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A GENERALIZATION OF MAYNARD'S RESULTS ON THE BRUN-TITCHMARSH THEOREM TO NUMBER FIELDS

Jeoung-Hwan Ahn and Soun-Hi Kwon

Abstract. Maynard proved that there exists an effectively computable constant q_1 such that if $q \geq q_1$, then $\frac{\log q}{\sqrt{q}\phi(q)}\mathrm{Li}(x) \ll \pi(x;q,m) < \frac{2}{\phi(q)}\mathrm{Li}(x)$ for $x \geq q^8$. In this paper, we will show the following. Let δ_1 and δ_2 be positive constants with $0 < \delta_1, \delta_2 < 1$ and $\delta_1 + \delta_2 > 1$. Assume that $L \neq \mathbb{Q}$ is a number field. Then there exist effectively computable constants c_0 and d_1 such that for $d_L \geq d_1$ and $x \geq \exp\left(326 n_L^{\delta_1} (\log d_L)^{1+\delta_2}\right)$, we have

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \leq \left(1 - c_0 \frac{\log d_L}{d_L^{7.072}}\right) \frac{|C|}{|G|}\mathrm{Li}(x).$$

1. Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. For a prime ideal $\mathfrak{p}$ of K which is unramified in L we let $\left[\frac{L/K}{\mathfrak{p}}\right]$ be the conjugacy class of Frobenius automorphisms corresponding to the prime ideals $\mathfrak{P}$ of L lying above $\mathfrak{p}$. For each conjugacy class C of G we let $\pi_C(x)$ be the number of prime ideals $\mathfrak{p}$ of K unramified in L such that $\left[\frac{L/K}{\mathfrak{p}}\right] = C$ and $N_{K/\mathbb{Q}}\,\mathfrak{p} \leq x$. The Chebotarev density theorem states that

$$\pi_C(x) \sim \frac{|C|}{|G|}\mathrm{Li}(x)$$

as $x \to \infty$, where $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$ as $x \to \infty$ (see [16] and [9]).

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In [9] Lagarias and Odlyzko proved the following theorem. For a number field F we let d_F denote the absolute value of the discriminant of F and let $n_F = [F : \mathbb{Q}]$.

Theorem 1.1 (Effective version of the Chebotarev density theorem). *Let $L \neq \mathbb{Q}$ and β_0 be the possible exceptional zero of $\zeta_L(s)$ with $1-1/(4\log d_L) \leq \beta_0 \leq 1$. There exist absolute effectively computable constants c_1 and c_2 such that if*

$$x \geq \exp\left(10 n_L (\log d_L)^2\right),$$

then

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \leq \frac{|C|}{|G|}\mathrm{Li}(x^{\beta_0}) + c_1 x \exp\left(-c_2\left(\frac{\log x}{n_L}\right)^{\frac{1}{2}}\right),$$

where the β_0 term is present only when β_0 exists.

The explicit error term is known in [18], [19], and [4]. This effective version of the Chebotarev density theorem says that if $x \geq \exp\left(10n_L(\log d_L)^2\right)$, then

$$\pi_C(x) \leq (2+o(1))\frac{|C|}{|G|}\mathrm{Li}(x).$$

If $K = \mathbb{Q}$ and $L = \mathbb{Q}\left(e^{2\pi i/q}\right)$, the conjugacy classes of G correspond to the residue classes modulo q, and the Chebotarev density theorem is the prime number theorem for arithmetic progressions. Let $\pi(x; q, m)$ be the number of primes less than or equal to x which are congruent to $m \pmod{q}$ for positive coprime integers m, q. Montgomery and Vaughan [12] proved the following theorem.

Theorem 1.2 (Brun-Titchmarsh theorem). *For $x > q$ we have*

$$\pi(x; q, m) \leq \frac{2}{1 - \log q/\log x}\frac{x}{\phi(q)\log x}.$$

The term $2/(1-\log q/\log x)$ of Brun-Titchmarsh theorem is also $2+o(1)$ if q is fixed and $x \to \infty$. Maynard [11] proved the following theorem.

Theorem 1.3 (Maynard). *There exists an effectively computable constant q_1 such that for $q \geq q_1$ and $x \geq q^8$ we have*

$$\frac{\log q}{\sqrt{q}\phi(q)}\mathrm{Li}(x) \ll \pi(x; q, m) < \frac{2}{\phi(q)}\mathrm{Li}(x).$$

In this paper, we show the following.

Theorem 1.4. *Let δ_1 and δ_2 be positive constants with $0 < \delta_1, \delta_2 < 1$ and $\delta_1 + \delta_2 > 1$. Assume that $L \neq \mathbb{Q}$ is a number field.*

(i) *There exist effectively computable constants c_0 and d_1 such that for $d_L \geq d_1$ and $x \geq \exp\left(326 n_L^{\delta_1}(\log d_L)^{1+\delta_2}\right)$, we have*

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \leq \left(1 - c_0\frac{\log d_L}{d_L^{7.072}}\right)\frac{|C|}{|G|}\mathrm{Li}(x).$$

(ii) *Suppose that* $\zeta_L(s)$ *has no real zero in the interval*

$$\left[1-\left(n_L^{\delta_1}(\log d_L)^{1+\delta_2}\right)^{-1},1\right].$$

Then for all ϵ *sufficiently small, there exists an effectively computable constant* d_2 *such that for* $d_L \geq d_2$ *and* $x \geq \exp\left(326 n_L^{\delta_1}(\log d_L)^{1+\delta_2}\right)$, *we have*

$$\left|\pi_C(x)-\frac{|C|}{|G|}\mathrm{Li}(x)\right| \leq \left(1-\frac{\epsilon}{2}\right)\frac{|C|}{|G|}\mathrm{Li}(x).$$

For comparison, Thorner and Zaman [15] proved the following theorem.

Theorem 1.5 (Thorner and Zaman)**.** *Let* L/K *be a Galois extension of number fields with Galois group* G *and let* C *be any conjugacy class of* G. *Let* H *be an abelian subgroup of* G *such that* $H \cap C$ *is nonempty. For a character* χ *in the dual group* $\widehat{H}$, *let* $\mathfrak{f}_\chi$ *be the conductor of* χ. *If* F *is the subfield of* L *fixed by* H *and* $Q = \max\{N_{F/\mathbb{Q}}\mathfrak{f}_\chi : \chi \in \widehat{H}\}$, *then*

$$\pi_C(x) < \left\{2 + O\left([F:\mathbb{Q}]x^{-\frac{1}{166[F:\mathbb{Q}]+327}}\right)\right\}\frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \gg d_F^{695}Q^{522} + d_F^{232}Q^{367}[F:\mathbb{Q}]^{290[F:\mathbb{Q}]}$ *provided that* $d_F\, Q\, [F:\mathbb{Q}]^{[F:\mathbb{Q}]}$ *is sufficiently large. If any of the following conditions also hold, then the error term can be omitted*:

- *There exists a sequence of number fields* $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n = F$ *such that* F_{j+1}/F_j *is a normal extension for all* $j = 0, 1, \ldots, n-1$.
- $(2[F:\mathbb{Q}])^{2[F:\mathbb{Q}]^2} \ll d_F Q^{1/2}$.
- $x \gg [F:\mathbb{Q}]^{334[F:\mathbb{Q}]^2}$.

The range of x in Theorem 1.4 is narrower than that of x in Theorem 1.5. However, the upper bound for $\pi_C(x)$ in Theorem 1.4 is better than that in Theorem 1.5.

For the lower bound for $\pi_C(x)$, Zaman [20] proved the following theorem.

Theorem 1.6 (Zaman)**.** *Let* L/F *be a Galois extension of number fields with Galois group* G *and let* $C \subseteq G$ *be a conjugacy class. Then*

$$\pi_C(x) \gg \frac{1}{d_L^{19}}\frac{|C|}{|G|}\mathrm{Li}(x)$$

for $x \geq d_L^{35}$ *and* d_L *is sufficiently large.*

The range of x in Theorem 1.4 is narrower than that of x in Theorem 1.6. However, the lower bound for $\pi_C(x)$ in Theorem 1.4 is better than that in Theorem 1.6. See also Theorem 3.1 in [14].

For much larger x, Kadiri and Wong [7] proved the following theorem.

Theorem 1.7. *Assume that* $L \neq \mathbb{Q}$. *Then for* $x \geq \exp\left(d_L^{11.7}\right)$,

$$\pi_C(x) \geq 0.4849 \frac{|C|}{|G|} \frac{x}{\log x}.$$

This improves significantly the result in [3]. The range of x in Theorem 1.4 is explicit and depends only on n_L and d_L. In the proof of Theorem 1.4 the possibility of the existence of the exceptional zero of $\zeta_L(s)$ makes difficulties. We will use the Deuring-Heilbronn phenomenon which asserts that if the exceptional zero exists, then the other zeros cannot lie very close to $s = 1$. Our argument relies mainly on Corollary 3.8 to Theorem 3.7 (Deuring-Heilbronn phenomenon).

