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FOURIER TRANSFORM OF ANISOTROPIC MIXED-NORM HARDY SPACES WITH APPLICATIONS TO HARDY–LITTLEWOOD INEQUALITIES

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Abstract. Let $\vec{p} \in (0,1]^n$ be an n-dimensional vector and A a dilation. Let $H_A^{\vec{p}}(\mathbb{R}^n)$ denote the anisotropic mixed-norm Hardy space defined via the radial maximal function. Using the known atomic characterization of $H_A^{\vec{p}}(\mathbb{R}^n)$ and establishing a uniform estimate for corresponding atoms, the authors prove that the Fourier transform of $f \in H_A^{\vec{p}}(\mathbb{R}^n)$ coincides with a continuous function F on $\mathbb{R}^n$ in the sense of tempered distributions. Moreover, the function F can be controlled pointwisely by the product of the Hardy space norm of f and a step function with respect to the transpose matrix of A. As applications, the authors obtain a higher order of convergence for the function F at the origin, and an analogue of Hardy–Littlewood inequalities in the present setting of $H_A^{\vec{p}}(\mathbb{R}^n)$.

1. Introduction

Let $\vec{p} := (p_1, \dots, p_n) \in (0,\infty)^n$ be an n-dimensional vector and A a dilation. The anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ was introduced in [18]. The main purpose of this paper is to study the Fourier transform on $H_A^{\vec{p}}(\mathbb{R}^n)$ associated with $\vec{p} \in (0,1]^n$. The question of the Fourier transform on classical Hardy spaces $H^p(\mathbb{R}^n)$ was put forward originally by Fefferman and Stein [12], which is an important topic in the real-variable theory of $H^p(\mathbb{R}^n)$. Applying entire functions of exponential type, Coifman [10] first characterized the Fourier transform $\widehat{f}$ of $f \in H^p(\mathbb{R})$. The related conclusions in higher dimensions were studied in [2, 11, 13, 25]. Particularly, the following estimate was given by Taibleson and Weiss [25]: for any given $p \in (0,1]$, the Fourier transform of $f \in H^p(\mathbb{R}^n)$

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coincides with a continuous function F on $\mathbb{R}^n$, which satisfies that there exists a positive constant $C_{(n,p)}$ such that, for any $x \in \mathbb{R}^n$,

$$|F(x)| \le C_{(n,p)} \|f\|_{H^p(\mathbb{R}^n)} |x|^{n(1/p-1)}. \tag{1}$$

Moreover, the estimate (1) illustrates the following inequality as a generalization of the well-known Hardy–Littlewood inequality for Hardy spaces, that is, for any fixed $p \in (0,1]$, there exists a positive constant K such that, for each $f \in H^p(\mathbb{R}^n)$,

$$\left[\int_{\mathbb{R}^n} |x|^{n(p-2)} |F(x)|^p \, dx\right]^{1/p} \le K \|f\|_{H^p(\mathbb{R}^n)}, \tag{2}$$

where F is as in (1); see [23, p. 128].

On the other hand, the theory of classic Hardy spaces $H^p(\mathbb{R}^n)$ has a wide range of applications in many mathematical fields such as harmonic analysis and partial differential equations; see, for instance, [12, 21, 23, 24]. Inspired by the notable work of Calderón and Torchinsky [3] on parabolic Hardy spaces, there were various generalizations of classic Hardy spaces; see, for instance, [1,8, 14,18,26–28]. In particular, Bownik [1] introduced the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$, where $p \in (0,\infty)$ and A is a dilation, which is actually a generalization of both the isotropic Hardy space and the parabolic Hardy space. In addition, via the atomic characterization of $H^p_A(\mathbb{R}^n)$, Bownik and Wang [2] extended both inequalities (1) and (2) to the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$. Recently, the analogous results were proved in the new setting of Hardy spaces associated with ball quasi-Banach function spaces and the anisotropic mixed-norm Hardy space $H^{\vec{p}}_{\vec{a}}(\mathbb{R}^n)$, where

$$\vec{a} := (a_1, \ldots, a_n) \in [1,\infty)^n \quad \text{and} \quad \vec{p} := (p_1, \ldots, p_n) \in (0,1]^n;$$

see, respectively, [15, 16]. In addition, motivated by the previous work of [8, 12, 17], Huang et al. [18] introduced the anisotropic mixed-norm Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ with respect to $\vec{p} \in (0,\infty)^n$ and a dilation A, and investigated its various real-variable characterizations. For more information on mixed-norm function spaces, we refer the reader to [4–7, 9, 19, 20, 22].

Inspired by the known results about the Fourier transform of the aforementioned Hardy-type spaces (namely, $H^p(\mathbb{R}^n)$, $H^p_A(\mathbb{R}^n)$ and $H^{\vec{p}}_{\vec{a}}(\mathbb{R}^n)$), using the real-variable theory of the anisotropic mixed-normed Hardy space $H^{\vec{p}}_A(\mathbb{R}^n)$ from [18], in this paper, we extend the inequality (1) to the setting of anisotropic mixed-norm Hardy spaces $H^{\vec{p}}_A(\mathbb{R}^n)$ and also present some applications via our main result.

As a preliminary, in Section 2, we present definitions of dilations, mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ and anisotropic mixed-norm Hardy spaces.

