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THE PSEUDO ORBIT TRACING PROPERTY AND EXPANSIVENESS ON UNIFORM SPACES

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Abstract. Uniform space is a generalization of metric space. The main purpose of this paper is to extend several results contained in [5, 6] which have for an expansive homeomorphism with the pseudo orbit tracing property(POTP in short) on a compact metric space (X, d) for an expansive homeomorphism with the POTP on a compact uniform space $(X,\mathcal{U})$.

we characterize stable and unstable sets, sink and source and saddle, recurrent points for an expansive homeomorphism which has the POTP on a compact uniform space $(X,\mathcal{U})$.

1. Introduction and preliminaries

In 1987, Jerzy Ombach had characterized results concering the class of expansive homeomorphisms having the pseudo orbit tracing property on a compact metric space (X, d). Uniform space $(X,\mathcal{U})$ is a generalization of metric space (X, d).

The main purpose of this paper is to extend some main theorems contained in [5] for an expansive homeomorphism with the POTP on a compact uniform space $(X,\mathcal{U})$.

We prove here Theorem 2.3, Theorem 3.10 and Theorem 4.1 for an expansive homeomorphism $f : (X,\mathcal{U}) \to (X,\mathcal{U})$ which has the POTP on a compact uniform space $(X,\mathcal{U})$.

We now introduce a definition of uniform space.

Definition 1.1. A uniform structure or uniformity $\mathcal{U}$ on a set X is a collection of subsets of $X \times X$ satisfying the following properties:

(1) Each member of $\mathcal{U}$ contains the diagonal $\triangle$.

(2) If $\alpha \in \mathcal{U}$ and $\alpha \subset \beta \subset X \times X$, then $\beta \in \mathcal{U}$.

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(3) If α and β are members of $\mathcal{U}$, then $\alpha \cap \beta$ is also a member of $\mathcal{U}$.
(4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} = \{(y, x) \,|\, (x, y) \in \alpha\} \in \mathcal{U}$.
(5) For any $\alpha \in \mathcal{U}$, there is $\beta \in \mathcal{U}$ such that $\beta^2 = \beta \circ \beta \subset \alpha$, where

$$\beta \circ \beta = \{(x, y) \,|\, \text{there exists } \; z \in X \, with (x, z) \in \beta, (x, y) \in \beta\}.$$

The set X equipped with a uniformity $\mathcal{U}$ is called a uniform space and an element of $\mathcal{U}$ called an entourage of X. An entourage α of X is called symmetric if $\alpha^{-1} = \alpha$.

REMARK 1.2. If X is a uniform space, then the topology $\Im$ on X induced by $\mathcal{U}$ is the collection of all subsets U of X such that for each $x \in U$ there is $\alpha \in \mathcal{U}$ with $\alpha[x] = \{y \,|\, (x, y) \in \alpha\} \subset U$, and a uniform space X is Hausdorff if and only if $\bigcap_{\alpha \in \mathcal{U}} \alpha \in \triangle_X$.

More details for uniform spaces is contained in [4].

DEFINITION 1.3. Let X be an uniform space and $f : X \to X$ be a homeomorphism. Let α and β be entourages of X. A sequence of points $\{x_n\}_{n \in \mathbb{Z}}$ is called a β-pseudo orbit for f if $(f(x_n), x_{n+1}) \in \beta$ for all in $n \in \mathbb{Z}$. A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is α-traced if there is a point $x \in X$ such that $(f^n(x), x_n) \in \alpha$ for all in $n \in \mathbb{Z}$.

The homeomorphism f has the *pseudo orbit tracing property*(POTP in short) if for any entourage α there is an entourage β such that any β-pseudo orbit is α-traced. A homeomorphism f is said to be *expansive*(EXPS in short) if there is an entourage α_f such that $(f^n(x), f^n(y)) \in \alpha_f$ for all $n \in \mathbb{Z}$ and $x, y \in X$ implies $x = y$.

2. Stable and unstable sets

To prove main theorems we need the following Lemma 2.1 and Lemma 2.2, which we give without proofs.

LEMMA 2.1. *Let $f : X \to X$ be a homeomorphism on uniform space X and let $n \in \mathbb{Z} - \{0\}$. Then $f : X \to X$ has POTP if and only if $f^n : X \to X$ has POTP.*

Also $f : X \to X$ is EXPS if and only if $f^n : X \to X$ is EXPS.

Let $f : X \to X$ be a homeomorphism on uniform space X. For $N > 0$, we denoted by

$$V_N = \{(x, y) \in X^2 \,|\, (f^n(x), f^n(y)) \in \alpha \, \text{for all} \, |n| \leq N \, \text{and entourage} \, \alpha\}$$

LEMMA 2.2. *Let a homeomorphism $f : X \to X$ be expansive on uniform space X. Then*

(1) for every entourage α, there is $N \in \mathbb{N}$ such that $V_N \subset \alpha$,

(2) for every $N \in \mathbb{N}$, there is entourage α such that $\alpha \subset V_N$.