In the following we write

$$\mathcal{L} := n_L^{\delta_1} \left(\log d_L\right)^{1+\delta_2}.$$

2. Proof of Theorem 1.4

Let

$$\theta_C(t) := \sum_{\substack{\mathfrak{p} \text{ unramified in } L/K \\ N\mathfrak{p} \leq t,\ \left[\frac{L/K}{\mathfrak{p}}\right] = C}} \log N\mathfrak{p}.$$

Using partial summation arguments we have, for $x \geq 2$

$$\pi_C(x) = \frac{\theta_C(x)}{\log x} + \int_{2^-}^{x} \frac{\theta_C(t)}{t(\log t)^2}\, dt.$$

Let

$$\psi_C(t) := \sum_{\substack{\mathfrak{p} \text{ unramified in } L/K,\ m \in \mathbb{N} \\ N\mathfrak{p}^m \leq t,\ \left[\frac{L/K}{\mathfrak{p}}\right]^m = C}} \log N\mathfrak{p}.$$

We note that

$$\theta_C(t) = \psi_C(t) + O(n_K t^{1/2})$$

(see [9, (9.7)]). Then for $x \geq 2$ we have, for any constant $A > 0$,

$$\pi_C(x) = \frac{\psi_C(x) + O(n_K x^{1/2})}{\log x} + \int_{e^{A\mathcal{L}}}^{x} \frac{\psi_C(t) + O(n_K t^{1/2})}{t(\log t)^2} dt + \int_{2}^{e^{A\mathcal{L}}} \frac{\theta_C(t)}{t(\log t)^2} dt.$$

This yields

$$\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x) = \frac{\psi_C(x) - \frac{|C|}{|G|}x}{\log x} + \int_{e^{A\mathcal{L}}}^{x} \frac{\psi_C(t) - \frac{|C|}{|G|}t}{t(\log t)^2} dt$$
$$+ O\left(n_K \frac{x^{1/2}}{\log x} + n_K \frac{e^{A\mathcal{L}}}{\mathcal{L}}\right).$$

In order to prove Theorem 1.4 we use the following.

Proposition 2.1. *Assume that* $L \neq \mathbb{Q}$ *is a number field.*

(i) *We suppose that* $\zeta_L(s)$ *has a real zero* β_0 *in the interval* $[1-\mathcal{L}^{-1},1]$. *Let*

$$\lambda_0 := (1-\beta_0)\log d_L. \tag{1}$$

Then there exists an effectively computable constant d_3 *such that for* $d_L \geq d_3$ *and* $t \geq e^{\mathcal{L}}$ *we have*

$$\left|\psi_C(t) - \frac{|C|}{|G|}t\right| \leq (1-\lambda_0)\frac{|C|}{|G|}t.$$

(ii) *We suppose that* $\zeta_L(s)$ *has no real zero in the interval* $[1-\mathcal{L}^{-1},1]$. *Then for all* ϵ *sufficiently small, there exists an effectively computable constant* d_2 *such that for* $d_L \geq d_2$ *and* $t \geq e^{325\mathcal{L}}$ *we have*

$$\left|\psi_C(t) - \frac{|C|}{|G|}t\right| \leq (1-\epsilon)\frac{|C|}{|G|}t.$$

See also [11, Proposition 3.5]. We will show Proposition 2.1(i) and (ii) in Sections 3 and 4 below, respectively. We use two different kernel functions, one in the case that $\zeta_L(s)$ has a real zero in the interval $[1-\mathcal{L}^{-1},1]$ and the other when it does not. Assuming the Proposition 2.1 we will show Theorem 1.4.

2.1. Case I: $\zeta_L(s)$ has a real zero β_0 in the interval $[1-\mathcal{L}^{-1},1]$

If $x \geq e^{326\mathcal{L}}$, then we have

$$\begin{aligned} n_K\frac{x^{1/2}}{\log x} + n_K\frac{e^{\mathcal{L}}}{\mathcal{L}} &\leq \frac{|C|}{|G|}\left(\frac{n_L}{\log x}x^{1/2} + \frac{n_L}{\mathcal{L}}e^{\mathcal{L}}\right) \ (\text{as } n_K = n_L/|G| \leq n_L|C|/|G|) \\ &\ll \frac{|C|}{|G|}x^{1/2}\left(\text{as } n_L \ll \log d_L \leq \mathcal{L} \ll \log x \text{ and } e^{\mathcal{L}} \leq x^{1/326}\right). \end{aligned}$$

According to [6, Corollary 1.3.1]

$$1-\beta_0 \gg d_L^{-7.072} \tag{2}$$

for d_L sufficiently large, so $d_L^{-7.072}\log d_L \ll \lambda_0 < 1/2$ (see also [8, Corollary 5.2], [2, Corollary 7.4], [13, Lemma 3], and [1, Theorem 1]). Thus, for $x \geq e^{326\mathcal{L}}$ we have

$$\lambda_0 \mathrm{Li}(x) \gg \frac{\log d_L}{d_L^{7.072}}\frac{x}{\log x} \gg \frac{x^{1/3}}{d_L^{7.072}}x^{1/2} \gg d_L^{\frac{326}{3}n_L^{\delta_1}(\log d_L)^{\delta_2}-7.072}x^{1/2}$$

since $x \geq d_L^{326 n_L^{\delta_1}(\log d_L)^{\delta_2}}$. Thus, for $x \geq e^{326\mathcal{L}}$ we have

$$\frac{\lambda_0}{2}\frac{|C|}{|G|}\mathrm{Li}(x) \gg d_L^{\frac{326}{3}n_L^{\delta_1}(\log d_L)^{\delta_2}-7.072}\left(n_K\frac{x^{1/2}}{\log x} + n_K\frac{e^{\mathcal{L}}}{\mathcal{L}}\right).$$

Therefore we have, for $x \geq e^{326\mathcal{L}}$

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right| \leq \frac{\left|\psi_C(x) - \frac{|C|}{|G|}x\right|}{\log x} + \int_{e^{\mathcal{L}}}^{x}\frac{\left|\psi_C(t) - \frac{|C|}{|G|}t\right|}{t(\log t)^2}dt + \frac{\lambda_0}{2}\frac{|C|}{|G|}\mathrm{Li}(x)$$

$$\leq (1-\lambda_0)\frac{|C|}{|G|}\frac{x}{\log x} + (1-\lambda_0)\frac{|C|}{|G|}\int_{e^{\mathcal{L}}}^{x}\frac{t}{t(\log t)^2}dt$$
$$+ \frac{\lambda_0}{2}\frac{|C|}{|G|}\mathrm{Li}(x)$$
$$\leq \left(1-\frac{\lambda_0}{2}\right)\frac{|C|}{|G|}\mathrm{Li}(x)$$

provided that d_L is sufficiently large.

2.2. Case II : $\zeta_L(s)$ has no real zero in the interval $[1-\mathcal{L}^{-1}, 1]$

If $x \geq e^{326\mathcal{L}}$, then we have

$$n_K\frac{x^{1/2}}{\log x} + n_K\frac{e^{325\mathcal{L}}}{\mathcal{L}}$$
$$\leq \frac{|C|}{|G|}\left(\frac{n_L}{\log x}x^{1/2} + \frac{n_L}{\mathcal{L}}e^{325\mathcal{L}}\right) \text{ (as } n_K = n_L/|G| \leq n_L|C|/|G|)$$
$$\ll \frac{|C|}{|G|}x^{325/326} \left(\text{as } n_L \ll \log d_L \leq \mathcal{L} \ll \log x \text{ and } e^{325\mathcal{L}} \leq x^{325/326}\right).$$

Thus we have, for $x \geq e^{326\mathcal{L}}$

$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\right|$$
$$\leq \frac{\left|\psi_C(x) - \frac{|C|}{|G|}x\right|}{\log x} + \int_{e^{325\mathcal{L}}}^{x}\frac{\left|\psi_C(t) - \frac{|C|}{|G|}t\right|}{t(\log t)^2}dt + O\left(\frac{|C|}{|G|}x^{325/326}\right)$$
$$\leq (1-\epsilon)\frac{|C|}{|G|}\frac{x}{\log x} + (1-\epsilon)\frac{|C|}{|G|}\int_{e^{325\mathcal{L}}}^{x}\frac{t}{t(\log t)^2}dt + \frac{\epsilon}{2}\frac{|C|}{|G|}\mathrm{Li}(x)$$
$$\leq \left(1-\frac{\epsilon}{2}\right)\frac{|C|}{|G|}\mathrm{Li}(x)$$

provided that d_L is sufficiently large.

3. Proof of point (i) of Proposition 2.1

We assume that $\zeta_L(s)$ has a real zero β_0 in the interval $[1-\mathcal{L}^{-1}, 1]$. We will use Theorem 7.1 of [9]. Following [9], we let

$$F_C(s) := -\frac{|C|}{|G|}\sum_{\phi}\overline{\phi}(g)\frac{L'}{L}(s, \phi, L/K)$$

with $g \in C$, where ϕ runs over the irreducible characters of G and $L(s, \phi, L/K)$ is the Artin L-function associated to ϕ. Using the orthogonality relations for characters we have the Dirichlet series expansion

$$F_C(s) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\Theta(\mathfrak{p}^m)(\log N\mathfrak{p})(N\mathfrak{p})^{-ms}$$

for $\Re s > 1$, where $\mathfrak{p}$ runs over the prime ideals of K, $0 \leq \Theta(\mathfrak{p}^m) \leq 1$, and for $\mathfrak{p}$ unramified in L

$$\Theta(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } \left[\frac{L/K}{\mathfrak{p}}\right]^m = C, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that $F_C(s)$ can be written in terms of Hecke L-functions (see [5], [10], and [9, Section 4]). We have

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \frac{L'}{L}(s, \chi, E),$$

where E is the fixed field of the cyclic group $\langle g \rangle$, and χ are certain primitive Hecke characters satisfying $\chi(\mathcal{P}) = \chi\left(\left[\frac{L/E}{\mathcal{P}}\right]\right)$ for all prime ideals $\mathcal{P}$ of E unramified in L and $L(s, \chi, E)$ are certain Hecke L-functions attached to the field E. We will use $L(s, \chi)$ to denote $L(s, \chi, E)$.