Section 3 is aimed at proving the main result (see Theorem 3.1 below), namely, the Fourier transform $\widehat{f}$ of $f \in H^{\vec{p}}_A(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions. To this end, applying Lemmas 3.2 and 3.4, we first obtain a uniform pointwise estimate for atoms (see Lemma 3.3 below). Then, we use some real-variable characterizations from [18], especially atom decompositions, to show Theorem 3.1. Meanwhile, we also get a pointwise inequality of the continuous function F, which indicates the necessity of vanishing moments of anisotropic mixed-norm atoms in some sense (see Remark 3.7(ii) below).

As applications, in Section 4, we present some consequences of Theorem 3.1. First, the above function F has a higher order convergence at the origin; see (19) below. Moreover, we prove that the term

$$|F(\cdot)| \min\left\{[\rho_*(\cdot)]^{1-\frac{1}{p_-}-\frac{1}{p_+}}, [\rho_*(\cdot)]^{1-\frac{2}{p_+}}\right\}$$

is L^{p_+}-integrable, and this integral can be uniformly controlled by a positive constant multiple of the Hardy space norm of f; see (25) below. The above result is actually a generalization of the Hardy–Littlewood inequality from classic Hardy spaces to the setting of anisotropic mixed-norm Hardy spaces.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbf{0}$ be the *origin* of $\mathbb{R}^n$. For a given multi-index $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. We use C to denote a positive constant which is independent of the main parameters, but may vary in different setting. The *symbol* $g \lesssim h$ means $g \le Ch$ and, if $g \lesssim h \lesssim g$, then we write $g \sim h$. If $f \le Ch$ and $h = g$ or $h \le g$, we then write $f \lesssim h \sim g$ or $f \lesssim h \lesssim g$, *rather than* $f \lesssim h = g$ or $f \lesssim h \le g$. In addition, for any set $E \subset \mathbb{R}^n$, we denote its *characteristic function* by $\mathbf{1}_E$, the set $\mathbb{R}^n \setminus E$ by $E^{\complement}$ and its *n-dimensional Lebesgue measure* by $|E|$. For any $s \in \mathbb{R}$, we use $\lfloor s \rfloor$ (resp., $\lceil s \rceil$) to denote the *largest* (resp., *least*) *integer not* greater (resp., *less*) *than* s.

2. Preliminaries

In this section, we give the definitions of dilations, mixed-norm Lebesgue spaces and anisotropic mixed-norm Hardy spaces. The following definition is originally from [1].

Definition 1. We call A a *dilation* if A is a real $n \times n$ matrix A and satisfies the following condition:

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

where $\sigma(A)$ denotes the *set of all eigenvalues of* A. We denote the eigenvalues of A by $\lambda_1, \ldots, \lambda_n$, which satisfies $1 < |\lambda_1| \le \cdots \le |\lambda_n|$. Here and thereafter, let λ_- and λ_+ be two numbers such that $1 < \lambda_- < |\lambda_1| \le \cdots \le |\lambda_n| < \lambda_+$.

By [1, p. 5, Lemma 2.2], for a given dilation A, there exists an open set in $\mathbb{R}^n$ which is called an *ellipsoid*, denoted by Δ, and has the following property: $|\Delta| = 1$, and we can find a constant $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$. For any given $i \in \mathbb{Z}$, we denote $A^i\Delta$ by B_i. It is easy to check that $\{B_i\}_{i\in\mathbb{Z}}$ is a family of open sets around the origin, $B_i \subset rB_i \subset B_{i+1}$ and $|B_i| = b^i$ with $b := |\det A|$. For any given dilation A, the notation $\mathfrak{B}$ is the set of all *dilated balls*, namely,

$$\mathfrak{B} := \{x + B_i : x \in \mathbb{R}^n,\ i \in \mathbb{Z}\}. \tag{3}$$

The next two definitions were introduced by Bownik [1].

Definition 2. A measurable mapping $\rho : \mathbb{R}^n \to [0,\infty)$ is called a *homogeneous quasi-norm*, with respect to a dilation A, if

(i) $\rho(x) \geq 0$, and $\rho(x) = 0 \Rightarrow x = 0$;
(ii) for any $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$;
(iii) for any x, $y \in \mathbb{R}^n$, $\rho(x + y) \leq c[\rho(x) + \rho(y)]$, where c is a positive constant independent of x and y.

It is easy to verify that the following *step homogeneous quasi-norm* is a homogeneous quasi-norm.

Definition 3. A *step homogeneous quasi-norm* ρ with respect to a dilation A, is defined by setting, for each $x \in \mathbb{R}^n$,

$$\rho(x) := \begin{cases} b^i & \text{when} \quad x \in B_{i+1}\backslash B_i, \\ 0 & \text{when} \quad x = \mathbf{0}. \end{cases}$$

In [1, p. 5, Lemma 2.4], it was proved that any two homogeneous quasi-norms associated with a fixed dilation A are equivalent. For convenience, in what follows, we always use the step homogeneous quasi-norm.

A C^∞ complex-valued function ϕ on $\mathbb{R}^n$ is called a *Schwartz function* if, for every pair of $k \in \mathbb{Z}_+$ and multi-index $\gamma \in \mathbb{Z}_+^n$, the following inequality

$$\|\phi\|_{\gamma,k} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^k \, |\partial^\gamma \phi(x)| < \infty$$

holds true. The set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$. Indeed, $\{\|\cdot\|_{\gamma,k}\}_{\gamma\in\mathbb{Z}_+^n,\, k\in\mathbb{Z}_+}$ is a family of semi-norms, which induces a topology and makes $\mathcal{S}(\mathbb{R}^n)$ to be a topological vector space. We denote the dual space of $\mathcal{S}(\mathbb{R}^n)$ by $\mathcal{S}'(\mathbb{R}^n)$, equipped with the weak-$*$ topology.