Let (x_n) and (y_n) be sequences on uniform space X.

If for any entourage α, there exists $n \in \mathbb{N}$ such that $(x_n, y_n) \in \alpha$, then we denote $(x_n, y_n) \to 0$ as $n \to \infty$.

Let $f : X \to X$ be a homeomorphism on uniform space X. Let $x \in X$ and an entourage α.

We define the local stable set and local unstable set of x by

(a) $W^s_\alpha = \{y \in X \,|\, (f^n(x), f^n(y)) \in \alpha \text{ for all } n \geq 0\}$,

(b) $W^u_\alpha = \{y \in X \,|\, (f^n(x), f^n(y)) \in \alpha \text{ for all } n \leq 0\}$.

And also, for $x \in X$ the stable set and unstable set are defined by

(c) $W^s = \{y \in X \,|\, (f^n(x), f^n(y)) \to 0 \text{ for all } n \to \infty\}$,

(d) $W^u = \{y \in X \,|\, (f^n(x), f^n(y)) \to 0 \text{ for all } n \to -\infty\}$.

THEOREM 2.3. *Let X be a compact uniform space and $f : X \to X$ be a homeomorphism. Assume that f is an expansive with the pseudo orbit tracing property.*

Then there is entourage α_0 such that for every entourage $\alpha \subset \alpha_0$, there is entourage β and continuous function $h : \beta \to X$ such that

(A) *for every $x, y \in X$, $W^s_\alpha(x) \cap W^u_\alpha(y)$ consists at most one point ;*
for $(x, y) \in \beta$, $W^s_\alpha(x) \cap W^u_\alpha(y) = \{h(x, y)\}$,

(B) *for every $x \in X$,*
$W^s_\alpha(x) \cap \beta[x] = \{y \,|\, y = h(x, y), (x, y) \in \beta\}$,
$W^u_\alpha(x) \cap \beta[x] = \{y \,|\, y = h(y, x), (x, y) \in \beta\}$,

(C) $W^s_\alpha(x) \subset W^s(x), W^u_\alpha(x) \subset W^u$,

(D) $W^s(x) = \cup_{n=0}^{\infty} f^{-n}(W^s_\alpha(f^n(x)))$,
$W^u(x) = \cup_{n=0}^{\infty} f^{n}(W^u_\alpha(f^{-n}(x)))$.

Proof. Let α_f be entourage for an expansive homeomorphism f. There exists entourage α_0 such that $\alpha_0^3 \subset \alpha_f$.

Let $\alpha \subset \alpha_0$. By the uniform continuity of f, there exists entourage $\beta_0 \subset \alpha$ such that $(x, y) \in \beta_0$ implies $(f(x), f(y)) \in \alpha$.

Since $f \in$ POTP, there is entourage β such that every β-pseudo orbit is β_0-traced. Fix $(x, y) \in \beta$.

Consider a sequence $\{x_n\}$ defined as

$$x_n = \begin{cases} f^n(y) & \text{for } n < 0 \\ f^n(x) & \text{for } n \geq 0, \end{cases}$$

which is a β-pseudo orbit.

By the POTP of f, it is β_0-traced by some point $p \in X$.

We claim that there is only one such point. Suppose that p and p' are two such points. From $(f^n(p), x_n), (f^n(p'), x_n) \in \beta$ for all $n \in \mathbb{Z}$, we have $(f^n(p), f^n(p')) \in \beta_0^2 \subset \alpha^2 \subset \alpha_0^2 \subset \alpha_0^3 \subset \alpha_f$ for all $n \in \mathbb{Z}$. Thus we get $p = p'$.

Next, define a map $h : \beta \to X$ by $h(x, y) = p$ for each $(x, y) \in \beta$.

For each $(x, y) \in \beta$, we obtain the following

$$(f^n(x), f^n(h(x, y))) \in \beta_0 \subset \alpha \, \text{for all} \, n \geq 0,$$

$$(f^n(y), f^n(h(x, y))) \in \beta_0 \subset \alpha \, \text{for all} \, n < 0.$$

In particular, $(f^{-1}(y), f^{-1}(h(x, y))) \in \beta_0$. By the choice of β_0, we get $(y, h(x, y)) \in \alpha$. This means that $(f(f^{-1}(y)), f(f^{-1}(h(x, y)))) = (y, h(x, y)) \in \alpha$. Hence $(f^n(y), f^n(h(x, y))) \in \alpha$ for all $n \leq 0$, so that $h(x, y) \in W^s_\alpha(x) \cap W^u_\alpha(y)$.

