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MODIFIED KOSZUL COMPLEXES IN A QUANTUM SPACE RING

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Abstract. In this article, we define a modified Koszul complex, which we call a quantized Koszul complex, on a quantum space ring, and we also prove that it is an acyclic complex.

1. Backgrounds and preliminaries

In [2], Koh explained how some of properties of the sheaf cohomology on the projective schemes could be understood from some properties of graded modules; Serre Duality was derived as a consequence of the graded version of the Local Duality in the polynomial ring. He also introduced a quantum space ring $R = k[x_1, \cdots, x_n]_{q_{ij}}$ (here, k is a field,and $q_{ij} \in k - \{0\}$), and claimed that 'Serre Duality for R' would hold by a similar argument: $H^i(X, M) \cong Hom(Ext^{n-i}(M^{\sim}, D^{\sim}), k)$ for all $i \geq 0$ where $D = R(-n)$, and M is a finitely generated R-module ([3]).

In this article, we don't establish Serre Duality for R, but we modify a Koszul complex which is known to be an important tool for understanding local cohomology modules: Local cohomology modules can be explained as limits of Koszul complexes ([1,6]).

We first recall the definition of a Koszul complex ([5]). Let A be a ring and $x_1, \cdots, x_n \in A$. We define a complex $K_\bullet$ as follows: Let $K_0 = A$, and for $1 \leq p \leq n$, $K_p = \oplus A_{e_{i_1 \cdots i_p}}$ be the free A-module of rank $\binom{n}{p}$ with a basis $\{e_{i_1 \cdots i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$. The p-th differential map $d : K_p \to K_{p-1}$ is defined by

$$d(e_{i_1 \cdots i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1 \cdots \hat{i_r} \cdots i_p}$$

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(for $p = 1$, $d(e_i) = x_i$). This complex is called the Koszul complex, denoted by $K_\bullet(x_1, \cdots, x_n)$.

EXAMPLE 1.1. *For $x_1, x_2, x_3 \in A$, (i) $d(e_1) = x_1$, $d(e_2) = x_2$, (ii) $d(e_{12}) = x_1e_2 - x_2e_1$, $d(e_{13}) = x_1e_3 - x_3e_1$, $d(e_{23}) = x_2e_3 - x_3e_2$, and (iii) $d(e_{123}) = x_1e_{23} - x_2e_{13} + x_3e_{12}$. Thus the Koszul complex of x_1, x_2, x_3 is*

$$K_\bullet(x_1, x_2, x_3) : 0 \to R \xrightarrow{d_3} R^3 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R \to 0,$$

where $d_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $d_2 = \begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix}$, *and* $d_3 = \begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}$.

We note that $d_2 = \begin{bmatrix} -d_1^{(2)} & | & x_1 \cdot \mathbf{I_2} \\ -- & & -- \\ 0 & | & d_2^{(2)} \end{bmatrix}$, *where $d_t^{(2)}$ is a t-th differential map of a Koszul complex of x_2, x_3.*

Like the above example, we may understand a Koszul complex with the maps represented by matrices as follows; the proof can be done by using an induction.

FACT 1.2. Let $x_1, \cdots, x_n$ be in a commutative ring A, and let $d_\ell^{(1)}$ be the ℓ-th differential map of a Koszul complex of $x_1, \cdots, x_n$. We denote by $d_\ell^{(2)}$ the ℓ-th differential map of a Koszul complex of $x_2, \cdots, x_n$. Then for $1 \leq \ell \leq n$,

$$d_\ell^{(1)} = \begin{bmatrix} -d_{\ell-1}^{(2)} & | & x_1 \cdot \mathbf{I} \\ --- & & --- \\ 0 & | & d_\ell^{(2)} \end{bmatrix},$$

where $\mathbf{I}$ is an identity matrix of a proper dimension.