For an n-dimensional vector $\vec{p} := (p_1, \ldots, p_n) \in (0,\infty]^n$, let

$$p_- := \min_{i\in\{1,\ldots,n\}} \{p_i\}, \quad p_+ := \max_{i\in\{1,\ldots,n\}} \{p_i\}, \quad \text{and} \quad \underline{p} := \min\{p_-, 1\}. \tag{4}$$

Definition 4. Let $\vec{p} := (p_1, \ldots, p_n) \in (0,\infty]^n$. The *mixed-norm Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} \cdots dx_n\right)^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \ldots, n\}$.

Definition 5. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$. The *radial maximal function* $M_\varphi(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$, with respect to φ, is defined by

$$M_\varphi(f)(x) := \sup_{k\in\mathbb{Z}} |f * \varphi_k(x)|, \quad \forall\, x \in \mathbb{R}^n,$$

here and thereafter, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, $\varphi_k(\cdot) := b^k \varphi(A^k \cdot)$.

Definition 6. Let $\vec{p} \in (0,\infty)^n$ and φ be as in Definition 5. The *anisotropic mixed-norm Hardy space* $H_A^{\vec{p}}(\mathbb{R}^n)$ is defined by setting

$$H_A^{\vec{p}}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : M_\varphi(f) \in L^{\vec{p}}(\mathbb{R}^n) \right\}.$$

Moreover, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$, let $\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} := \|M_\varphi(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}$.

3. Fourier transforms of $H_A^{\vec{p}}(\mathbb{R}^n)$

In this section, we study the Fourier transform $\widehat{f}$ of $f \in H_A^{\vec{p}}(\mathbb{R}^n)$. We first present the notion of Fourier transforms.

For a given Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define its *Fourier transform* as follows:

$$\mathscr{F}\varphi(x) = \widehat{\varphi}(x) := \int_{\mathbb{R}^n} \varphi(t) e^{-2\pi \imath t\cdot x}\,dt, \quad \forall\, x \in \mathbb{R}^n,$$

where $\imath := \sqrt{-1}$ and $t \cdot x := \sum_{k=1}^n t_k x_k$ for any $t := (t_1,\ldots,t_n)$, $x := (x_1,\ldots,x_n) \in \mathbb{R}^n$. Furthermore, we can also define the *Fourier transform* of $f \in \mathcal{S}'(\mathbb{R}^n)$, also denoted by $\mathscr{F}f$ or $\widehat{f}$, that is, for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \mathscr{F}f, \varphi \rangle = \left\langle \widehat{f}, \varphi \right\rangle := \langle f, \widehat{\varphi} \rangle.$$

We now give the main result of this paper.

Theorem 3.1. *Let $\vec{p} \in (0,1]^n$. Then, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that*

$$\widehat{f} = F \quad in \quad \mathcal{S}'(\mathbb{R}^n),$$

and there exists a positive constant C, depending only on A and $\vec{p}$, such that, for any $x \in \mathbb{R}^n$,

$$|F(x)| \le C\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \max\left\{ [\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1} \right\}, \tag{5}$$

here and thereafter, ρ_* is as in Section 2 with A replaced by its transposed matrix A^*.

Recall that, for a given measurable set $E \subset \mathbb{R}^n$, the *Lebesgue space* $L^p(E)$, $0 < p < \infty$, is the set of all the measurable functions satisfying that

$$\|f\|_{L^p(E)} := \left[\int_E |f(x)|^p\,dx \right]^{1/p} < \infty,$$

and $L^\infty(E)$ is the set of all the measurable functions satisfying that

$$\|f\|_{L^\infty(E)} := \operatorname*{ess\,sup}_{x\in E} |f(x)| < \infty.$$

The dilation operator D_A is defined by setting, for any measurable function f on $\mathbb{R}^n$,

$$D_A(f)(\cdot) := f(A\cdot).$$

Then, for any $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, the following identity

$$\widehat{f}(x) = b^k \left(D^k_{A^*} \mathscr{F} D^k_A f\right)(x)$$

can be easily verified.

Next, we present some notions appearing in the real-variable characterizations of anisotropic mixed-norm Hardy spaces; see [18].

Definition 7. Let $\vec{p} \in (0,\infty)^n$, $r \in (1,\infty]$ and

$$s \in \left[\left\lfloor \left(\frac{1}{p_-} - 1\right) \frac{\ln b}{\ln \lambda_-}\right\rfloor, \infty\right) \cap \mathbb{Z}_+, \tag{6}$$

where p_- is as in (4).

(I) A measurable function a on $\mathbb{R}^n$ is called an *anisotropic* $(\vec{p}, r, s)$*-atom* (simply, a $(\vec{p}, r, s)$*-atom*) if
  (i) supp $a \subset B$, where $B \in \mathfrak{B}$ with $\mathfrak{B}$ as in (1);
  (ii) $\|a\|_{L^r(\mathbb{R}^n)} \le \frac{|B|^{1/r}}{\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)}}$;
  (iii) $\int_{\mathbb{R}^n} a(x) x^\gamma \, dx = 0$ for any $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \le s$.