We claim that $h(x, y) = p$ is unique in $W^s_\alpha(x) \cap W^u_\alpha(y)$.

Assume that $p, p' \in W^s_\alpha(x) \cap W^s_\alpha(y)$ with $h(x, y) = p, h(x, y) = p'$. Then for every $n \geq 0$, by $(f^n(p), f^n(x)) \in \alpha^{-1} = \alpha$ and $(f^n(x), f^n(p')) \in \alpha$, we get $(f^n(p), f^n(p')) \in \alpha^2 \subset \alpha_0^2 \subset \alpha_0^3 \subset \alpha_f$. Also for every $n \leq 0$, by $(f^n(p), f^n(y)) \in \alpha^{-1} = \alpha$ and $(f^n(y), f^n(p')) \in \alpha$, we have $(f^n(p), f^n(p')) \in \alpha^2 \subset \alpha_0^2 \subset \alpha_0^3 \subset \alpha_f$. This means that $p = p'$ by expansiveness.

To prove the continuity of the mapping h we use Lemma 2.1. For each neighborhood U of $h(x, y)$, there exists entourage γ such that $\gamma[h(x, y)] \subset U$. By Lemma 2.2, we get $V_N \subset \gamma$ for some $N \in \mathbb{N}$. Since $f^{-N}, \cdots, f^N$ are uniformly continuous, there is entourage δ such that $(U, V) \in \delta$ implies $(f^k(U), f^k(V)) \in \gamma$ for all $|k| \leq \mathbb{N}$.

Let $(x', y') \in \delta[x] \times \delta[y]$. From $(x, x') \in \delta$ and $(y, y') \in \delta$, we obtain $(f^n(x), f^n(x')) \in \gamma$ for all $0 \leq n \leq N$ and $(f^n(y), f^n(y')) \in \gamma$ for all $-N \leq n \leq 0$.

By $(f^n(h(x, y)), f^n(x)) \in \alpha^{-1} = \alpha$ and $(f^n(x'), f^n(h(x', y'))) \in \alpha$ for all $n \geq 0$, it follows that $(f^n(h(x, y)), f^n(h(x', y'))) \in \alpha^3 \subset \alpha_0^3 \subset \alpha$ for all $0 \leq n \leq N$.

Also by $(f^n(h(x, y)), f^n(y)) \in \alpha^{-1} = \alpha$ and $(f^n(y'), f^n(h(x', y'))) \in \alpha$ for all $n \geq 0$, it follows that $(f^n(h(x, y)), f^n(h(x', y'))) \in \alpha^3 \subset \alpha_0^3 \subset \alpha$ for all $-N \leq n \leq 0$.

Thus $(f^n(h(x, y)), f^n(h(x', y')) \in \alpha_f$ for all $|n| \leq N$, so that

$(h(x, y), h(x', y')) \in V_N \subset \gamma$. This means that $h(x', y') \in \gamma[h(x, y)] \subset U$, i.e., h is continuous. Therefore (A) is proved.

To show (B), let us note first $y = h(x, y) \in W^s_\alpha(x)$ and for $y \in W^s_\alpha(x)$, we know $y \in W^s_\alpha(x) \cap W^u_\alpha(y)$. From $(x, y) \in \beta$, it follows that $y = h(x, y)$. The proof of the remaining case is similar.

Next we prove (C). Let $y \in W^s_\alpha(x)$. For each entourage γ, there exists $N \in \mathbb{N}$ such that $V_N \subset \gamma$ by Lemma 2.1. Put $n \geq N$. From $k + n \geq 0$ for all $|k| \leq N$, it follows that $(f^k(f^n(x)), f^k(f^n(y))) = (f^{k+n}(x), f^{k+n}(y)) \in \alpha \subset \alpha_f$ for all $|k| \leq N$. Thus $(f^n(x), f^n(y)) \in V_N \subset \gamma$ for all $n \geq N$, so that $(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. Consequently, we conclude that $y \in W^s(x)$. By similar method, we get $W^u_\alpha(x) \subset W^u(x)$.

Finally, the proof of (D) is trivial. □

3. Sink, source and saddle

In this section we characterize three types(sink, source and saddle) of behaviour of orbits near a point on a compact uniform space $(X, \mathcal{U})$

We assume in the sequel that an expansive homeomorphism $f : X \to X$ which has the POTP on a compact uniform space $X = (X, \mathcal{U})$ and $\alpha \subset \alpha_0$, where α_0 is given satisfying the Theorem 2.3.

Define maps L^+, L^- from X into 2^X by setting for each $x \in X$,

$$L^+(x) = \{y \mid f^{n_k}(x) \to y \text{ for some } n_k \to \infty\},$$

$$L^-(x) = \{y \mid f^{n_k}(x) \to y \text{ for some } n_k \to -\infty\}.$$

For each $x \in X$, the set $L^+(x)$ is called its positive limit set, and the set $L^-(x)$ is called its negative limit set.