Let $t \geq 2$ and

$$k_1(s) := \frac{t^s}{s}.$$

For any $\sigma_0 > 1$ and $T \geq 2$ we let

$$I_C(t, T) := \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F_C(s) k_1(s)\, ds.$$

Choosing $\sigma_0 = 1 + (\log t)^{-1}$ we obtain

$$\psi_C(t) - I_C(t, T) \ll \log t \log d_L + n_K \log t + n_K t T^{-1} (\log t)^2$$

(see [9, (3.18)]). Let

$$I_\chi(t, T) := \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'}{L}(s, \chi) k_1(s)\, ds$$

and

$$I_\chi(t, T, U) := \frac{1}{2\pi i} \int_{B_{T,U}} \frac{L'}{L}(s, \chi) k_1(s)\, ds$$

with $U = j + 1/2$ for some non-negative integer j, where $B_{T,U}$ is the positively oriented rectangle with vertices at $\sigma_0 - iT$, $\sigma_0 + iT$, $-U + iT$, and $-U - iT$.

Proposition 3.1. *Let $n_\chi(y)$ denote the number of zeros $\rho = \beta + i\gamma$ of Hecke L-function $L(s, \chi, E)$ in the rectangle $0 \leq \beta \leq 1$ and $|\gamma - y| \leq 1$. Then*

$$n_\chi(y) \ll \log\left(d_E N f(\chi)\right) + n_E \log\left(|y| + 2\right),$$

where $f(\chi)$ is the conductor of χ.

Proof. See [9, Lemma 5.4]. □

By using the zero density estimate of Proposition 3.1, in Section 6 of [9] it is proved that

$$R_\chi(t, T, U) := I_\chi(t, T, U) - I_\chi(t, T)$$

is small. Evaluating $I_\chi(t,T,U)$ by Cauchy's residue theorem and sums over zeros by using the density of zeros in Proposition 3.1 the following theorem is proved.

Theorem 3.2 (Lagarias and Odlyzko)**.** *If $t \geq 2$ and $T \geq 2$, then*

$$\psi_C(t) - \frac{|C|}{|G|}t + S(t,T) \ll R_0(t,T),$$

where

$$S(t,T) := \frac{|C|}{|G|}\sum_\chi \overline{\chi}(g)\left(\sum_{\substack{\rho \\ |\Im\rho|<T}} \frac{t^\rho}{\rho} - \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \frac{1}{\rho}\right)$$

and

$$R_0(t,T) := \frac{|C|}{|G|}\left[\frac{t\log t + T}{T}\log d_L + n_L \log t + \frac{n_L t \log t \log T}{T}\right] + \log t \log d_L + \frac{n_K t (\log t)^2}{T}.$$

The inner sums in the definition of $S(t,T)$ are over the nontrivial zeros ρ of $L(s,\chi)$.

Let

$$R_1(t) := \log d_L + n_L \log t + n_L \log t \log d_L,$$

$$R_2(t,T) := \frac{t\log t \log d_L}{T} + \frac{n_L t \log t \log T}{T} + \frac{n_L t(\log t)^2}{T},$$

and

$$R(t,T) := \frac{|C|}{|G|}\left[R_1(t) + R_2(t,T)\right].$$

Since $n_K|G|/|C| \leq n_L$, we have

$$R_0(t,T) \leq R(t,T).$$

Thus, if $t \geq 2$ and $T \geq 2$, then we have

$$\left|\psi_C(t) - \frac{|C|}{|G|}t\right| \leq \frac{|C|}{|G|}\frac{t^{\beta_0}}{\beta_0} + \frac{|C|}{|G|}\sum_\chi \sum_{\substack{\rho \\ \rho\neq\beta_0, |\Im\rho|<T}} \left|\frac{t^\rho}{\rho}\right| + \frac{|C|}{|G|}\sum_\chi \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \left|\frac{1}{\rho}\right| + O(R(t,T)).$$

Lemma 3.3. *Let $\epsilon > 0$. If d_L is sufficiently large and $t \geq e^{\mathcal{L}}$, then we have*

$$\frac{|C|}{|G|}\frac{t^{\beta_0}}{\beta_0} = \frac{|C|}{|G|}t\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right) + O^*\left(\epsilon\frac{|C|}{|G|}\lambda_0 t\right),$$

where $f(t) = O^*(g(t))$ means $|f(t)| \leq g(t)$.

Proof. We have

$$\frac{t^{\beta_0}}{\beta_0} = t\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right)\left(1+O\left(\frac{\lambda_0}{\log d_L}\right)\right)$$
$$= t\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right) + O\left(\frac{1}{\log d_L}\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right)\lambda_0 t\right).$$

Since $\lambda_0 \gg d_L^{-7.072}\log d_L$, we have $\lambda_0 \geq d_L^{-8}\log d_L$ for d_L sufficiently large and

$$\frac{1}{\log d_L}\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right) \leq \frac{1}{\log d_L}\exp\left(-\frac{\log t}{d_L^8}\right).$$

Let $f(y) := \frac{1}{\log y}\exp\left(-\frac{\log t}{y^8}\right)$. We have then

$$f'(y) = \frac{1}{y(\log y)^2}\exp\left(-\frac{\log t}{y^8}\right)\left(\frac{\log y^8}{y^8}\log t - 1\right).$$

Let $y_0 > 0$ be the critical point of f so that

$$\frac{\log y_0^8}{y_0^8} = \frac{1}{\log t}.$$

Then we have

$$8 - \frac{\log\log t}{\log y_0} = \frac{\log 8 + \log\log y_0}{\log y_0}.$$

Note that if d_L is sufficiently large, then $e^{\mathcal{L}}$ is sufficiently large. Thus $t \geq e^{\mathcal{L}}$ is sufficiently large, which implies that y_0 is sufficiently large. We have then

$$0 \leq 8 - \frac{\log\log t}{\log y_0} \leq 1.$$

Thus, we have

$$\frac{1}{\log d_L}\exp\left(-\frac{\log t}{d_L^8}\right) \leq \frac{1}{\log y_0}\exp\left(-\frac{1}{\log y_0^8}\right) \leq \frac{8}{\log\log t}\exp\left(-\frac{7}{8\log\log t}\right)$$

for d_L sufficiently large and $t \geq e^{\mathcal{L}}$.

Hence, we have

$$\frac{|C|}{|G|}\frac{t^{\beta_0}}{\beta_0} = \frac{|C|}{|G|}t\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right) + O^*\left(\epsilon\frac{|C|}{|G|}\lambda_0 t\right).$$ □

Lemma 3.4. *Let $\epsilon > 0$. If d_L is sufficiently large and $t \geq e^{\mathcal{L}}$, then we have*

$$\frac{|C|}{|G|}\sum_{\chi}\sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}}\left|\frac{1}{\rho}\right| \leq \epsilon\frac{|C|}{|G|}\lambda_0 t.$$

Proof. We have

$$\sum_{\chi}\sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}}\left|\frac{1}{\rho}\right| \leq \sum_{\chi}\sum_{\substack{\rho \\ \rho\neq 1-\beta_0, |\rho|<\frac{1}{2}}}\left|\frac{1}{\rho}\right| + \frac{1}{1-\beta_0}.$$

Moreover, we have

$$\sum_{\chi} \sum_{\substack{\rho \\ \rho\neq 1-\beta_0, |\rho|<\frac{1}{2}}} \left|\frac{1}{\rho}\right| \ll (\log d_L)^2 \tag{3}$$

(see the proof of Theorem 9.2 of [9] in page 459). Since $(1-\beta_0)^{-1} \ll d_L^{7.072}$, we have

$$\sum_{\chi} \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \left|\frac{1}{\rho}\right| \ll d_L^{7.072}. \tag{4}$$

Since $\lambda_0 \gg d_L^{-7.072} \log d_L$ and $t \geq e^{\mathcal{L}} = d_L^{n_L^{\delta_1} (\log d_L)^{\delta_2}}$, we have

$$d_L^{7.072} = \frac{d_L^{7.072}}{\lambda_0 t}\lambda_0 t \ll \frac{d_L^{14.144}}{t \log d_L}\lambda_0 t \ll \frac{1}{t^{1-14.144/\left(n_L^{\delta_1}(\log d_L)^{\delta_2}\right)}}\lambda_0 t \leq \epsilon \lambda_0 t$$

for d_L sufficiently large and $t \geq e^{\mathcal{L}}$, hence

$$\frac{|C|}{|G|}\sum_{\chi} \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \left|\frac{1}{\rho}\right| \leq \epsilon \frac{|C|}{|G|}\lambda_0 t.$$