(II) The *anisotropic mixed-norm atomic Hardy space* $H^{\vec{p},r,s}_A(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s)$-atoms $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

Furthermore, for any $f \in H^{\vec{p},r,s}_A(\mathbb{R}^n)$, let

$$\|f\|_{H^{\vec{p},r,s}_A(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where the infimum is taken over all the decompositions of f as above.

By an argument similar to that used in proof [2, Lemma 4], we immediately obtain Lemma 3.2, which will be used to prove Lemma 3.3 below; the details are omitted.

Lemma 3.2. *Let $\vec{p}$, r and s be as in Definition 7. Assume that a is a $(\vec{p}, r, s)$-atom supported in $x_0 + B_{i_0}$ with some $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathbb{Z}$. Then there exists* *a positive constant C, depending only on A and s, such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$ and $x \in \mathbb{R}^n$,*

$$\left|\partial^\alpha \left(\mathscr{F} D_A^{i_0} a\right)(x)\right| \le C b^{-i_0/r} \|a\|_{L^r(\mathbb{R}^n)} \min\left\{1, |x|^{s-|\alpha|+1}\right\}.$$

Applying Lemma 3.2, we obtain a uniform estimate for $(\vec{p}, r, s)$-atoms as follows, which plays a key role in the proof of Theorem 3.1.

Lemma 3.3. *Let $\vec{p} \in (0,1]^n$, $r \in (1,\infty]$ and s be as in (6). Then there exists a positive constant C such that, for any $(\vec{p}, r, s)$-atom a and $x \in \mathbb{R}^n$,*

$$|\widehat{a}(x)| \le C \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1},\ [\rho_*(x)]^{\frac{1}{p_+}-1}\right\}, \tag{7}$$

where ρ_* is as in Theorem 3.1.

The following inequalities will be used to prove Lemma 3.3, which are just [1, p. 11, Lemma 3.2].

Lemma 3.4. *Let A be a given dilation. There exists a positive constant C such that, for any $x \in \mathbb{R}^n$,*

$$\frac{1}{C}[\rho(x)]^{\ln \lambda_-/\ln b} \le |x| \le C[\rho(x)]^{\ln \lambda_+/\ln b} \qquad \text{when } \rho(x) \in (1,\infty),$$

and

$$\frac{1}{C}[\rho(x)]^{\ln \lambda_+/\ln b} \le |x| \le C[\rho(x)]^{\ln \lambda_-/\ln b} \qquad \text{when } \rho(x) \in [0,1],$$

where λ_- and λ_+ are as in Section 2.

We now give the proof of Lemma 3.3.

Proof of Lemma 3.3. Let a be a $(\vec{p}, r, s)$-atom supported in $x_0 + B_{i_0}$ with some $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathbb{Z}$. Without loss of generality, we may assume $x_0 = \mathbf{0}$. By Lemma 3.2 with $\alpha = (\overbrace{0, \dots, 0}^{n \text{ times}})$, we find that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |\widehat{a}(x)| &= \left|b^{i_0}\left(D_{A^*}^{i_0} \mathscr{F} D_A^{i_0} a\right)(x)\right| = \left|b^{i_0}\left(\mathscr{F} D_A^{i_0} a\right)\left((A^*)^{i_0} x\right)\right| \\ &\lesssim b^{i_0} b^{-i_0/r} \|a\|_{L^r(\mathbb{R}^n)} \min\left\{1, \left|(A^*)^{i_0} x\right|^{s+1}\right\} \\ &\lesssim b^{i_0} \left\|\mathbf{1}_{B_{i_0}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \min\left\{1, \left|(A^*)^{i_0} x\right|^{s+1}\right\}. \end{aligned} \tag{8}$$

Next, we show that

$$\left\|\mathbf{1}_{B_{i_0}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \lesssim \max\left\{b^{-\frac{i_0}{p_-}},\ b^{-\frac{i_0}{p_+}}\right\}. \tag{9}$$

Indeed, there exists a $K \in \mathbb{Z}$ large enough such that, if $i_0 \in (K,\infty) \cap \mathbb{Z}$, then

$$\left\|\mathbf{1}_{B_{i_0}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |\mathbf{1}_{B_{i_0}}|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} \cdots dx_n\right)^{\frac{1}{p_n}}$$

$$\geq \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |\mathbf{1}_{B_{i_0}}|^{p_+} dx_1\right)^{\frac{p_+}{p_+}} \cdots dx_n\right)^{\frac{1}{p_+}} = b^{\frac{i_0}{p_+}}.$$

On the other hand, if $i_0 \in (-\infty, K]$, by [18, Lemma 6.8], we conclude that, for any $\varepsilon \in (0,1)$,

$$\frac{\|\mathbf{1}_{B_K}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\mathbf{1}_{B_{i_0}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \lesssim b^{(K-i_0)\frac{1+\varepsilon}{p_-}}.$$

Letting $\varepsilon \to 0$, we have

$$\|\mathbf{1}_{B_{i_0}}\|^{-1}_{L^{\vec{p}}(\mathbb{R}^n)} \lesssim \frac{b^{\frac{K}{p_-}}}{\|\mathbf{1}_{B_K}\|_{L^{\vec{p}}(\mathbb{R}^n)}} b^{-\frac{i_0}{p_-}}.$$

Thus, (9) holds true. From this and (8), it follows that, for any $x \in \mathbb{R}^n$,

$$|\widehat{a}(x)| \lesssim b^{i_0} \max\left\{b^{-\frac{i_0}{p_-}}, b^{-\frac{i_0}{p_+}}\right\} \min\left\{1, \left|(A^*)^{i_0} x\right|^{s+1}\right\}. \tag{10}$$

We next prove (7) by considering two cases: $\rho_*(x) \leq b^{-i_0}$ and $\rho_*(x) > b^{-i_0}$.