PROPOSITION 3.1. *If* $\mathrm{int}W^s(x) \neq \emptyset$, *then* $x \in \mathrm{int}W^s(x)$.

Proof. Let $y \in \mathrm{int}W^s(x)$. There exists entourage α_1 such that $\alpha_1^2 \subset \alpha$, $\alpha_1[y] \subset W^s(x)$, where α is given in Theorem 2.3.

By the POTP of f, every β-pseudo orbit is α-traced for some entourage β. Since f is uniformly continuous, there exists entourage β_0 such that

(1)
$$(u, v) \in \beta_0 \text{ implies } (f(u), f(v)) \in \beta.$$

Choose entourage γ such that $\gamma^2 \subset \beta_0$. By $y \in W^s(x)$, there is $N \in \mathbb{N}$ such that $(f^n(x), f^n(y)) \in \gamma$ for all $n > N$.

By the continuity of f^N, there exists neighborhood U of x such that $f^N(x') \in \gamma[f^N(x)]$ for all $x' \in U$. We claim that $U \subset W^s(x)$. Let $x' \in U$. Since $(f^N(x'), f^N(x)) \in \gamma^{-1}$ and $(f^N(x), f^N(y)) \in \gamma$, we have $(f^N(x'), f^N(y)) \in \gamma^2 \subset \beta_0$.

By (1), the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} f^n(y) & \text{for } n \leq N \\ f^n(x) & \text{for } n > N, \end{cases}$$

is a β-pseudo orbit. Hence it is α_1-traced by some point $y' \in X$. This implies in particular.

(2)

$$(y', y) \in \alpha,$$

(3)

$$(f^n(y'), f^n(x')) \in \alpha_1 \text{ for all } n > N.$$

(2) implies $y' \in \alpha_1[y] \subset W^s(x)$ and there is N' such that $(f^n(y'), f^n(x)) \in \alpha_1$ for all $n \geq N'$ which together with (3) imply $(f^n(x), f^n(x')) \in \alpha_1^2 \subset \alpha$ for all $n \geq M$, where $M = \max\{N, N'\}$. Thus $x' \in f^{-M}(W^s_\alpha(f^M(x))) \subset W^s(x)$. Hence the proof is completed. □

PROPOSITION 3.2. *Let α be any entourage. If $x \in \text{int}W^s(x)$, then $x \in \text{int}W^s_\alpha(x)$.*

Proof. Let $x \in \text{int}W^s(x)$ and α be entourage. It is enough to have $N \in \mathbb{N}$ and a neighborhood U_1 of x such that $(f^n(x), f^n(y)) \in \alpha$ for all $n > N$ and all $y \in U_1$.

Since f^n are continuous for all $0 \leq n \leq N$, there exists neighborhood U_2 of x such that $(f^n(x), f^n(y)) \in \alpha$ for all $0 \leq n \leq N$ and all $y \in U_2$.

Then $U = U_1 \cap U_2$ is a neighborhood of x and $U \subset W^s_\alpha(x)$. Assume that is not true, that is,

(4) for any number $n \in \mathbb{N}$ and any neighborhood U of x there is entourage α and $y \in U$ such that $(f^n(x), f^n(y)) \notin \alpha$.

By $x \in \text{int}W^s(x)$, there exists entourage α_1 such that $\alpha_1^2 \subset \alpha$ and $\alpha_1[x] \subset W^s(x)$. From the POTP of f, there is entourage β such that every β-pseudo orbit is α_1-traced.

Since f is uniformly continuous, there exists entourage γ such that $(u, v) \in \gamma$ implies $(f(u), f(v)) \in \beta$. □

To complete the proof of Proposition 3.2, we need the following.

LEMMA 3.3. *There is a β-pseudo orbit (z_n) sequences $(l_i)_{i=1}^\infty$, $(n_i)_{i=2}^\infty$ in $\mathbb{N}$ with $0 = l_1 < n_2 < l_2 < n_3 < \cdots$ such that*

(5)

$$(f^{n_i}(x), z_{n_i}) \notin \alpha,$$

(6)

$$z_{l_i} = f^{l_i}(x) \text{ for all } i.$$

Proof. Define $z_n = f^n(x)$ for $n \leq 0$ and Put $l_1 = 0$. Assume that we have already defined numbers n_i, l_i for $i \leq j$ and points z_n for $n \leq l_j$ such that (5) and (6) hold and $(f^n(z_n), z_{n+1}) \in \beta$ for $n < l_j$.