For an A-module M, we define $K_\bullet(\underline{x}, M) = K_\bullet(\underline{x}) \otimes M$. The Koszul complex $K_\bullet(\underline{x}, M)$ has homology groups $H_p(K_\bullet(\underline{x}, M))$, which we abbreviate to $H_p(\underline{x}, M)$. The ideal $(\underline{x}) = (x_1, \cdots, x_n)$ annihilates the homology groups $H_p(\underline{x}, M)$. A Koszul complex plays an important role in a commutative algebra, for examples, it is known ([4]) that (1) if $x_1, \cdots, x_n$ is an M-sequence, then $H_p(\underline{x}, M) = 0$ for $p > 0$ and $H_0(\underline{x}, M) = M/xM$, and (2) if $I = (y_1, \cdots, y_n)$ is an ideal of A and $M \neq IM$, then $\text{depth}(I, M) = n - \sup\{i : H_i(\underline{y}, M) \neq 0\}$, which is called 'depth sensitivity' of the Koszul complex.

2. Main theorems

In [2], Koh introduced the definition of the left spectrum of a non-commutative ring due to Rosenberg ([5]), and as a special case, he took a quantum space ring $R = k[x_1, \cdots, x_n]_{q_{ij}}$ (here, k is a field) with $x_i x_j = q_{ij} x_j x_i$ for $q_{ij} \in k - \{0\}$. He thought that like a commutative case, there would be an equvalence between the category of quasi-coherent sheaves on $Proj(R)$ and the category of graded R-modules mod the subcategory which is generated by the modules of finite length. He believed that Serre Duality would hold for $Proj(R)$.

In this section, we define a quantized Koszul complex, and pove that it is acyclic, which may be a first step to eastablish Serre Duality for $Proj(R)$.

Let $K_\bullet(x_1, \cdots, x_n)$ be a Koszul complex of $x_1, \cdots, x_n$ in $k[x_1, \cdots, x_n]$, and $d_p : K_P \to K_{p-1}$ be the p-th differential map of $K_\bullet(x_1, \cdots, x_n)$. We know that d_p is represented by an $\binom{n}{p} \times \binom{n}{p-1}$ matrix (Fact 1.2). Let $b(p)_{ij}$ be an (ij)-th entry of d_p, where $b(p)_{ij} = x_{t(p)_{ij}}, -x_{t(p)_{ij}}$, or 0, and $1 \le t(p)_{ij} \le n$.

Let $R = k[x_1, \cdots, x_n]_{q_{ij}}$. We define a complex $Q_\bullet(x_1, \cdots, x_n)$ as follows: set $Q_0 = R$, and $Q_p = 0$ if p is not in the range $0 \le p \le n$. For $1 \le p \le n$, let $Q_p = \oplus R_{e_{i_1 \cdots i_p}}$ be the free R-module of rank $\binom{n}{p}$ with a basis $\{e_{i_1 \cdots i_p} : 1 \le i_1 < \cdots < i_p \le n\}$. The p-th differential map $\partial_p : Q_p \to Q_{p-1}$ of $Q_\bullet$ is defined by an $\binom{n}{p} \times \binom{n}{p-1}$ matrix with an (ij)-th entry $a(p)_{ij} b(p)_{ij}$, where $b(p)_{ij}$ is an (ij)-th entry of d_p of a p-th differential map of $K_\bullet$ as above, and $a(p)_{1j} = 1$ (i.e., for $i = 1$),

$$a(p)_{ij} = \prod_{r=1}^{i-1} c(p,r)_{ij}, \text{ where } c(p,r)_{ij} = \{ \begin{array}{cl} q_{t(p)_{rj} t(p)_{ij}} & \text{if } b(p)_{ij} \neq 0 \\ 1 & \text{if } b(p)_{ij} = 0 \end{array} .$$

Definition 2.1. The complex $(Q_\bullet(x_1, \cdots, x_n), \partial_\bullet)$, which is defined in the above, is called a quantized Koszul complex of $x_1, \cdots, x_n$ in R.