Case 1: $\rho_*(x) \leq b^{-i_0}$. In this case, note that $\rho_*((A^*)^{i_0} x) \leq 1$. From (10), Lemma 3.4 and the fact that

$$1 - \frac{1}{p_+} + (s+1)\frac{\ln \lambda_-}{\ln b} \geq 1 - \frac{1}{p_-} + (s+1)\frac{\ln \lambda_-}{\ln b} > 0,$$

we deduce that, for any $x \in \mathbb{R}^n$ satisfying $\rho_*(x) \leq b^{-i_0}$,

$$\begin{aligned} |\widehat{a}(x)| &\lesssim b^{i_0} \max\left\{b^{-\frac{i_0}{p_-}}, b^{-\frac{i_0}{p_+}}\right\} \left[\rho_*\left((A^*)^{i_0} x\right)\right]^{(s+1)\frac{\ln \lambda_-}{\ln b}} \\ &\sim \max\left\{b^{i_0[1-\frac{1}{p_-}+(s+1)\frac{\ln \lambda_-}{\ln b}]}, b^{i_0[1-\frac{1}{p_+}+(s+1)\frac{\ln \lambda_-}{\ln b}]}\right\} [\rho_*(x)]^{(s+1)\frac{\ln \lambda_-}{\ln b}} \\ &\lesssim \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\}. \end{aligned} \tag{11}$$

This shows (7) for Case 1.

Case 2: $\rho_*(x) > b^{-i_0}$. In this case, note that $\rho_*((A^*)^{i_0} x) > 1$. Using (10), Lemma 3.4 again and the fact that

$$\frac{1}{p_-} - 1 \geq \frac{1}{p_+} - 1 \geq 0,$$

it is easy to see that, for any $x \in \mathbb{R}^n$ satisfying $\rho_*(x) > b^{-i_0}$,

$$\begin{aligned} |\widehat{a}(x)| &\lesssim b^{i_0} \max\left\{b^{-\frac{i_0}{p_-}}, b^{-\frac{i_0}{p_+}}\right\} \sim \max\left\{b^{(1-\frac{1}{p_-})i_0}, b^{(1-\frac{1}{p_+})i_0}\right\} \\ &\lesssim \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\}, \end{aligned}$$

which completes the proof of (7) and hence of Lemma 3.3. ☐

Lemma 3.5. *Let $\vec{p} \in (0,1]^n$. Then, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$,*

$$\sum_{i\in\mathbb{N}} |\lambda_i| \le \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where $\underline{p}$ is as in (4).

Proof. Observe that, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\gamma \in (0,1]$,

$$\left(\sum_{i=1}^{\infty} |\lambda_i| \right)^{\gamma} \le \sum_{i=1}^{\infty} |\lambda_i|^{\gamma}. \tag{12}$$

By this and the inverse Minkovski inequality, we know that

$$\begin{aligned} \left\| \left\{ \sum_{i=1}^{\infty} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} &\ge \left\| \sum_{i=1}^{\infty} \frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ &\ge \left\| \sum_{i=1}^{N} \frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \ge \sum_{i=1}^{N} |\lambda_i|. \end{aligned}$$

Letting $N \to \infty$, we obtain the desired inequality as in Lemma 3.5. □

To show Theorem 3.1, we also need the following atomic characterizations of $H_A^{\vec{p}}(\mathbb{R}^n)$, which is just [18, Theorem 4.7].

Lemma 3.6. *Let $\vec{p} \in (0,\infty)^n$, $r \in (\max\{p_+, 1\}, \infty]$ with p_+ as in* (4)*, s be as in* (6) *and*

$$N \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{\min\{p_-, 1\}} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right\rfloor + 2, \infty \right)$$

with p_- as in (4)*. Then $H_A^{\vec{p}}(\mathbb{R}^n) = H_A^{\vec{p},r,s}(\mathbb{R}^n)$ with equivalent quasi-norms.*

We now prove Theorem 3.1.

Proof of Theorem 3.1. Let $\vec{p} \in (0,1]^n$, $r \in (\max\{p_+, 1\}, \infty]$, s be as in (6) and $f \in H_A^{\vec{p}}(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} > 0$. Then, by Lemma 3.6, we find that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s)$-atoms $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$, such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{13}$$

and

$$\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}. \tag{14}$$

Taking the Fourier transform on the both sides of (13), we have

$$\widehat{f} = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n). \tag{15}$$

Note that a function $f \in L^1(\mathbb{R}^n)$ implies that $\widehat{f}$ is well defined in $\mathbb{R}^n$, so does $\widehat{a_i}$ for any $i \in \mathbb{N}$. From Lemmas 3.3 and 3.5, and (14), it follows that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\sum_{i\in\mathbb{N}} |\lambda_i||\widehat{a_i}(x)| &\lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\} \\ &\lesssim \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\} < \infty. \end{aligned}\tag{16}$$