Since f^{l_j} is continuous, there is a neighborhood $U \subset W^s(x)$ of x such that $(f^{l_i}(x), f^{l_i}(p)) \in \gamma$ for all $p \in U$. By (4), $(f^{n_{j+1}}(x), f^{n_{j+1}}(y)) \notin \alpha$ for some $n_{j+1} > l_i$ and for some $y \in U$. Because $y \in W^s(x)$, $(f^{l_{j+1}}(x), f^{l_{j+1}}(y)) \in \gamma$ for some $l_{j+1} > n_{j+1}$.

Put $z_n = f^n(y)$ for $l_j < n < l_{j+1}$ and $z_{l_{j+1}} = f^{l_{j+1}}(x)$. Since $(f^{l_j}(x), f^{l_j}(y)) \in \gamma$, we have $(f(f^{l_j}(x)), f(f^{l_j}(y))) = (f(z_{l_j}), f^{l_{j+1}}(y)) = (f(z_{l_j}), z_{l_{j+1}}) \in \beta$.

Also $(f^{l_{j+1}-1}(x), f^{l_{j+1}-1}(y)) \in \gamma$ implies $(f(f^{l_{j+1}-1}(y)), f(f^{l_{j+1}-1}(x))) = (f(z_{l_{j+1}-1}), f^{l_{j+1}}(x)) = (f(z_{l_{j+1}-1}), z_{l_{j+1}}) \in \beta$.

Therefore (5) and (6) are also satisfied for $j+1$. Thus the Lemma is proved. □

Now to finish the proof of Proposition 3.2, apply to Lemma 3.3. Take a point x', α_1-tracing the β-pseudo orbit (z_n). Then we get $(f^n(x'), z_n) \in \alpha_1$ for all $n \in \mathbb{Z}$.

In Particular, $(x', z_0) = (x', x) \in \alpha_1$. By $x' \in \alpha_1[x] \subset W^s(x)$, we get $(f^n(x'), f^n(x)) \to 0$ as $n \to \infty$. Since $(f^{n_i}(x), z_{n_i}) \in \alpha_1$, we conclude that $(f^{n_i}(x), z_{n_i}) \subset \alpha_1^2 \subset \alpha$.

We have a contradiction to the fact that $(f^{n_i}(x), z_{n_i}) \notin \alpha$ by (5). This contradiction proves that $x \in \text{int}W^s_\alpha(x)$.

PROPOSITION 3.4. *Let α_0, α_1 be entourages with $\alpha_1^4 \subset \alpha$ on compact uniform space X. If $x \in \text{int}W^s_{\alpha_1}(x)$, then $L^+(x)$ is a periodic orbit which equal to $\{y, f(y), \cdots, f^{k-1}(y)\}$ and $y \in \text{int}W^s(x)$ for all $y \in L^+(x)$.*

Proof. Let $x \in \text{int}W^s_{\alpha_1}(x)$. There exists entourage $\alpha_2 \subset \alpha_1$ such that $\alpha_2[x] \subset W^s_{\alpha_1}(x) \subset W^s(x)$.

By the compactness of X, it follows that $L^+(x) \neq \emptyset$. Also, it is known that $L^+(y) \subset L^+(x)$ for all $y \in L^+(x)$.

We claim that $L^+(y) = L^+(x)$ for all $y \in L^+(x)$. Let $z \in L^+(x)$. For each neighborhood U of z, it follows that $\beta_0[z] \subset U$ for some entourage $\beta_0 \subset \alpha_0$. Choose entourage β_1 such that $\beta_1^3 \subset \beta_0$. By $f \in POTP$, there is entourage γ such that every γ-pseudo orbit is β_1-traced.

Also, $y \in L^+(x)$ implys $(f^N(x), y) \in \gamma$ for some $N \in \mathbb{N}$.

Let us defined by

$$z_n = \begin{cases} f^n(x) & \text{for } n < N \\ f^{n-N}(y) & \text{for } n \geq N. \end{cases}$$

Since a sequence (z_n) is a γ-pseudo orbit, (z_n) is β_1-traced by some point $x' \in X$. Because $(x', z_0) = (x', x) \in \beta$, it follows that $x' \in \beta_1[x] \subset \beta_0[x] \subset \alpha_2[x] \subset W^s(x)$.

So we get the following, respectively,

$$(f^n(x), f^n(x')) \in \beta \text{ for large } n,$$

$$(f^n(x), z) \in \beta_1 \text{ for infinitely many } n, \text{ and}$$

$$(f^n(x')), f^{n-N}(y)) \in \beta_1 \text{ for } n \geq N.$$

Thus $(f^n(x), z) \in \beta_1^3 \subset \beta_0$, so $\beta^n(y) \in \beta_0[z] \subset U$ for infinitely many n. Therefore we conclude that $z \in L^+(y)$.

