTS and $(r, s) \in I \otimes I$. Then the concepts of fuzzy (r, s)-semiclosed and fuzzy (r, s)-semiopen coincide if and only if every intuitionistic fuzzy set in X is semi-generalized fuzzy (r, s)-closed.

Proof. Let C be an intuitionistic fuzzy set in X such that $C \subseteq D$, where D is fuzzy (r, s)-semiopen. By the assumption, D is fuzzy (r, s)-semiclosed, and hence $\mathrm{scl}(C, r, s) \subseteq \mathrm{scl}(D, r, s) = D$. Thus C is semi-generalized fuzzy (r, s)-closed.

Conversely, let A be a fuzzy (r, s)-semiopen set. By the hypothesis, A is semi-generalized fuzzy (r, s)-closed. Thus $\mathrm{scl}(A, r, s) \subseteq A$. Hence A is fuzzy (r, s)-semiclosed. Next let B be a fuzzy (r, s)-semiclosed set. Then B^c is fuzzy (r, s)-semiopen. Since B^c is semi-generalized fuzzy

(r, s)-closed, it may be seen as before that B^c is fuzzy (r, s)-semiclosed, and hence B is fuzzy (r, s)-semiopen. □

The following example shows that the union of two semi-generalized fuzzy (r, s)-closed sets need not be a semi-generalized fuzzy (r, s)-closed set.

EXAMPLE 4.10. Let $X = \{x, y, z\}$ and let A_1, A_2, and A_3 be intuitionistic fuzzy sets in X defined as

$$A_1(x) = (0.6, 0.3),\ A_1(y) = (0, 1),\ A_1(z) = (0, 1);$$

$$A_2(x) = (0, 1),\ A_2(y) = (0.6, 0.3),\ A_2(z) = (0, 1);$$

and

$$A_3(x) = (0.6, 0.3),\ A_3(y) = (0.6, 0.3),\ A_3(z) = (0, 1).$$

Define $\mathcal{T} : I(X) \to I \otimes I$ by

$$\mathcal{T}(A) = (\mathcal{T}_1(A), \mathcal{T}_2(A)) = \begin{cases} (1, 0) & \text{if } A = \underline{0}, \underline{1}, \\ (\frac{1}{2}, \frac{1}{3}) & \text{if } A = A_1, A_2, A_3, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a SoIFT on X. Since A_1 and A_2 are fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed, A_1 and A_2 are semi-generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed sets. But $A_1 \cup A_2$ is not semi-generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed. For $A_1 \cup A_2 \subseteq A_3$ and A_3 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiopen but $\mathrm{scl}(A_1 \cup A_2, \frac{1}{2}, \frac{1}{3}) = \underline{1} \not\subseteq A_3$.

DEFINITION 4.11. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a SoIFTS. For each $(r, s) \in I \otimes I$ and for each $A \in I(X)$, the *semi-generalized fuzzy* (r, s)*-closure* is defined by

$$\mathrm{sgcl}(A, r, s) = \bigcap \{B \in I(X) \mid A \subseteq B, \\ B \text{ is semi-generalized fuzzy } (r, s)\text{-closed}\}.$$

THEOREM 4.12. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then $A \subseteq \mathrm{sgcl}(A, r, s) \subseteq \mathrm{scl}(A, r, s) \subseteq \mathrm{cl}(A, r, s)$.

Proof. Straightforward. □

5. Generalized fuzzy (r, s)-semiclosed sets

DEFINITION 5.1. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then A is said to be *generalized fuzzy (r, s)-semiclosed* if $\text{scl}(A, r, s) \subseteq B$ whenever $A \subseteq B$ and B is fuzzy (r, s)-open. The complement of a generalized fuzzy (r, s)-semiclosed set is called *generalized fuzzy (r, s)-semiopen*.

REMARK 5.2. It is clear that every generalized fuzzy (r, s)-closed set is generalized fuzzy (r, s)-semiclosed. However, the following example shows that the converse need not be true.

EXAMPLE 5.3. Let $X = \{x\}$ and let A_1 and A_2 be intuitionistic fuzzy sets in X defined as

$$A_1(x) = (0.1, 0.7), \; A_2(x) = (0.3, 0.6).$$

Define $\mathcal{T} : I(X) \to I \otimes I$ by

$$\mathcal{T}(A) = (\mathcal{T}_1(A), \mathcal{T}_2(A)) = \begin{cases} (1, 0) & \text{if } A = \underline{0}, \underline{1}, \\ (\frac{1}{2}, \frac{1}{3}) & \text{if } A = A_1, A_2, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a SoIFT on X. It is easy to see that A_2 is generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed. But A_2 is not generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed. For $A_2 \subseteq A_2$ and A_2 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-open but $\text{cl}(A_2, \frac{1}{2}, \frac{1}{3}) = A_2^c \not\subseteq A_2$.