Example 2.2. *The differential maps of a Koszul complex $K_\bullet(x_1, x_2, x_3)$ are* $d_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $d_2 = \begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix}$, *and* $d_3 = \begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}$. *Then when $\partial_1 = [a(1)_{ij} b(1)_{ij}]$ and $b(1)_{11} = x_1$, we can see that $b(1)_{21} = x_2$, $b(1)_{31} = x_3$. Also, $a(1)_{11} = 1$, $a(1)_{21} = c(1,1)_{21} = q_{12}$ ($t(1)_{11} = 1, t(1)_{21} = 2$), $a(1)_{31} = c(1,1)_{31} c(1,2)_{31} = q_{13} q_{23}$ ($t(1)_{11} = 1, t(1)_{31} =$*

$3, t(1)_{21} = 2, t(1)_{31} = 3)$. *Thus,* $\partial_1 = \begin{bmatrix} x_1 \\ q_{12}x_2 \\ q_{23}q_{13}x_3 \end{bmatrix}$. *If* $\partial_2 = [a(2)_{ij}b(2)_{ij}]$, *then* $a(2)_{21} = c(2,1)_{21} = q_{t(2)_{11}t(2)_{21}} = q_{23}$, $a(2)_{23} = c(2,1)_{23} = 1$ *since* $b(2)_{13} = 0$. *Also,* $a(2)_{32} = c(2,1)_{32}c(2,2)_{32} = q_{t(2)_{12}t(2)_{32}} = q_{13}$ *since* $b(2)_{22} = 0$, *and so* $c(2,2)_{32} = 1$. *We can check* $a(2)_{33} = c(2,1)_{33}c(2,2)_{33} = q_{t(2)_{23}t(2)_{33}} = q_{12}$, *and so on. Thus*

$$\partial_2 = \begin{bmatrix} -x_2 & x_1 & 0 \\ -q_{23}x_3 & 0 & x_1 \\ 0 & -q_{13}x_3 & q_{12}x_2 \end{bmatrix}, \text{ and } \partial_3 = \begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}.$$

In a similar manner, if $\partial_2^{(x_1\cdots x_4)}$ *is the 2nd differential map of a complex* $Q_\bullet(x_1, x_2, x_3, x_4)$, *then we can see that*

$$\partial_2^{(x_1\cdots x_4)} = \begin{bmatrix} -x_2 & x_1 & 0 & 0 \\ -q_{23}x_3 & 0 & x_1 & 0 \\ -q_{34}q_{24}x_4 & 0 & 0 & x_1 \\ 0 & -q_{13}x_3 & q_{12}x_2 & 0 \\ 0 & -q_{34}q_{14}x_4 & 0 & q_{12}x_2 \\ 0 & 0 & -q_{24}q_{14}x_4 & q_{23}q_{13}x_3 \end{bmatrix}.$$

Like Fact 1.2, we can formulate the maps in a quantized Koszul complex with the form of matrices as follows; we leave the proof for the reader.

PROPOSITION 2.3. *Let* $Q_\bullet(x_1, \cdots, x_n)$ *be a quantized Koszul complex, and* $\partial_p^{(x_1\cdots x_n)}$ *its p-th differential map. If* $\partial_\ell^{(x_2\cdots x_n)} = [a(\ell)_{ij}b(\ell)_{ij}]$ *is an* ℓ*-th differential map of a quantized Koszul complex of* $x_2, \cdots, x_n$, *then*

$$\partial_p^{(x_1\cdots x_n)} = \begin{bmatrix} -\partial_{p-1}^{(x_2\cdots x_n)} & | & x_1 \cdot \mathbf{I} \\ --- & & --- \\ 0 & | & \partial_p^{(*x_2\cdots x_n)} \end{bmatrix},$$

where $\partial_p^{(*x_2\cdots x_n)} = [q_{1t(p)_{ij}}a(p)_{ij}b(p)_{ij}]$.

REMARK 2.4. It is easy to show that $\partial_p^{(*x_2\cdots x_n)} = \partial_p^{(q_{12}x_2, q_{13}x_3, \cdots, q_{1n}x_n)}$, which is a p-th differential map of a quantized Koszul complex of $q_{12}x_2$, $q_{13}x_3$, $\cdots$, $q_{1n}x_n$.

Now, we prove that a quantized Koszul complex is really a complex.

THEOREM 2.5. *A quantized Koszul complex of* $x_1, \cdots, x_n$ *in a quantum space ring* R *is a complex, i.e.,* $\partial_p \cdot \partial_{p-1} = 0$.