Next we shall show that for each $y \in L^+(x)$, $y \in \text{int}W^s(y)$. Let $y \in L^+(x)$. Since f satisfy POTP, there exists entourage β such that every β-pseudo orbit is α_2-traced. Choose entourage γ such that $\gamma^2 \subset \beta$. Also $y \in L^+(x)$ implies $(f^N(x), y) \in \gamma$ for some $N \in \mathbb{N}$.

To show that $\gamma[y] \subset W^s(y)$, let $z \in \gamma[y]$. We define two sequence $(y_n), (z_n)$ by the following

$$y_n = \begin{cases} f^n(x) & \text{for } n < N \\ f^{n-N}(y) & \text{for } n \geq N \end{cases} \text{ and } z_n = \begin{cases} f^n(x) & \text{for } n < N \\ f^{n-N}(x) & \text{for } n \geq N. \end{cases}$$

Then this sequences are both β-pseudo orbits, and they are α_2-traced by points $y', z' \in X$, respectively. By $(y', y_0) = (y', x) \in \alpha_2$ and $(z', z_0) = (z', x) \in \alpha_2$, we have $y', z' \in \alpha_2[x] \subset W^s_{\alpha_1}(x)$.

we obtain for $n > 0$, respectively, the following

$$(f^{n+N}(z'), z_{n+N}) \in \alpha_2 \subset \alpha_1,$$

$$(f^{n+N}(x), f^{n+N}(z')) \in \alpha_1,$$

$$(f^{n+N}(x), f^{n+N}(y')) \in \alpha_1, \text{ and}$$

$$(f^{n+N}(y), y_{n+N}) \in \alpha_2 \subset \alpha_1.$$

So we have $(f^n(y), f^n(z)) = (y_{n+N}, z_{n+N}) \in \alpha_1^4 \subset \alpha$.

Thus $z \in W^s_\alpha(y) \subset W^s(y)$. Hence since $\gamma[y] \subset W^s(y)$, we obtain that $y \in \text{int}W^s(y)$.

By Proposition 3.1, any neighborhood of a limit point contains a periodic point. Let $p \in W^s(y)$ be a periodic point. We claim that $p = y$. For $p \neq y$ we would find another periodic point $q \in W^s(x), q \neq p$. Hence $W^s(p) = W^s(y) = W^s(q)$.

So $(f^n(p), f^n(q)) \to 0$ as $n \to \infty$, which is impossible. Thus we conclude that $L^+(x) = L^+(y) = \{y, f(y), \cdots, f^{k-1}(y)\}$, where $k > 0$ is a periodic of y. □

Proposition 3.5. *Let α be an entourage. $x \in \mathrm{int} W^s_\alpha(x)$ if and only if $W^s_\alpha(x) \cap H = \{x\}$ for some neighborhood H of x.*

Proof. Assume that $x \in \mathrm{int} W^s_\alpha(x)$. Take a neighborhood H of x such that $H \subset W^s_\alpha(x) \cap \beta[x]$, where β is chosen in terms of α as in Theorem 2.3. For $y \in W^s_\alpha \cap H$, We have $y \in W^s_\alpha(x) \cap W^u_\alpha(x) = \{h(x,x)\} = \{x\}$.

Suppose that $W^u_\alpha(x) \cap H = \{x\}$ for some neighborhood H of x.

The continuity of the mapping $h : \beta \to X$ in Theorem 2.3 provides a neighborhood U of x such that $y \in U$ implies $h(x,y) \in H$.

We may assume that α is symmetric. Since $x \in W^s_\alpha(y)$, we have $y \in W^s_\alpha(x)$ and so $U \subset W^s_\alpha(x)$. Thus $x \in \mathrm{int} W^s_\alpha(x)$. □

Proposition 3.6. *Let α, α_f be an entourages. Let $x \in \mathrm{int} W^s_\alpha(x)$ be a periodic point with periodic k and let $\alpha \subset \alpha_f$ (see Lemma 2.1). Then there is a neighborhood U of x such that*

(7)
$$f^k(U) \subset U,$$

(8)
$$\bigcap_{n=0}^{\infty} f^{n_k}(U) = \{x\}.$$

Proof. Let $x \in U_1 \subset \mathrm{int} W^s_\alpha(x)$ for some open U_1. For every $n \geq 0$, it follows that $f^{n_k}(U_1) \subset \alpha[x]$.

Define $U = \bigcup_{n=0}^{\infty} f^{n_k}(U_1)$. Then U is a neighborhood of x and (7) holds true. Furthermore, $U \subset \alpha[x]$. Let $x' \in \bigcap_{n=0}^{\infty} f^{n_k}(U_1)$. Then in particular $x' \in U$ and by (7), $x' \in f^{n_k}(U)$ for any $n < 0$. So for any $n \in \mathbb{Z}$, $x' \in f^{n_k}\alpha[x]$ and hence for all n, $(f_{n_k}(x'), (f_{n_k}(x)) = (f_{n_k}(x'), x) \in \alpha \subset \alpha_{f^k}$. So $x' = x$, and (8) is proved. □

The following theorem 3.7 summarizes and completes all previous results.