□

We choose

$$T = \frac{n_L (\log t)^3}{\lambda_0}. \tag{5}$$

Lemma 3.5. *Let $\epsilon > 0$. If d_L is sufficiently large and $t \geq e^{\mathcal{L}}$, then we have*

$$R(t,T) \leq \epsilon \frac{|C|}{|G|}\lambda_0 t.$$

Proof. We have

$$\begin{aligned} R_1(t) &\ll \frac{n_L \log t \log d_L}{\lambda_0 t}\lambda_0 t \\ &\ll \frac{d_L^{7.072}(\log t)^{(2+\delta_1+\delta_2)/(1+\delta_1+\delta_2)}}{t}\lambda_0 t \\ &\qquad \left(\text{for } \lambda_0^{-1} \ll \frac{d_L^{7.072}}{\log d_L} \text{ and } n_L \ll (\log t)^{1/(1+\delta_1+\delta_2)}\right) \\ &\ll \frac{(\log t)^{(2+\delta_1+\delta_2)/(1+\delta_1+\delta_2)}}{t^{1-7.072/\left(n_L^{\delta_1}(\log d_L)^{\delta_2}\right)}}\lambda_0 t \quad \left(\text{for } t \geq d_L^{n_L^{\delta_1}(\log d_L)^{\delta_2}}\right) \\ &\leq \frac{\epsilon}{2}\lambda_0 t \end{aligned}$$

for d_L sufficiently large and $t \geq e^{\mathcal{L}}$. Since $\log d_L \ll (\log t)^{1/(1+\delta_2)}$, $\log n_L \ll \log\log t$, and $\log \lambda_0^{-1} \ll \log d_L$, we have

$$R_2(t,T) = \frac{\log d_L}{n_L(\log t)^2}\lambda_0 t + \frac{\log n_L + 3\log\log t + \log \lambda_0^{-1}}{(\log t)^2}\lambda_0 t + \frac{1}{\log t}\lambda_0 t$$

$$\ll \frac{1}{(\log t)^{(1+2\delta_2)/(1+\delta_2)}}\lambda_0 t + \frac{\log\log t + (\log t)^{1/(1+\delta_2)}}{(\log t)^2}\lambda_0 t + \frac{1}{\log t}\lambda_0 t$$
$$\leq \frac{\epsilon}{2}\lambda_0 t$$

for d_L sufficiently large and $t \geq e^{\mathcal{L}}$. Thus, we have

$$R(t,T) \leq \epsilon \frac{|C|}{|G|}\lambda_0 t.$$

□

Let $\epsilon > 0$. From Theorem 3.2 and Lemmas 3.3-3.5, we have then, for $t \geq e^{\mathcal{L}}$

$$(6)\qquad \left|\psi_C(t) - \frac{|C|}{|G|}t\right| \leq \frac{|C|}{|G|}t\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right) + \frac{|C|}{|G|}\sum_{\chi}\sum_{\substack{\rho \\ \rho\neq\beta_0,|\Im\rho|<T}}\left|\frac{t^\rho}{\rho}\right| + O^*\left(\epsilon\frac{|C|}{|G|}\lambda_0 t\right)$$

provided that d_L is sufficiently large. Now we will show that

$$\sum_{\chi}\sum_{\substack{\rho \\ \rho\neq\beta_0,|\Im\rho|<T}}\left|\frac{t^\rho}{\rho}\right| \leq \epsilon\lambda_0 t.$$

We will use the following properties on the locations of the nontrivial zeros of $\zeta_L(s)$.

Proposition 3.6. *Assume that $L \neq \mathbb{Q}$. Let $\rho = \beta + i\gamma$ be a nontrivial zero of $\zeta_L(s)$ with $\rho \neq \beta_0$. Then, we have*

$$1 - \beta > \frac{1}{29.57\log\left(d_L\left(|\gamma|+2\right)^{n_L}\right)}.$$

Proof. See Lemma 2.3 of [8] and Proposition 6.1 of [2]. □

Theorem 3.7 (Deuring-Heilbronn phenomenon)**.** *Assume that $L \neq \mathbb{Q}$. There are positive, absolute, effectively computable constants c_3 and c_4 such that if $\zeta_L(\beta + i\gamma) = 0$ with $\beta + i\gamma \neq \beta_0$, then*

$$1 - \beta \geq \frac{c_3}{\log\left(d_L\left(|\gamma|+2\right)^{n_L}\right)}\log\left(\frac{c_4}{(1-\beta_0)\log\left(d_L\left(|\gamma|+2\right)^{n_L}\right)}\right).$$

Proof. See Theorem 5.1 of [8] and Theorem 7.3 of [2]. □

Corollary 3.8. *Assume that d_L is sufficiently large. Let $\rho = \beta + i\gamma$ be a zero of $\zeta_L(s)$ with $\rho \neq \beta_0$ and $|\gamma| \ll d_L^{c_5}$ for some positive constant c_5. If $\beta_0 = 1 - \lambda_0/\log d_L \geq 1 - \mathcal{L}^{-1}$, then there exists a positive constant c_6 such that*

$$(1-\beta)n_L\log d_L \geq c_6\log\left(\lambda_0^{-1}\right).$$

Proof. We have $\log\left(d_L(|\gamma|+2)^{n_L}\right) \leq c_7 n_L \log d_L$ for some constant $c_7 > 0$. We may assume that $c_7 > c_4/2$. From the fact that $1 - \beta_0 = \lambda_0/\log d_L \leq \mathcal{L}^{-1}$,

$d_L \geq 3^{n_L/2}$ ([2, p. 1421], [13, p. 140], and [8, p. 291]), and $\delta_1 + \delta_2 > 1$ it follows that

$$\begin{aligned}
&(1-\beta) n_L \log d_L \\
&\geq \frac{c_3}{c_7} \log(\lambda_0^{-1}) \left(1 - \frac{\log(c_7 n_L/c_4)}{\log(\lambda_0^{-1})}\right) \text{ (Theorem 3.7)} \\
&\geq \frac{c_3}{c_7} \log(\lambda_0^{-1}) \left(1 - \frac{\log(c_7 n_L/c_4)}{\delta_1 \log n_L + \delta_2 \log\log d_L}\right) \\
&\quad \left(\text{for } \lambda_0^{-1} \geq \frac{\mathcal{L}}{\log d_L} = n_L^{\delta_1} (\log d_L)^{\delta_2}\right) \\
&\geq \frac{c_3}{c_7} \log(\lambda_0^{-1}) \left(1 - \frac{\log(2c_7/(c_4 \log 3)) + \log\log d_L}{\delta_1 \log(2/\log 3) + (\delta_1 + \delta_2) \log\log d_L}\right) \\
&\quad \Bigg(\text{as } n_L \leq \frac{2}{\log 3} \log d_L \text{ and } f(x) = 1 - \frac{\log(c_7/c_4) + x}{\delta_1 x + \delta_2 \log\log d_L} \text{ is decreasing} \\
&\quad\quad \text{for sufficiently large } d_L\Bigg) \\
&\geq \frac{c_3}{c_7} \log(\lambda_0^{-1}) \left(1 - \frac{\log(2c_7/(c_4 \log 3)) + \log\log d_L}{(\delta_1 + \delta_2) \log\log d_L}\right) \\
&\geq c_6 \log(\lambda_0^{-1}) \text{ (for } \delta_1 + \delta_2 > 1)
\end{aligned}$$

with $c_6 = c_3(\delta_1 + \delta_2 - 1)/(2c_7(\delta_1 + \delta_2))$. □

It seems that the lower bound for $1 - \beta$ in Theorem 3.7 is best possible. A possible better version of the Deuring-Heilbronn phenomenon would yield a wider range of x in Theorem 1.4.

Lemma 3.9. *Assume that d_L is sufficiently large. Let $\rho = \beta + i\gamma$ be a nontrivial zero of $\zeta_L(s)$ with $\rho \neq \beta_0$ and $|\gamma| < T$.*

(i) *If $\log t \geq d_L$, then we have*

$$|t^\rho| \leq t \exp\left(-c_8 \frac{\log t}{(\log\log t)^2}\right)$$

for some constant $c_8 > 0$.

(ii) *Assume that $t \geq e^{\mathcal{L}}$ and $\beta_0 \geq 1 - \mathcal{L}^{-1}$. If $\log t \leq d_L$, then we have*

$$|t^\rho| \leq \lambda_0 t \exp\left(-c_9 \log\left(\lambda_0^{-1}\right)\right)$$

for some constant $c_9 > 3/\delta_2$.

Proof. (i) By (5)

$$\begin{aligned}
T = \frac{n_L (\log t)^3}{\lambda_0} &\ll \frac{n_L d_L^{7.072} (\log t)^3}{\log d_L} \quad \left(\text{for } {\lambda_0}^{-1} \ll \frac{d_L^{7.072}}{\log d_L}\right) \\
&\ll (\log t)^{10.072} \quad (\text{for } n_L \ll \log d_L \text{ and } d_L \leq \log t).
\end{aligned}$$

Thus we have

$$|t^\rho| = t \exp\left(-(1-\beta)\log t\right)$$

$$\leq t\exp\left(-\frac{\log t}{29.57\log d_L(T+2)^{n_L}}\right) \text{ (Proposition 3.6)}$$
$$\leq t\exp\left(-\frac{\log t}{29.57\log d_L\left(1+\frac{2}{\log 3}\log(T+2)\right)}\right) \left(\text{for } n_L\leq \frac{2}{\log 3}\log d_L\right)$$
$$\leq t\exp\left(-c_8\frac{\log t}{(\log\log t)^2}\right) \text{ (for } \log T\ll \log\log t)$$

for some constant $c_8>0$.