Proof. Let's use an induction on n. For $n = 1$, $0 \to R \xrightarrow{x_1} R \to 0$ is a complex. Also, for $n = 2$, $0 \to R \xrightarrow{\partial_2} R^2 \xrightarrow{\partial_1} R \to 0$ is a complex, where $\partial_2 = \begin{bmatrix} -x_2 & x_1 \end{bmatrix}$, and $\partial_1 = \begin{bmatrix} x_1 \\ q_{12}x_2 \end{bmatrix}$. Suppose that it is true for the elements whose number is less than n, i.e., $Q_\bullet(x_{i_1}, \cdots, x_{i_t})$ is a complex if $t < n$. Then we have $\partial_p^{(x_2\cdots x_n)} \cdot \partial_{p-1}^{(x_2\cdots x_n)} = 0$ for $1 \le p \le n$. We note that

$$\begin{aligned}
&\partial_p^{(x_1\cdots x_n)} \cdot \partial_{p-1}^{(x_1\cdots x_n)} \\
= &\begin{bmatrix} -\partial_{p-1}^{(x_2\cdots x_n)} & | & x_1 \cdot \mathbf{I} \\ --- & & --- \\ 0 & | & \partial_p^{(*x_2\cdots x_n)} \end{bmatrix} \begin{bmatrix} -\partial_{p-2}^{(x_2\cdots x_n)} & | & x_1 \cdot \mathbf{I} \\ --- & & --- \\ 0 & | & \partial_{p-1}^{(*x_2\cdots x_n)} \end{bmatrix}.
\end{aligned}$$

Since $\partial_{p-1}^{(x_2\cdots x_n)} \cdot \partial_{p-2}^{(x_2\cdots x_n)} = 0$ by induction hypothesis, it is enough to show that

(a) $\begin{bmatrix} -\partial_{p-1}^{(x_2\cdots x_n)} & | & x_1 \cdot \mathbf{I} \end{bmatrix} \begin{bmatrix} x_1 \cdot \mathbf{I} \\ --- \\ \partial_{p-1}^{(*x_2\cdots x_n)} \end{bmatrix} = 0$, and

(b) $\partial_p^{(*x_2\cdots x_n)} \cdot \partial_{p-1}^{(*x_2\cdots x_n)} = 0$.

For (a), we note that

$$\begin{aligned}
&i\text{-th row of } \begin{bmatrix} -\partial_{p-1}^{(x_2\cdots x_n)} & | & x_1 \cdot \mathbf{I} \end{bmatrix} \times j\text{-th column of } \begin{bmatrix} x_1 \cdot \mathbf{I} \\ --- \\ \partial_{p-1}^{(*x_2\cdots x_n)} \end{bmatrix} \\
= \;& -a(p-1)_{ij} b(p-1)_{ij} x_1 + x_1\{q_{1t(p-1)_{ij}} a(p-1)_{ij} b(p-1)_{ij}\} \\
= \;& -x_1\{q_{1t(p-1)_{ij}} a(p-1)_{ij} b(p-1)_{ij}\} + x_1\{q_{1t(p-1)_{ij}} a(p-1)_{ij} b(p-1)_{ij}\} \\
& (\text{ since } x_{t(p-1)_{ij}} x_1 = q_{1t(p-1)_{ij}} x_1 x_{t(p-1)_{ij}}) \\
= \;& 0
\end{aligned}$$

For (b), we use the facts that $\partial_p^{(*x_2\cdots x_n)} = \partial_p^{(q_{12}x_2, q_{13}x_3, \cdots, q_{1n}x_n)}$, and $k[x_1, \cdots, x_n]_{q_{ij}}$ is isomorphic to $k[q_{11}x_1, \cdots, q_{1n}x_n]_{q_{ij}}$. Then by an induction hypothesis,

$$\partial_p^{(*x_2\cdots x_n)} \cdot \partial_{p-1}^{(*x_2\cdots x_n)} = \partial_p^{(q_{12}x_2\cdots q_{1n}x_n)} \cdot \partial_{p-1}^{(q_{12}x_2\cdots q_{1n}x_n)} = 0.$$

In all, we have proved that $\partial_p \cdot \partial_{p-1} = 0$, which means that a quantized Koszul complex of $x_1, \cdots, x_n$ in a quantum space ring R is a complex. □