Theorem 3.7. *Let α, α_0 be entourages and $y \in L^+(x)$ for $x \in X$. Then for a point $x \in X$, the following are equivalent.*

(AS) $\mathrm{int} W^s(x) \neq \emptyset$,
(BS) $x \in \mathrm{int} W^s(x)$,
(CS) (x) *is open,*
(DS) $x \in \mathrm{int} W^s_\alpha(x)$,
(ES) $\mathrm{int} W^s_\alpha(x) \neq \emptyset$,
(FS) $\mathrm{int} W^u_\alpha(x) \cap H = \{x\}$ *for some neighborhood H of x,*
(GS) $\mathrm{int} W^u_\alpha(y) = \{y\}$,
(HS) $\mathrm{int} W^u(y) = \{y\}$,

(IS) $L^+(x) = \{y, f(y), \cdots, f^{k-1}(y)\}$ *is a periodic orbit.*
There is a neighborhood U *of* y *such that*
(9) $f^k(U) \subset U$,
(10) $\bigcap_{n=0}^{\infty} f^{n_k}(U) = \{x\}$.

Proof. (AS) $\Longrightarrow$ (BS) by Proposition 3.1.

To see that (BS) $\Longrightarrow$ (CS) consider a point $z \in W^s(x)$. Then $W^s(z) = W^s(x)$ contains x in its interior. By Proposition 3.1, $z \in \mathrm{int}W^s(z) = \mathrm{int}W^s(x)$ and hence $W^s_\alpha(x)$ is open. (CS) $\Longrightarrow$ (DS) Proposition 3.2.

(DS) $\Longrightarrow$ (ES) is trivial.

(ES) $\Longrightarrow$ (AS) by Theorem 2.3 (C).

(DS) $\Longrightarrow$ (FS) by Proposition 3.4.

Thus we have proved equivalences among conditions (AS) to (FS).

Now we shall prove (AS) $\Longrightarrow$ (IS). By Proposition 3.2, $x \in \mathrm{int}W^s_{\alpha_0}(x)$. By Proposition 3.2, $y \in \mathrm{int}W^s(y)$ and y is periodic. Once again by Proposition 3.2, $y \in \mathrm{int}W^s_\alpha(y)$ for small α_1. Proposition 3.5 completes the proof.

To see (IS) $\Longrightarrow$ (HS), take some $z \neq y$. Then for all n large enough, $z \notin f^{n_k}(U)$, as (9) implies the sequence $f^{n_k}(U)$ decrease. So $f^{-n_k}(z) \notin U$ for such n and $z \notin W^u(y)$. Hence $W^u(y) = \{y\}$.

(HS) $\Longrightarrow$ (GS) by Theorem 2.3 (C).

We prove (GS) $\Longrightarrow$ (DS). $W^u_\alpha(y) = \{y\}$ implies $W^u_\beta(y) = \{y\}$, where $\beta^2 \subset \alpha$ and by Proposition 3.4, $y \in \mathrm{int}W^s_\beta(y) = \{y\}$. As $y \in L^+(x)$, then there is a number N such that $f^N(x) \in \mathrm{int}W^s_\beta(y)$.

The continuity of f^n, $0 \leq n \leq N$, provides a neighborhood U of x such that for $z \in U$, $(f^n(x), f^n(z)) \in \alpha$ for $0 \leq n \leq N$ and $f^N(z) \in W^s_\beta(y)$. It is clear that $U \subset W^s_\alpha(x)$. So (DS) is true. The proof of Theorem 3.7 complete. □

Applying Theorem 3.7 to the inverseof f we immediately get.

Theorem 3.8. *Let* $x \in X$ *and let* α, α_0 *be entourages with* $\alpha \subset \alpha_0$, *and* $y \in L^-(x)$. *Then the following are equivalent.*
(AU) $\mathrm{int}W^u(x) \neq \emptyset$,
(BU) $x \in \mathrm{int}W^u(x)$,
(CU) $W^u(x)$ *is open,*
(DU) $x \in \mathrm{int}W^u_\alpha(x)$,
(EU) $\mathrm{int}W^u_\alpha(x) \neq \emptyset$,
(FU) $\mathrm{int}W^s_\alpha(x) \cap H = \{x\}$ *for some neighborhood* H *of* x,
(GU) $\mathrm{int}W^s_\alpha(y) = \{y\}$,
(HU) $\mathrm{int}W^s(y) = \{y\}$,

(IU) $L^{-}(x) = \{y, f(y), \cdots, f^{k-1}(y)\}$ *is a periodic point,*
There is a neighborhood U *of* y *such that*
$f^{-n}(U) \subset U$*, and* $\bigcap_{n=0}^{\infty} f^{-n_k}(U) = \{y\}$.