(ii) By (5)

$$T=\frac{n_L(\log t)^3}{\lambda_0}\ll\frac{n_L d_L^{7.072}(\log t)^3}{\log d_L} \left(\text{for } \lambda_0{}^{-1}\ll\frac{d_L^{7.072}}{\log d_L}\right)$$
$$\ll d_L^{10.072} \text{ (for } n_L\ll\log d_L \text{ and } \log t\leq d_L).$$

Therefore we have, for $t\geq e^{\mathcal{L}}$

$$|t^\rho|=t\exp\left(-(1-\beta)n_L\log d_L\frac{\log t}{n_L\log d_L}\right)$$
$$\leq t\exp\left(-c_6\log\left(\lambda_0^{-1}\right)\frac{\log t}{n_L\log d_L}\right) \text{ (Corollary 3.8)}$$
$$\leq \lambda_0 t\exp\left(-c_6\left(\frac{\log d_L}{n_L}\right)^{1-\delta_1}(\log d_L)^{\delta_1+\delta_2-1}\log(\lambda_0^{-1})+\log(\lambda_0^{-1})\right)$$
$$\left(\text{for } \log t\geq\mathcal{L}=n_L^{\delta_1}\left(\log d_L\right)^{1+\delta_2}\right)$$
$$\leq \lambda_0 t\exp\left(-c_9\log(\lambda_0^{-1})\right) \text{ (for } n_L\ll\log d_L \text{ and } \delta_1+\delta_2>1)$$

for some positive constant c_9. Moreover, we may assume that $c_9>3/\delta_2$ since d_L is sufficiently large. The inequality $c_9>3/\delta_2$ will be needed in the proof of Lemma 3.10 below. □

Lemma 3.10. *Let $\epsilon>0$. Assume that d_L is sufficiently large and $\beta_0\geq 1-\mathcal{L}^{-1}$. If $t\geq e^{\mathcal{L}}$, then we have*

$$\sum_{\chi}\sum_{\substack{\rho\\ \rho\neq\beta_0,|\Im\rho|<T}}\left|\frac{t^\rho}{\rho}\right|\leq\epsilon\lambda_0 t.$$

Proof. From the proof of Theorem 9.2 of [9, p. 459] we have

$$\sum_{\chi}\sum_{\substack{\rho\\ |\rho|\geq\frac12,|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\ll\log T\log(d_L T^{n_L}). \tag{7}$$

(i) Suppose that $\log t\geq d_L$. According to the proof of point (i) of Lemma 3.9 we have $T\ll(\log t)^{10.072}$. Since $n_L\ll\log d_L\leq\log\log t$ and $\log T\ll\log\log t$

we have

$$\sum_{\chi}\sum_{\substack{\rho\\|\rho|\geq\frac{1}{2},|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\ll\log T\log(d_LT^{n_L})\ll(\log\log t)^3.$$

Thus we have

$$\begin{aligned}&\sum_{\chi}\sum_{\substack{\rho\\\rho\neq\beta_0,|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\\&\leq\frac{1}{1-\beta_0}+\sum_{\chi}\sum_{\substack{\rho\\|\rho|<\frac{1}{2},\rho\neq1-\beta_0}}\left|\frac{1}{\rho}\right|+\sum_{\chi}\sum_{\substack{\rho\\|\rho|\geq\frac{1}{2},|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\\&\ll d_L^{7.072}+(\log d_L)^2+(\log\log t)^3\ \ \big(\text{for }(1-\beta_0)^{-1}\ll d_L^{7.072}\text{ and (3)}\big)\\&\ll(\log t)^{7.072}+(\log\log t)^2+(\log\log t)^3\ \ (\text{for }d_L\leq\log t)\\&\ll(\log t)^{7.072}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{\chi}\sum_{\substack{\rho\\\rho\neq\beta_0,|\Im\rho|<T}}\left|\frac{t^\rho}{\rho}\right|\\&\leq t\exp\left(-c_8\frac{\log t}{(\log\log t)^2}\right)\sum_{\chi}\sum_{\substack{\rho\\\rho\neq\beta_0,|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\ \ (\text{point (i) of Lemma 3.9})\\&\ll t\exp\left(-c_8\frac{\log t}{(\log\log t)^2}\right)(\log t)^{7.072}\\&=\exp\left(-c_8\frac{\log t}{(\log\log t)^2}\right)\frac{(\log t)^{7.072}}{\lambda_0}\lambda_0t\\&\ll\exp\left(-c_8\frac{\log t}{(\log\log t)^2}\right)\frac{d_L^{7.072}(\log t)^{7.072}}{\log d_L}\lambda_0t\ \ \left(\text{for }\lambda_0^{-1}\ll\frac{d_L^{7.072}}{\log d_L}\right)\\&\ll\exp\left(-c_8\frac{\log t}{(\log\log t)^2}\right)(\log t)^{14.144}\lambda_0t\ \ (\text{for }d_L\leq\log t)\\&\leq\epsilon\lambda_0t.\end{aligned}$$

(ii) Suppose that $\log t\leq d_L$. According to the proof of point (ii) of Lemma 3.9 we have $T\ll d_L^{10.072}$. From (7) we have

$$\sum_{\chi}\sum_{\substack{\rho\\\rho\neq\beta_0,|\rho|\geq\frac{1}{2},|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\ll\log T\log(d_LT^{n_L}).$$

Since $n_L\ll\log d_L$ and $\log T\ll\log d_L$ we have

$$\sum_{\chi}\sum_{\substack{\rho\\\rho\neq\beta_0,|\rho|\geq\frac{1}{2},|\Im\rho|<T}}\left|\frac{1}{\rho}\right|\ll\log T\log(d_LT^{n_L})\ll(\log d_L)^3.$$

Hence we have

$$\begin{aligned}
&\sum_{\chi} \sum_{\substack{\rho \\ \rho\neq\beta_0, |\rho|\geq\frac{1}{2}, |\Im\rho|<T}} \left|\frac{t^{\rho}}{\rho}\right| \\
&\leq \lambda_0 t \exp\left(-c_9 \log\left(\lambda_0^{-1}\right)\right) \sum_{\chi} \sum_{\substack{\rho \\ \rho\neq\beta_0, |\rho|\geq\frac{1}{2}, |\Im\rho|<T}} \left|\frac{1}{\rho}\right| \quad \text{(point (ii) of Lemma 3.9)} \\
&\ll \lambda_0 t \exp\left(-c_9\delta_2 \log\log d_L\right)\left(\log d_L\right)^3 \quad \left(\text{for } \lambda_0^{-1} \geq \mathcal{L}\left(\log d_L\right)^{-1} \geq \left(\log d_L\right)^{\delta_2}\right) \\
&= \lambda_0 t \exp\left(-(c_9\delta_2 - 3)\log\log d_L\right) \\
&\leq \lambda_0 t \exp\left(-(c_9\delta_2 - 3)\log\log\log t\right) \quad (\text{for } d_L \geq \log t \text{ and } c_9\delta_2 > 3) \\
&\leq \frac{\epsilon}{2}\lambda_0 t.
\end{aligned}$$

Moreover we have, for $t \geq e^{\mathcal{L}}$

$$\begin{aligned}
\sum_{\chi} \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \left|\frac{t^{\rho}}{\rho}\right| &\leq \sqrt{t} \sum_{\chi} \sum_{\substack{\rho \\ |\rho|<\frac{1}{2}}} \left|\frac{1}{\rho}\right| \\
&\ll \sqrt{t}\, d_L^{7.072} \quad (\text{for } (4)) \\
&= \frac{d_L^{7.072}}{\lambda_0\sqrt{t}}\lambda_0 t \\
&\ll \frac{d_L^{14.144}}{\sqrt{t}\log d_L}\lambda_0 t \quad \left(\text{for } \lambda_0 \gg \frac{\log d_L}{d_L^{7.072}}\right) \\
&\leq \frac{1}{t^{1/2 - 14.144/\left(n_L^{\delta_1}(\log d_L)^{\delta_2}\right)}}\lambda_0 t \quad \left(\text{for } t \geq d_L^{n_L^{\delta_1}(\log d_L)^{\delta_2}}\right) \\
&\leq \frac{\epsilon}{2}\lambda_0 t.
\end{aligned}$$

□

We can now complete the proof of point (i) of Proposition 2.1. Now we use the same ϵ in Lemmas 3.3, 3.4, 3.5, and 3.10. From (6) and Lemma 3.10 we have, for $t \geq e^{\mathcal{L}}$

$$\begin{aligned}
\left|\psi_C(t) - \frac{|C|}{|G|}t\right| &\leq \frac{|C|}{|G|}t\left(\exp\left(-\lambda_0\frac{\log t}{\log d_L}\right) + 2\epsilon\lambda_0\right) \\
&\leq \frac{|C|}{|G|}t\left(\exp\left(-n_L^{\delta_1}(\log d_L)^{\delta_2}\lambda_0\right) + 2\epsilon\lambda_0\right) \\
&\leq (1-\lambda_0)\frac{|C|}{|G|}t
\end{aligned}$$

provided that d_L is sufficiently large.