Therefore, for any $x \in \mathbb{R}^n$, the function

$$F(x) := \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(x) \tag{17}$$

is well defined pointwisely and

$$|F(x)| \lesssim \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\}.$$

We next show the continuity of the function F on $\mathbb{R}^n$. If we can prove that F is continuous on any compact subset of $\mathbb{R}^n$, then the continuity on $\mathbb{R}^n$ is obvious. Indeed, for any compact subset E, there exists a positive constant K, depending only on A and E, such that $\rho_*(x) \le K$ holds for every $x \in E$. By this and (16), we conclude that, for any $x \in E$,

$$\sum_{i\in\mathbb{N}} |\lambda_i||\widehat{a_i}(x)| \lesssim \max\left\{K^{\frac{1}{p_-}-1}, K^{\frac{1}{p_+}-1}\right\} \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} < \infty.$$

Thus, the summation $\sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(\cdot)$ converges uniformly on E. This, together with the fact that, for any $i \in \mathbb{N}$, $\widehat{a_i}(x)$ is continuous, implies that F is also continuous on any compact subset E and hence on $\mathbb{R}^n$.

Finally, to complete the proof of Theorem 3.1, by (15) and (17), we only need to show that

$$F = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n). \tag{18}$$

For this purpose, from Lemma 3.3 and the definition of Schwartz functions, we deduce that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $i \in \mathbb{N}$,

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} \widehat{a_i}(x)\varphi(x)\,dx\right| \\ &\le \sum_{k=1}^{\infty} \int_{(A^*)^{k+1}B_0^*\setminus(A^*)^k B_0^*} \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\} |\varphi(x)|\,dx \\ &\quad + \|\varphi\|_{L^1(\mathbb{R}^n)} \end{aligned}$$

$$\lesssim \sum_{k=1}^{\infty} b^k b^{k(\frac{1}{p_-}-1)} b^{-k(\lceil \frac{1}{p_-}-1\rceil+2)} + \|\varphi\|_{L^1(\mathbb{R}^n)}$$
$$\sim \sum_{k=1}^{\infty} b^{-k} + \|\varphi\|_{L^1(\mathbb{R}^n)},$$

where B_0^* is the unit dilated ball with respect to A^*. This implies that there exists a positive constant C such that $\left|\int_{\mathbb{R}^n} \widehat{a_i}(x)\varphi(x)\,dx\right| \le C$ holds true uniformly for any $i \in \mathbb{Z}$. Combining this, Lemma 3.5 and (14), we have

$$\lim_{I\to\infty} \sum_{i=I+1}^{\infty} |\lambda_i| \left|\int_{\mathbb{R}^n} \widehat{a_i}(x)\varphi(x)\,dx\right| \lesssim \lim_{I\to\infty} \sum_{i=I+1}^{\infty} |\lambda_i| = 0.$$

Therefore, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle F, \varphi\rangle = \lim_{I\to\infty} \left\langle \sum_{i=1}^{I} \lambda_i \widehat{a}_i,\, \varphi \right\rangle.$$

This finishes the proof of (18) and hence of Theorem 3.1. □

Remark 3.7. (i) When $\vec{p} = (p, \dots, p) \in (0, 1]^n$, the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ in Theorem 3.1 coincides with the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [1], and the inequality (5) becomes

$$|F(x)| \le C\|f\|_{H_A^p(\mathbb{R}^n)}[\rho_*(x)]^{\frac{1}{p}-1}$$

with C as in (5). In this case, Theorem 3.1 is just [2, Theorem 1].

(ii) Let $f \in H_A^{\vec{p}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. By the inequality (5) with $x = \mathbf{0}$, we obtain $F = \widehat{f}$ and $\widehat{f}(\mathbf{0}) = 0$. Thus, the function $f \in H_A^{\vec{p}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has a vanishing moment, which illustrates the necessity of the vanishing moment of atoms in some sense.

(iii) Very recently, in [15, Theorem 2.4], Huang et al. obtained a result similar to Theorem 3.1 in the setting of the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, where

$$\vec{a} := (a_1, \dots, a_n) \in [1, \infty)^n \quad \text{and} \quad \vec{p} := (p_1, \dots, p_n) \in (0, 1]^n.$$

We should point out that if

$$A := \begin{pmatrix} 2^{a_1} & 0 & \cdots & 0 \\ 0 & 2^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{a_n} \end{pmatrix},$$

then $H_A^{\vec{p}}(\mathbb{R}^n)=H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ with equivalent quasi-norms. In this sense, Theorem 3.1 covers [15, Theorem 2.4] as a special case.

4. Applications

As applications of Theorem 3.1, we first prove the function F given in Theorem 3.1 has a higher order convergence at the origin. Then we extend the Hardy–Littlewood inequality to the setting of anisotropic mixed-norm Hardy spaces.

We embark on the proof of the first desired result.

Theorem 4.1. *Let $\vec{p} \in (0,1]^n$. Then, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that $\widehat{f} = F$ in $\mathcal{S}'(\mathbb{R}^n)$ and*

$$\lim_{|x|\to 0^+} \frac{F(x)}{[\rho_*(x)]^{\frac{1}{p_+}-1}} = 0. \tag{19}$$

Proof. Let $\vec{p} \in (0,1]^n$, $r \in (\max\{p_+, 1\}, \infty]$, s be as in (6) and $f \in H_A^{\vec{p}}(\mathbb{R}^n)$. Then, by Lemma 3.6, we find that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

and

$$\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}. \tag{20}$$

Furthermore, from the proof of Theorem 3.1, it follows that the function

$$F(x) = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(x), \quad \forall x \in \mathbb{R}^n, \tag{21}$$

is continuous and satisfies that $\widehat{f} = F$ in $\mathcal{S}'(\mathbb{R}^n)$.