Assume in the sequel that x is a non-isolated point in the space X and $y \in L^{+}(X)$. Let us note the situations described in Theorem 3.7 and Theorem 3.8 exclude each other. To see this, compare conditions (DS) and (FU) . There is just on other interesting situation.

It is enough to combine the negation of a condition from (AS) to (IS) with the negation of a condition from (AU) to (IU) to get the description of this situation.

Taking advantage of (FS) to (FU), for example, we get

(Y)

any neighborhood of x contains points

$$x_S \in W^s_\alpha(x) - \{x\},\ x_U \in W^u_\alpha(x) - \{x\},\ x_S \neq x_U.$$

By Theorem 2.3, $x_S \neq x_U$ implies $W^s_\alpha(x) \cap W^u_\alpha(x) = \{x\}$.

LEMMA 3.9. *If a point x is periodic, then condition (Y) is equivalent to*

(X)

any neighborhood of x contains points

$$x_S \in W^s(x) - \{x\},\ x_U \in W^u(x) - \{x\}, x_S \neq x_U.$$

Proof. $(Y) \Longrightarrow (X)$ by Theorem 2.3 (C).

To see $(Y) \Longrightarrow (X)$, take advantage of (HS) $\Longleftrightarrow$ (FS) and (HU) $\Longleftrightarrow$ (FU) in the case $y = x \in L^{+}(x) = L^{-}(x)$. □

Now, it makes sense to call a periodic point a sink, if the situation described in Theorem 3.7 occurs, a source, if the situation described in Theorem 3.8 occurs, a saddle, if the situation described in Lemma 3.9 occurs.

Similarities to the hyperbolic case are clear. Thus we have

THEOREM 3.10. *Any periodic point is either a sink or a source or a saddle.*

4. Recurrent points

Let $(X,\mathcal{U})$ be an uniform space and $f : X \to X$ be a homeomorphism. A *fixed point* of f is a point x such that $f(x) = x$; a *periodic point* is a fixed point for an iterate of f. we denote by $Per(f)$ the set of periodic points.

The *negative and positive limit set* of f are defined by $L^-(f) = \bigcup_{x\in X} L^-(x)$ and $L^+(f) = \bigcup_{x\in X} L^+(f)$.

The *nonwandering set* is defined by $\Omega(f) = \{x \in X \mid$ for all neighborhood U of x, there exists $n \neq 0$ such that $f^n(U) \cap U \neq \emptyset\}$.

And the *chain recurrent set* is defined by $CR(f) = \{x \in X \mid$ for all entourage α, there exists α-chain from x to itself $\}$, where an α-chain from x tor itself is a finite α-pseudo orbit $\{x_0, \cdots, x_n\}$, $x = x_0 = x_n, n > 0$.

Then by definition $\Omega(f)$ and $CR(f)$ are closed invariant sets, and $Per(f) \subset L^-(f) \cap L^+(f) \subset L^-(f) \cup L^+(f) \subset \Omega(f) \subset CR(f)$.

THEOREM 4.1. *Let $(X,\mathcal{U})$ be compact uniform space. Let f be an expansive homeomorphism for which f has POTP. Then*

$$\overline{Per(f)} = \overline{L^-(f)} = \overline{L^+(f)} = \Omega(f) = CR(f).$$

Proof. We must show that $CR(f) \subset \overline{Per(f)}$. Let $x \in CR(f)$ and entourage α_f for expansiveness of f. Fixed a entourage α such that $\alpha^2 \subset \alpha_f$

POTP provided a β-corresponding to this α, i.e., for any entourage α there is entourage β such that any β-pseudo orbit is α-traced. There is a β-chain $\{x_1, x_2, \cdots, x_N\}$ from x to itself.

The sequence (x'_n), $n \in \mathbb{Z}$ defined by $x'_n = x_i$ if $n = i(\mathrm{mod}\,\mathrm{N})$ is a β-pseudo orbit.

It is α-traced by some point $y \in X$, but also by $f^N(y)$. By $(f^n(y), x'_n) \in \alpha$ and $(f^n(f^N(y)), x'_n)) \in \alpha$ for $n \in \mathbb{Z}$, it follows that $(f^n(F^N(y)), f^n(y)) \in \alpha^2 \subset \alpha_f$.

Thus $f^N(y) = y$ and so $y \in Per(f)$. Since $(y, x'_0) = (y, x) \in \alpha$, we get $y \in \alpha[x]$. Consequently, we conclude that $x \in \overline{Per(f)}$.

□

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