THEOREM 5.4. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If A is semi-generalized fuzzy (r, s)-closed, then A is generalized fuzzy (r, s)-semiclosed.

Proof. Let B be a fuzzy (r, s)-open set such that $A \subseteq B$. Since B is fuzzy (r, s)-semiopen and A is semi-generalized fuzzy (r, s)-closed, we have $\text{scl}(A, r, s) \subseteq B$. Thus A is generalized fuzzy (r, s)-semiclosed. □

The following example shows that the converse of the above theorem need not be true.

EXAMPLE 5.5. Let $(X, \mathcal{T})$ be the SoIFTS as described in Example 4.2. Let B be an intuitionistic fuzzy set in X such that $B(x) = (0, 1), B(y) = (0.7, 0.2)$. Then B is generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed. However, B is not semi-generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-closed, because $B \subseteq A_4$ and A_4 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiopen but $\text{scl}(B, \frac{1}{2}, \frac{1}{3}) = \underline{1} \not\subseteq A_4$.

THEOREM 5.6. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If $A \subseteq B \subseteq \mathrm{scl}(A, r, s)$ and A is generalized fuzzy (r, s)-semiclosed, then B is generalized fuzzy (r, s)-semiclosed.

Proof. Similar to the proof of Theorem 4.6. □

THEOREM 5.7. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. If $\mathrm{sint}(A, r, s) \subseteq B \subseteq A$ and A is generalized fuzzy (r, s)-semiopen, then B is generalized fuzzy (r, s)-semiopen.

Proof. It is obvious. □

THEOREM 5.8. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then A is generalized fuzzy (r, s)-semiopen if and only if $B \subseteq \mathrm{sint}(A, r, s)$ whenever $B \subseteq A$ and B is fuzzy (r, s)-closed.

Proof. Similar to the proof of Theorem 4.8. □

The following example shows that the intersection of two generalized fuzzy (r, s)-semiclosed sets need not be a generalized fuzzy (r, s)-semiclosed set.

EXAMPLE 5.9. Let $(X, \mathcal{T})$ be the SoIFTS as described in Example 3.3. It is clear that A_2 and A_3 are generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed. But $A_2 \cap A_3 = A_1$ is not generalized fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiclosed, because $A_1 \subseteq A_1$ and A_1 is fuzzy $(\frac{1}{2}, \frac{1}{3})$-open but $\mathrm{scl}(A_1, \frac{1}{2}, \frac{1}{3}) = \underline{1} \not\subseteq A_1$.

DEFINITION 5.10. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a SoIFTS. For each $(r, s) \in I \otimes I$ and for each $A \in I(X)$, the *generalized fuzzy* (r, s)*-semiclosure* is defined by

$$\mathrm{gscl}(A, r, s) = \bigcap \{B \in I(X) \mid A \subseteq B, \\ B \text{ is generalized fuzzy } (r, s)\text{-semiclosed}\}.$$

If A is generalized fuzzy (r, s)-semiclosed, then $\mathrm{gscl}(A, r, s) = A$. The converse is not true, because the intersection of generalized fuzzy (r, s)-semiclosed sets need not be generalized fuzzy (r, s)-semiclosed.

THEOREM 5.11. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then $\mathrm{gscl}(A, r, s) \subseteq \mathrm{sgcl}(A, r, s)$.

Proof. The result follows from Theorem 5.4. □

By Theorem 4.12 and Theorem 5.11, we have the following result.

THEOREM 5.12. Let A be an intuitionistic fuzzy set in a SoIFTS $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(r, s) \in I \otimes I$. Then

$$\begin{aligned} A \subseteq \mathrm{gscl}(A, r, s) &\subseteq \mathrm{sgcl}(A, r, s) \\ &\subseteq \mathrm{scl}(A, r, s) \subseteq \mathrm{cl}(A, r, s). \end{aligned}$$

We have investigated three different types of closed sets in intuitionistic fuzzy topology. They have minor differences from each other. However, these differences will cause the difference in the continuity. In a later work, we will discuss the continuity.

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