4. Proof of point (ii) of Proposition 2.1

We now assume that $\zeta_L(s)$ has no real zero in the interval $[1-\mathcal{L}^{-1}, 1]$. Let a be a constant with $a > 1$ and let $l \in \mathbb{N}$. Set $b := a^{1/l}$. We define

$$k_2(s) := \frac{t^s - 1}{s}\left(\frac{b^s - 1}{s\log b}\right)^l$$

for $t \geq 2$. We let

$$\widehat{k_2}(u) := \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} k_2(s)u^{-s}\,ds$$

be its inverse Mellin transform.

Lemma 4.1. *The support of $\widehat{k_2}$ is contained in the interval $[0, at]$. In particular, $0 \leq \widehat{k_2}(u) \leq 1$ and $\widehat{k_2}(u) \equiv 1$ for $a \leq u \leq t$.*

Proof. For $j \geq 1$, define

$$\omega(u) = \frac{l\log t}{\log a}1_{[0, \frac{\log a}{l\log t}]}(u),\ g_0(u) = 1_{[0,1]}(u),\ \text{and } g_j(u) = \int_{\mathbb{R}} \omega(\tau)g_{j-1}(u-\tau)\,d\tau.$$

Since $\int_{\mathbb{R}} \omega(u)\,du = 1$, the support of g_l is contained in the interval $\left[0, \frac{\log(at)}{\log t}\right]$, $0 \leq g_l(u) \leq 1$, and $g_l(u) \equiv 1$ for $\frac{\log a}{\log t} \leq u \leq 1$. The result follows from the fact that $\widehat{k_2}(u) = g_l\left(\frac{\log u}{\log t}\right)$. See also Lemma 3.2 of [17]. □

For our subsequent arguments we need the following lemmas.

Lemma 4.2. *If $z = x + iy \in \mathbb{C}$ with $x > 0$ and $y \in \mathbb{R}$, then*

$$\left|\frac{1-e^{-z}}{z}\right| \leq 1.$$

Proof. See [15, (2.10)]. □

Lemma 4.3. (i) *If $s = x > 0$, then*

$$0 < k_2(s) \leq \frac{a^x}{x}t^x.$$

In particular, $0 < k_2(1) \leq at$.

(ii) *If $s = -m$ with positive integer m, then*

$$0 < k_2(s) \leq \frac{1}{m^{l+1}}\left(\frac{l}{\log a}\right)^l.$$

(iii) *Let $s = x + iy \in \mathbb{C}$ with $x > 0$ and $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq l$. Then*

$$|k_2(s)| \leq \frac{2a^x t^x}{|s|}\left(\frac{2l}{|s|\log a}\right)^\alpha.$$

(iv) *If $s = x + iy \in \mathbb{C}$ with $x > 0$ and $|s| \geq 1/2$, then*

$$|k_2(s)| \leq 4a^x t^x.$$

(v) *If* $s = x + iy \in \mathbb{C}$ *with* $x > 0$ *and* $|s| \le 1/2$, *then*

$$|k_2(s)| \le \sqrt{a}t^{1/2}\log t.$$

Proof. (i) From Lemma 4.2 we have

$$0 < k_2(s) = \frac{t^x - 1}{x} a^x \left(\frac{1 - b^{-x}}{x \log b}\right)^l \le \frac{a^x}{x} t^x.$$

(ii) We have

$$0 < k_2(s) = \frac{1 - t^{-m}}{m} \left(\frac{1 - b^{-m}}{m \log b}\right)^l \le \frac{1}{m^{l+1}} \left(\frac{l}{\log a}\right)^l.$$

(iii) From Lemma 4.2 we have

$$|k_2(s)| = t^x \frac{|1 - t^{-s}|}{|s|} a^x \left|\frac{1 - b^{-s}}{s \log b}\right|^l \le \frac{2a^x t^x}{|s|} \left|\frac{1 - b^{-s}}{s \log b}\right|^\alpha \le \frac{2a^x t^x}{|s|} \left(\frac{2l}{|s| \log a}\right)^\alpha.$$

(iv) From Lemma 4.2 we have

$$|k_2(s)| = t^x \frac{|1 - t^{-s}|}{|s|} a^x \left|\frac{1 - b^{-s}}{s \log b}\right|^l \le 4a^x t^x.$$

(v) From Lemma 4.2 we have

$$|k_2(s)| = t^x \log t \left|\frac{1 - t^{-s}}{s \log t}\right| a^x \left|\frac{1 - b^{-s}}{s \log b}\right|^l \le \sqrt{a} t^{1/2} \log t.$$

□

Lemma 4.4. *Let* $\epsilon > 0$. *If* d_L *is sufficiently large and* $t \ge e^{325\mathcal{L}}$ *we have*

$$\left|\psi_C(t) - \sum_{N\mathfrak{p}^m \le t} \Theta(\mathfrak{p}^m) \log N\mathfrak{p}\right| \le \epsilon \frac{|C|}{|G|} t.$$

Proof. From the arguments in page 424 of [9] we have

$$\left|\psi_C(t) - \sum_{N\mathfrak{p}^m \le t} \Theta(\mathfrak{p}^m) \log N\mathfrak{p}\right| \le 2 \log t \log d_L.$$

Thus we have

$$\begin{aligned}
&\left|\psi_C(t) - \sum_{N\mathfrak{p}^m \le t} \Theta(\mathfrak{p}^m) \log N\mathfrak{p}\right| \\
&\le 2 \log t \log d_L \\
&\le \frac{2(\log t)^2}{325 n_L^{\delta_1} (\log d_L)^{\delta_2}} \frac{|G|}{|C|} \frac{|C|}{|G|} \frac{t}{t} \quad \left(\text{for } \log d_L \le \frac{\log t}{325 n_L^{\delta_1} (\log d_L)^{\delta_2}}\right) \\
&\le \frac{2}{325 (\log d_L)^{\delta_1 + \delta_2 - 1}} \left(\frac{n_L}{\log d_L}\right)^{1 - \delta_1} \frac{(\log t)^2}{t} \frac{|C|}{|G|} t \quad \left(\text{for } \frac{|G|}{|C|} \le n_L\right)
\end{aligned}$$

$$\le \epsilon \frac{|C|}{|G|} t \ \ (\text{for } n_L \ll \log d_L).$$

□

Let

$$I(t) := \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s) k_2(s) ds.$$

We have

$$I(t) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \Theta(\mathfrak{p}^m)(\log N\mathfrak{p}) \widehat{k_2}(N\mathfrak{p}^m),$$

where $\mathfrak{p}$ runs over all prime ideals of K.

For any given $\epsilon > 0$ we let $a := 1 + \epsilon/n_L$. Then we have $1 < a < 3/2$. To compute an upper bound for

$$\left| I(t) - \sum_{N\mathfrak{p}^m \le t} \Theta(\mathfrak{p}^m) \log N\mathfrak{p} \right|$$

we will use the following lemma.

Lemma 4.5. (i) *For $x > 1$,*

$$\pi(x) < c_{10} \frac{x}{\log x}$$

with $c_{10} = 1.25506$, where $\pi(x)$ is the number of primes p with $p \le x$.

(ii) *For $x > 1$,*

$$S(x) \le \frac{2c_{10}}{\log 2} \sqrt{x},$$

where $S(x)$ is the number of prime powers p^h with $h \ge 2$ and $p^h \le x$.

(iii) *For $x > y > 1$,*

$$\pi(x) - \pi(x-y) \le \frac{2y}{\log y}.$$

Proof. (i) and (ii) are (1) and (2) [2, Lemma 3.2], respectively. For the proof of (iii), see Theorem 2 and (1.12) in [12]. □

Lemma 4.6. *Let $\epsilon > 0$. If d_L is sufficiently large and $t \ge e^{325\mathcal{L}}$ we have*

$$\left| I(t) - \sum_{N\mathfrak{p}^m \le t} \Theta(\mathfrak{p}^m) \log N\mathfrak{p} \right| \le 5\epsilon \frac{|C|}{|G|} t.$$

Proof. We have

$$\left| I(t) - \sum_{N\mathfrak{p}^m \le t} \Theta(\mathfrak{p}^m) \log N\mathfrak{p} \right| \le \sum_{t < N\mathfrak{p}^m \le at} \Theta(\mathfrak{p}^m)(\log N\mathfrak{p}) \widehat{k_2}(N\mathfrak{p}^m)$$

$$\le n_K \int_t^{at} \log u \, d\, M(u)$$

$$\leq \frac{|C|}{|G|} n_L \int_t^{at} \log u \, d\, M(u),$$

where $M(u) = \left|\{p^h \,|\, p \text{ prime}, h \geq 1, \text{ and } p^h \leq u\}\right|$. Note that

$$\int_t^{at} \log u \, d\, M(u) = \int_t^{at} \log u \, d\, \pi(u) + \int_t^{at} \log u \, d\, S(u).$$