Thus, to show Theorem 4.1, we only need to prove (19) holds true for the function F as in (21). To do this, observe that, for any $(\vec{p}, r, s)$-atom a supported in $x_0 + B_{k_0}$ with some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$, when $\rho_*(x) \le b^{-k_0}$, (11) holds true. This, together with the fact that

$$1 - \frac{1}{p_+} + (s+1)\frac{\ln \lambda_-}{\ln b} > 0,$$

implies that

$$\lim_{|x|\to 0^+} \frac{|\widehat{a}(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}} = 0. \tag{22}$$

For any $x \in \mathbb{R}^n$, we get the following inequality by (21):

$$\frac{|F(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}} \le \sum_{i\in\mathbb{N}} |\lambda_i| \frac{|\widehat{a_i}(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}}. \tag{23}$$

Moreover, by (7) and the fact $\sum_{i\in\mathbb{N}}|\lambda_i|<\infty$, we know that the dominated convergence theorem can be applied to the right side of (23). Combining this and (22), we deduce that

$$\lim_{|x|\to 0^+}\frac{F(x)}{[\rho_*(x)]^{\frac{1}{p_+}-1}}=0,$$

which completes the proof of Theorem 4.1. □

Remark 4.2. (i) Similarly to Remark 3.7(i), if $\vec{p}=(p,\dots,p)\in(0,1]^n$, then the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ in Theorem 4.1 coincides with the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [1]. In this case, Theorem 4.1 is just [2, Corollary 6].

(ii) By Theorem 4.1 and Lemma 3.4, we have

$$\lim_{|x|\to 0^+}\frac{F(x)}{|x|^{\frac{\ln b}{\ln\lambda_+}(\frac{1}{p_+}-1)}}=0. \tag{24}$$

Observe that, when $\vec{p}=(p,\dots,p)\in(0,1]^n$ and $A=d\,\mathrm{I}_{n\times n}$ for some $d\in\mathbb{R}$ with $|d|\in(1,\infty)$, here and thereafter, $\mathrm{I}_{n\times n}$ denotes the *unit matrix* of order n, the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ comes back to the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman and Stein [12]. In this case, $\frac{\ln b}{\ln\lambda_+}=n$ and $p_+=p$, and hence (24) is just the well-known result on $H^p(\mathbb{R}^n)$ (see [23, p. 128]).

As another application of Theorem 3.1, we extend the Hardy–Littlewood inequality to the setting of anisotropic mixed norm Hardy spaces in the following theorem.

Theorem 4.3. *Let $\vec{p}\in(0,1]^n$. Then, for any $f\in H_A^{\vec{p}}(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that $\widehat{f}=F$ in $\mathcal{S}'(\mathbb{R}^n)$ and*

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}|F(x)|^{p_+}\min\left\{[\rho_*(x)]^{p_+-\frac{p_+}{p_-}-1},\,[\rho_*(x)]^{p_+-2}\right\}dx\right)^{\frac{1}{p_+}}\\ &\quad\le C\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)},\end{aligned}\tag{25}$$

where C is a positive constant depending only on A and $\vec{p}$.

Proof. Let $\vec{p}\in(0,1]^n$ and $f\in H_A^{\vec{p}}(\mathbb{R}^n)$. Then, by Lemma 3.6, we find that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of $(\vec{p},2,s)$-atoms $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}}\subset\mathfrak{B}$ such that

$$f=\sum_{i\in\mathbb{N}}\lambda_i a_i\quad\text{in}\quad\mathcal{S}'(\mathbb{R}^n),$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}\lesssim\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)}<\infty. \tag{26}$$

To prove Theorem 4.3, it suffices to show that (25) holds true for the function F as in (21). For this purpose, by the fact that $\underline{p} \le p_+ \le 1$, the inverse Minkovski inequality and (26), we have

$$\begin{aligned}
\left(\sum_{i\in\mathbb{N}} |\lambda_i|^{p_+}\right)^{1/p_+} &= \left(\sum_{i\in\mathbb{N}} \left\| \frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{p_+}\right)^{1/p_+} \\
&= \left(\sum_{i\in\mathbb{N}} \left\| \frac{|\lambda_i|^{p_+} \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}^{p_+}} \right\|_{L^{\vec{p}/p_+}(\mathbb{R}^n)}\right)^{1/p_+} \\
&\le \left\| \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{p_+} \right\|_{L^{\vec{p}/p_+}(\mathbb{R}^n)}^{1/p_+} \\
&= \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{p_+} \right\}^{1/p_+} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\
&\le \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\
&\lesssim \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)}. \qquad (27)
\end{aligned}$$

On another hand, from (21), the fact that $p_+ \in (0,1]$, (12) and the Fatou lemma, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |F(x)|^{p_+} \min\left\{ [\rho_*(x)]^{p_+ - \frac{p_+}{p_-} - 1},\ [\rho_*(x)]^{p_+ - 2} \right\} dx \\
\le\ & \sum_{i\in\mathbb{N}} |\lambda_i|^{p_+} \int_{\mathbb{R}^n} \left[|\widehat{a_i}(x)| \min\left\{ [\rho_*(x)]^{1 - \frac{1}{p_-} - \frac{1}{p_+}},\ [\rho_*(x)]^{1 - \frac{2}{p_+}} \right\} \right]^{p_+} dx. \qquad (28)
\end{aligned}$$