A complex $G_\bullet$ of a ring A-mdules is said to be acyclic if $H_i(G_\bullet) = 0$ for $i > 0$. For example, a quantized Koszul complex $Q_\bullet(x_1, x_2)$ is acyclic: $0 \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \to 0$, where $\partial_2 = \begin{bmatrix} -x_2 & x_1 \end{bmatrix}$, and $\partial_1 = \begin{bmatrix} x_1 \\ q_{12}x_2 \end{bmatrix}$. Let (r_1, r_2) be any element of the kernel of ∂_1. Then $r_1x_1 + r_2q_{12}x_2 = 0$, and so $r_2q_{12} \in \langle x_1 \rangle$. If $r_2q_{12} = r_2^*x_1$ for some $r_2^* \in R$, then we have

$$0 = r_1x_1 + r_2q_{12}x_2 = r_1x_1 + r_2^*x_1x_2 = r_1x_1 + r_2^*q_{21}x_2x_1 = (r_1 + r_2^*q_{21}x_2)x_1,$$

and so $r_1 + r_2^*q_{21}x_2 = 0$. Thus,

$$\begin{aligned} \partial_2(r_2^*q_{21}) &= (r_2^*q_{21}) \begin{bmatrix} -x_2 & x_1 \end{bmatrix} \\ &= (-(r_2^*q_{21})x_2, (r_2^*q_{21})x_1) = (r_1, r_2q_{12}q_{21}) = (r_1, r_2), \end{aligned}$$

i.e., $Q_\bullet(x_1, x_2)$ is exact at Q_2.

The following theorem shows that a quantized Koszul complex is acyclic.

THEOREM 2.6. *A quantized Koszul complex of $x_1, \cdots, x_n$ in a quantum space ring R is acyclic.*

Proof. We use an induction on the number of x_i. For x_1, x_2, we have proved in the above. We assume that it is acyclic for the elements x_i whose number is less than n, for example, $(Q_\bullet(x_2, \cdots, x_n), \partial_\bullet^{(x_2 \cdots x_n)})$ is acyclic.

We first show that there is a short exact sequence of complexes as follows:

Let $\mathfrak{C}_\bullet = (Q_\bullet(x_2, \cdots, x_n), \partial_\bullet^{(*x_2 \cdots x_n)})$, $\mathfrak{D}_\bullet = (Q_\bullet(x_1, \cdots, x_n), \partial_\bullet^{(x_1 \cdots x_n)})$, and $\mathfrak{F}_\bullet = (Q_\bullet(x_2, \cdots, x_n), \partial_\bullet^{(x_2 \cdots x_n)})$. Then we have

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{C}_{p-1} & \xrightarrow{f_{p-1}} & \mathfrak{D}_{p-1} & \xrightarrow{g_{p-1}} & \mathfrak{F}_{p-2} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{C}_p & \xrightarrow{f_p} & \mathfrak{D}_p & \xrightarrow{g_p} & \mathfrak{F}_{p-1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

where if $\mathfrak{C}_p = R^{s_2}$, $\mathfrak{D}_p = R^{s_1+s_2}$, and $\mathfrak{F}_{p-1} = R^{s_1}$, then $f_p : \mathfrak{C}_p \to \mathfrak{D}_p$ is defined by $f_p(r_1, \cdots, r_{s_2}) = (0, \cdots, 0, r_1, \cdots, r_{s_2})$, and $g_p : \mathfrak{D}_p \to \mathfrak{F}_{p-1}$ is defined by $g_p(r_1, \cdots, r_{s_1}, r_{s_1+1}, \cdots, r_{s_1+s_2}) = (r_1, \cdots, r_{s_1})$. By chasing a diagram, we can see that it is a short exact sequence of complexes.