Then we have

$$\begin{aligned}
&\int_t^{at} \log u \, d\, \pi(u) \\
&= \left(\pi(at)\log(at) - \pi(t)\log t\right) - \int_t^{at} \frac{\pi(u)}{u} \, d\, u \\
&\leq \left(\pi(at) - \pi(t)\right)\log t + \pi(at)\log a \\
&\leq 2(a-1)t \left(\frac{\log t}{\log(a-1) + \log t} \right) + c_{10} \frac{at}{\log(at)} \log a \\
&\quad \text{(points (iii) and (i) of Lemma 4.5)} \\
&\leq \frac{2\epsilon}{n_L} t \left(\frac{\log t}{\log t + \log \epsilon - c_{11} \log\log t} \right) + \frac{3c_{10}\log(3/2)}{650 n_L^{\delta_1} (\log d_L)^{1+\delta_2}} t \\
&\quad \left(\text{for } \log n_L \ll \log\log t, \ a = 1 + \frac{\epsilon}{n_L} < \frac{3}{2}, \text{ and } \log t \geq 325 n_L^{\delta_1} (\log d_L)^{1+\delta_2} \right) \\
&\leq \frac{3\epsilon}{n_L} t + \frac{3c_{10}\log(3/2)}{650(\log d_L)^{\delta_1+\delta_2}} \left(\frac{n_L}{\log d_L} \right)^{1-\delta_1} \frac{t}{n_L} \\
&\leq \frac{4\epsilon}{n_L} t \ \text{ (for } n_L \ll \log d_L)
\end{aligned}$$

for some positive constant c_{11}. Moreover, we have

$$\begin{aligned}
\int_t^{at} \log u \, d\, S(u) &= \left(S(at)\log(at) - S(t)\log t\right) - \int_t^{at} \frac{S(u)}{u} \, d\, u \\
&\leq S(3t/2)\log(3t/2) \frac{n_L}{t} \frac{t}{n_L} \quad \left(\text{for } a < \frac{3}{2} \right) \\
&\ll \frac{n_L \log t}{\sqrt{t}} \frac{t}{n_L} \quad \text{(point (ii) of Lemma 4.5)} \\
&\ll \frac{(\log t)^{(\delta_1+\delta_2+2)/(\delta_1+\delta_2+1)}}{\sqrt{t}} \frac{t}{n_L} \quad \left(\text{for } n_L \ll (\log t)^{1/(1+\delta_1+\delta_2)} \right) \\
&\leq \epsilon \frac{t}{n_L}.
\end{aligned}$$

Thus the result follows. □

Let $l := 2(81 n_L + 162)$.

Lemma 4.7. *Let $\epsilon > 0$. If d_L is sufficiently large and $t \geq e^{325\mathcal{L}}$ we have*

$$\left| I(t) - \frac{|C|}{|G|}t \right| \leq \epsilon \frac{|C|}{|G|}t + \frac{|C|}{|G|} \sum_{\chi} \sum_{\rho} |k_2(\rho)|,$$

where ρ runs through all the nontrivial zeros of $L(s, \chi)$.

Proof. By Cauchy's residue theorem, we have

$$\frac{|G|}{|C|} I(t) = k_2(1) - k_2(0) \sum_{\chi} \overline{\chi}(g) \left(a(\chi) - \delta(\chi)\right) - \sum_{m=1}^{\infty} k_2(-2m) \sum_{\chi} \overline{\chi}(g) a(\chi)$$
$$- \sum_{m=1}^{\infty} k_2(-2m+1) \sum_{\chi} \overline{\chi}(g) b(\chi) - \sum_{\chi} \overline{\chi}(g) \sum_{\rho} k_2(\rho),$$

where $a(\chi)$ and $b(\chi)$ are non-negative integers such that $a(\chi) + b(\chi) = n_E$. Since $k_2(1) \leq at$ by the point (i) of Lemma 4.3, $k_2(0) = \log t$, $\left| \sum_{\chi} \overline{\chi}(g) \left(a(\chi) - \delta(\chi)\right) \right| \leq n_L$, and

$$\left| \sum_{m=1}^{\infty} k_2(-2m) \sum_{\chi} \overline{\chi}(g) a(\chi) + \sum_{m=1}^{\infty} k_2(-2m+1) \sum_{\chi} \overline{\chi}(g) b(\chi) \right|$$
$$\leq n_L \sum_{m=1}^{\infty} k_2(-m)$$
$$\leq n_L \left(\frac{l}{\log a} \right)^l \sum_{m=1}^{\infty} \frac{1}{m^{l+1}}$$

by the point (ii) of Lemma 4.3, we have

$$\left| I(t) - \frac{|C|}{|G|}t \right| \leq \frac{|C|}{|G|} \left[\frac{\epsilon}{n_L} t + O\left(n_L \log t + n_L \left(\frac{l}{\log a} \right)^l \right) + \sum_{\chi} \sum_{\rho} |k_2(\rho)| \right].$$

Since $n_L \ll (\log t)^{1/(\delta_1 + \delta_2 + 1)}$ we have

$$n_L \log t \ll (\log t)^{(\delta_1 + \delta_2 + 2)/(\delta_1 + \delta_2 + 1)}.$$

Moreover, we have

$$n_L \left(\frac{l}{\log a} \right)^l \leq n_L \left(\frac{2l}{a-1} \right)^l \left(\text{for } \frac{1}{\log a} \leq \frac{2}{a-1} \text{ for } 1 < a < \frac{3}{2} \right)$$
$$\leq n_L \left(\frac{648 n_L^2}{\epsilon} \right)^{324 n_L} \quad (\text{for } l = 2(81 n_L + 162) \leq 324 n_L)$$
$$\leq \exp(c_{12} n_L \log n_L)$$
$$\leq \exp(c_{13} (\log t)^{1/(\delta_1 + \delta_2 + 1)} \log \log t) \ \left(\text{for } n_L \ll (\log t)^{1/(\delta_1 + \delta_2 + 1)} \right)$$

for some positive constants c_{12} and c_{13}. Thus the result follows. □

To compute

$$\sum_{\chi}\sum_{\rho}|k_2(\rho)|$$

we will use Proposition 3.1 and the log-free zero density estimate of [15, Theorem 4.5]. Define

$$N(\sigma,T,\chi):=\sharp\{\rho=\beta+i\gamma : L(\rho,\chi)=0, \sigma<\beta<1, |\gamma|\le T\}$$

and

$$N(\sigma,T):=\sum_{\chi}N(\sigma,T,\chi)$$

for $0<\sigma<1$ and $T\ge 1$.

Proposition 4.8. *There is a constant $c_{14}>0$ such that*

$$N(\sigma,T)\le c_{14}\left(e^{162\mathcal{L}}T^{81n_L+162}\right)^{1-\sigma}.$$

Proof. It follows from [15, Theorem 4.5]. Note that our $\mathcal{L}$ is larger than $\mathcal{L}$ of [15, (4.1)]. □

Lemma 4.9. *Let $\epsilon>0$ and $T:=\epsilon^{-1}$. If d_L is sufficiently large and $t\ge e^{325\mathcal{L}}$ we have*

$$\sum_{\chi}\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|>T}}|k_2(\rho)|\le 12c_{14}\epsilon t.$$

Proof. Let $T_1\ge 1$. Let $\rho=\beta+i\gamma$ with $\beta=1-\lambda/\mathcal{L}$ and $T_1\le|\gamma|\le 2T_1$. We have

$$\begin{aligned}
\left(\frac{4l}{\log a}\right)^{l} &\le \left(\frac{8l}{a-1}\right)^{l} \quad \left(\text{for } \frac{1}{\log a}\le\frac{2}{a-1} \text{ for } 1<a<\frac{3}{2}\right)\\
&\le \left(\frac{2592n_L^2}{\epsilon}\right)^{324n_L} \quad (\text{for } l=2(81n_L+162)\le 324n_L)\\
&\le \exp(c_{15}n_L\log n_L)\\
&\le \exp(c_{16}(\log t)^{1/(\delta_1+\delta_2+1)}\log\log t) \quad \left(\text{for } n_L\ll(\log t)^{1/(\delta_1+\delta_2+1)}\right)\\
&\le t^{1/325}
\end{aligned}$$

for some positive constants c_{15} and c_{16}. Then

$$\begin{aligned}
|k_2(\rho)| &\le \frac{2a^{\beta}t^{\beta}}{|\rho|}\left(\frac{2l}{|\rho|\log a}\right)^{l(1-\beta)} \quad (\text{point (iii) of Lemma 4.3})\\
&\le \frac{2a}{T_1}t\left(\frac{4l}{t^{1/l}\log a}\right)^{l(1-\beta)}(2T_1)^{-l(1-\beta)}\\
&\qquad \left(\text{for } t^{\beta}=t\left(\frac{1}{t^{1/l}}\right)^{l(1-\beta)} \text{ and } |\rho|\ge T_1\right)
\end{aligned}$$

$$\le \frac{2a}{T_1} t^{-324(1-\beta)/325} (2T_1)^{-l(1-\beta)} t \quad \left(\text{for } \left(\frac{4l}{\log a}\right)^l \le t^{1/325}\right)$$
$$= \frac{2a}{T_1} t^{-324\lambda/(325\mathcal{L})} (2T_1)^{-l\lambda/\mathcal{L}}\, t.$$

Thus, we have

$$\sum_\chi \sum_{\substack{\rho=\beta+i\gamma \\ T_1 \le |\gamma| \le 2T_1}} |k_2(\rho)|$$
$$\le \frac{2a}{T_1} t \int_0^{\mathcal{L}} t^{-324\lambda/(325\mathcal{L})} (2T_1)^{-l\lambda/\mathcal{L}}\, d\, N(1-\lambda/\mathcal{L}, 2T_1)$$
$$\le \frac{2c_{14}a}{T_1} t \left[\frac{t^{-162/325}}{(2T_1)^{81n_L+162}}\right]$$
$$+ \frac{2c_{14}a}{T_1} t \left[\frac{324\log t}{325\mathcal{L}} + \frac{l\log(2T_1)}{\mathcal{L}}\right] \int_0^{\mathcal{L}} \frac{t^{-162\lambda/(325\mathcal{L})}}{(2T_1)^{(81n_L+162)\lambda/\mathcal{L}}}\, d\,\lambda$$

(Proposition 4.8)

$$= \frac{2c_{14}a}{T_1} t \left[\frac{t^{-162/325}}{(2T_1)^{81n_L+162}} + 2\left(1 - \frac{t^{-162/325}}{(2T_1)^{81n_L+162}}\right)\right]$$
$$\le \frac{6c_{14}}{T_1} t \ \text{ (for } 1 < a < 3/2\text{)}.$$

Hence, we have

$$\sum_\chi \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>T}} |k_2(\rho)| \le \sum_{m=0}^{\infty} \sum_\chi \sum_{\substack{\rho=\beta+i\gamma \\ 2^mT \le |\gamma| \le 2^{m+1}T}} |k_2(\rho)| \le 6c_{14} t \sum_{m=0}^{\infty} \frac{1}{2^m T} = 12 c_{14}\epsilon t.$$

□

Lemma 4.10. *Let* $\epsilon > 0$, $T := \epsilon^{-1}$, *and*

$$R := \frac{\mathcal{L}}{29.57(\log d_L + n_L \log(\epsilon^{-1}+2))}.$$

If d_L *is sufficiently large and* $t \ge e^{325\mathcal{L}}$ *we have*

$$\sum_\chi \sum_{\substack{\rho=\beta+i\gamma \\ 0<\beta\le 1-R/\mathcal{L}, |\gamma|\le T}} |k_2(\rho)| \le 2\epsilon t.$$

Proof. Note that $R \gg n_L^{\delta_1} (\log d_L)^{\delta_2}$ since $n_L \ll \log d_L$. Let $\rho = \beta + i\gamma$ with $\beta = 1 - \lambda/\mathcal{L}$ and $|\gamma| \le T = \epsilon^{-1}$. Firstly, we have

$$\sum_\chi \sum_{\substack{\rho=\beta+i\gamma \\ 0<\beta\le 1-R/\mathcal{L}, |\gamma|\le T, |\rho|\ge 1/2}} |k_2(\rho)|$$
$$\le 4at \int_R^{\mathcal{L}} t^{-\lambda/\mathcal{L}}\, d\, N(1-\lambda/\mathcal{L}, \epsilon^{-1}) \ \text{ (point (iv) of Lemma 4.3)}$$

$$\le 4c_{14}at\left[\frac{(\epsilon^{-1})^{81n_L+162}}{t^{163/325}}+\frac{\log t}{\mathcal{L}}\int_R^{\mathcal{L}}\frac{(\epsilon^{-1})^{(81n_L+162)\lambda/\mathcal{L}}}{t^{163\lambda/(325\mathcal{L})}}\,d\lambda\right]\ \text{(Proposition 4.8)}$$
$$\le 6c_{14}t\left[\frac{(\epsilon^{-1})^{81n_L+162}}{t^{163/325}}+\frac{325\log t}{163\log t-325(81n_L+162)\log(\epsilon^{-1})}\frac{(\epsilon^{-1})^{(81n_L+162)R/\mathcal{L}}}{t^{163R/(325\mathcal{L})}}\right]$$
$$\left(\text{for } 1<a<\frac{3}{2} \text{ and } n_L \ll (\log t)^{1/(\delta_1+\delta_2+1)}\right).$$

Moreover, we have

$$\begin{aligned}
&\frac{(\epsilon^{-1})^{(81n_L+162)R/\mathcal{L}}}{t^{163R/(325\mathcal{L})}}\\
&= \exp\left(-\frac{163R\log t}{325\mathcal{L}}+\frac{(81n_L+162)R\log(\epsilon^{-1})}{\mathcal{L}}\right)\\
&\ll \exp\left(-\frac{163R\log t}{325\mathcal{L}}\right)\ \left(\text{as } \frac{(81n_L+162)R\log(\epsilon^{-1})}{\mathcal{L}} \text{ is bounded above}\right)\\
&\le \exp\left(-c_{17}n_L^{\delta_1}(\log d_L)^{\delta_2}\right)\ \left(\text{for } \log t\ge 325\mathcal{L} \text{ and } R\gg n_L^{\delta_1}(\log d_L)^{\delta_2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{(\epsilon^{-1})^{81n_L+162}}{t^{163/325}} &= \exp\left(-\frac{163}{325}\log t+(81n_L+162)\log(\epsilon^{-1})\right)\\
&\le \exp\left(-\frac{163}{325}\log t+c_{18}\log(\epsilon^{-1})(\log t)^{1/(\delta_1+\delta_2+1)}\right)\\
&\quad \left(\text{for } n_L\ll(\log t)^{1/(\delta_1+\delta_2+1)}\right)
\end{aligned}$$

for some positive constants c_{17} and c_{18}. Hence, we have

$$\sum_{\chi}\sum_{\substack{\rho=\beta+i\gamma\\ 0<\beta\le 1-R/\mathcal{L},|\gamma|\le T,|\rho|\ge 1/2}}|k_2(\rho)|\le \epsilon t.$$

Secondly, we have

$$\begin{aligned}
\sum_{\chi}\sum_{\substack{\rho\\|\rho|\le 1/2}}|k_2(\rho)| &\ll t^{1/2}\log t\sum_{\chi}\sum_{\substack{\rho\\|\rho|\le 1/2}}1\ \text{(point (v) of Lemma 4.3)}\\
&\ll t^{1/2}\log t\log d_L\ \text{(Proposition 3.1 and } n_L\ll\log d_L)\\
&\ll \frac{(\log t)^{(2+\delta_2)/(1+\delta_2)}}{t^{1/2}}t\ \left(\text{for } \log d_L\ll(\log t)^{1/(1+\delta_2)}\right)\\
&\le \epsilon t.
\end{aligned}$$

Thus the result follows. □

Lemma 4.11. *Let* $\epsilon>0$, $T:=\epsilon^{-1}$, *and*

$$R:=\frac{\mathcal{L}}{29.57(\log d_L+n_L\log(\epsilon^{-1}+2))}.$$

If d_L is sufficiently large and $t \geq e^{325\mathcal{L}}$ we have

$$\sum_{\chi} \sum_{\substack{\rho=\beta+i\gamma \\ 1-R/\mathcal{L}<\beta<1, |\gamma|\leq T}} |k_2(\rho)| \leq (1-(10+12c_{14})\epsilon)t.$$

Proof. From Proposition 3.6 and the definition of R we have

$$\sum_{\chi} \sum_{\substack{\rho=\beta+i\gamma \\ 1-R/\mathcal{L}<\beta<1, |\gamma|\leq T}} |k_2(\rho)| = 0 \text{ or } k_2(\beta_0).$$

Thus we have

$$\sum_{\chi} \sum_{\substack{\rho=\beta+i\gamma \\ 1-R/\mathcal{L}<\beta<1, |\gamma|\leq T}} |k_2(\rho)| \leq k_2(\beta_0),$$

where β_0 is the exceptional real zero of $\zeta_L(s)$ such that $1-\beta_0 \geq \mathcal{L}^{-1}$. Thus the result follows from the following inequality

$$\begin{aligned} |k_2(\beta_0)| &\leq 3t^{\beta_0} \ \text{(for } 1<a<3/2,\ \beta_0>1/2, \text{ and point (i) of Lemma 4.3)} \\ &= 3t\exp(-(1-\beta_0)\log t) \\ &\leq 3e^{-325}t \ \text{(for } (1-\beta_0)\log t \geq 325) \\ &\leq (1-(10+12c_{14})\epsilon)t. \end{aligned}$$

□

Now we use the same ϵ in Lemmas 4.4, 4.6, 4.7, 4.9, 4.10, and 4.11. Gathering Lemmas 4.4, 4.6, 4.7, 4.9, 4.10, and 4.11 we obtain, for $t \geq e^{325\mathcal{L}}$

$$\left|\psi_C(t) - \frac{|C|}{|G|}t\right| \leq (1-\epsilon)\frac{|C|}{|G|}t$$

provided that d_L is sufficiently large.

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Jeoung-Hwan Ahn
Institute of Basic Science
Korea University
02841 Seoul, Korea
Email address: jh-ahn@korea.ac.kr

Soun-Hi Kwon
Department of Mathematics Education
Korea University
02841 Seoul, Korea
Email address: sounhikwon@korea.ac.kr