Next, from this short exact sequence of complexes, we have a long exact sequence of homologies

$$\begin{aligned}\cdots \to H_p(q_{12}x_2, \cdots, q_{1n}x_n, R) &\to H_p(x_1, \cdots, x_n, R)\\ &\to H_{p-1}(x_2, \cdots, x_n, R)\\ &\xrightarrow{x_1} H_{p-1}(q_{12}x_2, \cdots, q_{1n}, R) \to \cdots ,\end{aligned}$$

where $H_{p-1}(x_2, \cdots, x_n, R) \xrightarrow{x_1} H_{p-1}(q_{12}x_2, \cdots, q_{1n}, R)$ is defined by a multiplication by x_1. Indeed, for an element $(r_1, \cdots, r_n)$ in the kernel of $\partial_{p-1}^{(x_2,\cdots x_n)}$,

$$\begin{aligned}& g_p(r_1, \cdots, r_{s_1}, 0, \cdots, 0)\\ = \ & (r_1, \cdots, r_{s_1})\\ & \partial_p^{(x_1\cdots x_n)}(r_1, \cdots, r_{s_1}, 0, \cdots, 0)\\ = \ & (-\partial_p^{(x_2\cdots x_n)}(r_1, \cdots, r_{s_1}), x_1r_1, \cdots, x_1r_{s_1})\\ = \ & (0, \cdots, 0, x_1r_1, \cdots, x_1r_{s_1}) \in \mathfrak{D}_{p-1}\\ & \partial_{p-1}^{(*x_2\cdots x_n)}(x_1r_1, \cdots, x_1r_{s_1})\\ = \ & \partial_{p-1}^{(q_{12}x_2\cdots q_{1n}x_n)}(x_1r_1, \cdots, x_1r_{s_1})\\ = \ & \partial_{p-1}^{(x_2\cdots x_n)}(r_1, \cdots, r_{s_1}) \text{ (since } q_{1j}q_{j1} = 1)\\ = \ & 0,\end{aligned}$$

which means that $x_1 \cdot (r_1, \cdots, r_{s_1})$ is in the kernel of $\partial_{p-1}^{(*x_2\cdots x_n)}$.

For $p > 1$, we have an exact sequence

$$\begin{aligned}\cdots \to H_p(q_{12}x_2, \cdots, q_{1n}x_n, R) &\to H_p(x_1, \cdots, x_n, R)\\ &\to H_{p-1}(x_2, \cdots, x_n, R) \to \cdots .\end{aligned}$$

By the induction hypothesis, $H_{p-1}(x_2, \cdots, x_n, R) = 0$. Thus we know that $H_p(q_{12}x_2, \cdots, q_{1n}, R)$ is also 0 since $k[x_1, \cdots, x_n]_{q_{ij}}$ is isomorphic to $k[q_{11}x_1, \cdots, q_{1n}x_n]_{q_{ij}}$. Hence $H_p(x_1, \cdots, x_n, R) = 0$.

For $p = 1$, we have an exact sequence

$$\begin{aligned}\cdots \to H_1(x_1, \cdots, x_n, R) &\longrightarrow H_0(x_2, \cdots, x_n, R)\\ &\xrightarrow{x_1} H_0(q_{12}x_2, \cdots, q_{1n}x_n, R) \to \cdots .\end{aligned}$$

We note that

$$\begin{aligned}& H_0(x_2, \cdots, x_n, R) \cong R/(x_2, \cdots, x_n), \text{ and}\\ & H_0(q_{12}x_2, \cdots, q_{1n}x_n, R) \cong R/(q_{12}x_2, \cdots, q_{1n}x_n).\end{aligned}$$

We know that $H_1(x_1, \cdots, x_n, R)$ is also 0 since x_1 is a nonzero divisor of $R/(x_2, \cdots, x_n)$. This completes the proof. □

References

[1] A. Grothendieck, *Local Cohomology,* Lecture Notes in Math **41**, Springer-Verlag, 1967.

[2] J. Koh, *Remarks on local cohomology modules,* preprint.

[3] K. Lee, *Local duality on a quantum space ring,* in preparation.

[4] H. Matsumura, *Commutative ring theory,* Camb. Stud. Adv. Math. **8**, Cambridge Univ. Press, Cambridge, 1992.

[5] A. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras,* Mathematics and its Applications. **330**, Kluwer Academic Publishers Group, Dordrecht, 1995.

[6] J. Strooker, *Homological questions in local algebra,* London Math. Soc. Lecture Note Series. **145**, Cambridge Univ. Press, Cambridge, 1990.

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