Next, we devote to proving the following uniform estimate for all $(\vec{p}, 2, s)$-atoms, namely,

$$\left(\int_{\mathbb{R}^n} \left[|\widehat{a}(x)| \min\left\{ [\rho_*(x)]^{1 - \frac{1}{p_-} - \frac{1}{p_+}},\ [\rho_*(x)]^{1 - \frac{2}{p_+}} \right\} \right]^{p_+} dx \right)^{1/p_+} \le M, \qquad (29)$$

where M is a positive constant independent of a. Assume that (29) holds true for the moment. Combining this, (27) and (28), we conclude that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |F(x)|^{p_+} \min\left\{ [\rho_*(x)]^{p_+ - \frac{p_+}{p_-} - 1},\ [\rho_*(x)]^{p_+ - 2} \right\} dx \right)^{1/p_+} \\
\le\ & M \left(\sum_{i\in\mathbb{N}} |\lambda_i|^{p_+} \right)^{1/p_+} \lesssim \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)}.
\end{aligned}$$

This is the desired conclusion (25).

Thus, the rest of the whole proof is to show the assertion (29). Indeed, for any $(\vec{p},2,s)$-atom a supported in a dilated ball $x_0+B_{i_0}$ with some $x_0\in\mathbb{R}^n$ and $i_0\in\mathbb{Z}$, it is easy to see that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},\,[\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\
&\lesssim\left(\int_{(A^*)^{-i_0+1}B_0^*}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},\,[\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\
&\quad+\left(\int_{((A^*)^{-i_0+1}B_0^*)^{\complement}}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},\,[\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\
&=:\mathrm{I}_1+\mathrm{I}_2,
\end{aligned}$$

where B_0^* is the unit dilated ball with respect to A^*.

Let θ be a fixed positive constant such that

$$1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}-\theta\geq 1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}-\theta>0.$$

Then, to deal with I_1, by (11), we know that

$$\begin{aligned}
\mathrm{I}_1&\lesssim b^{i_0[1+(s+1)\frac{\ln\lambda_-}{\ln b}]}\max\left\{b^{-\frac{i_0}{p_-}},\,b^{-\frac{i_0}{p_+}}\right\}\left(\int_{(A^*)^{-i_0+1}B_0^*}\right.\\
&\qquad\left.\left[\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}},\,[\rho_*(x)]^{1-\frac{2}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\
&\lesssim b^{i_0[1+(s+1)\frac{\ln\lambda_-}{\ln b}]}\max\left\{b^{-\frac{i_0}{p_-}},\,b^{-\frac{i_0}{p_+}}\right\}\\
&\qquad\times\min\left\{b^{-i_0[1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}-\theta]},\,b^{-i_0[1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}-\theta]}\right\}\\
&\qquad\times\left(\int_{(A^*)^{-i_0+1}B_0^*}[\rho_*(x)]^{\theta p_+-1}\,dx\right)^{1/p_+}\\
&\sim b^{i_0\theta}\left[\sum_{k\in\mathbb{Z}\setminus\mathbb{N}}b^{-i_0+k}(b-1)b^{(-i_0+k)(\theta p_+-1)}\right]^{1/p_+}\sim\left(\frac{b-1}{1-b^{-\theta p_+}}\right)^{\frac{1}{p_+}}.
\end{aligned}$$

As for the estimate of I_2, by the Hölder inequality, the Plancherel theorem, the fact that $0<p_-\leq p_+\leq 1$ and the size condition of a, we obtain

$$\mathrm{I}_2\lesssim\left\{\int_{((A^*)^{-i_0+1}B_0^*)^{\complement}}|\widehat{a}(x)|^2\,dx\right\}^{\frac{1}{2}}\left\{\int_{((A^*)^{-i_0+1}B_0^*)^{\complement}}\right.$$

$$
\begin{aligned}
&\left[\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},\, [\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{\frac{2p_+}{2-p_+}}\, dx\Bigg\}^{\frac{2-p_+}{2p_+}}\\
&\lesssim \|a\|_{L^2(\mathbb{R}^n)}\left\{\sum_{k\in\mathbb{N}} b^{-i_0+k}(b-1)\right.\\
&\qquad\left.\times\left[\min\left\{b^{(-i_0+k)(1-\frac{1}{p_-}-\frac{1}{p_+})},\, b^{(-i_0+k)(1-\frac{2}{p_+})}\right\}\right]^{\frac{2p_+}{2-p_+}}\right\}^{\frac{2-p_+}{2p_+}}\\
&\lesssim \|a\|_{L^2(\mathbb{R}^n)}\left\{b^{-i_0}\left[\min\left\{b^{-i_0(1-\frac{1}{p_-}-\frac{1}{p_+})},\, b^{-i_0(1-\frac{2}{p_+})}\right\}\right]^{\frac{2p_+}{2-p_+}}\right\}^{\frac{2-p_+}{2p_+}}\\
&\lesssim \max\left\{b^{i_0(\frac{1}{2}-\frac{1}{p_-})},\, b^{i_0(\frac{1}{2}-\frac{1}{p_+})}\right\}\min\left\{b^{-i_0(\frac{1}{2}-\frac{1}{p_-})},\, b^{-i_0(\frac{1}{2}-\frac{1}{p_+})}\right\}\\
&\sim 1.
\end{aligned}
$$

This finishes the proof of (29) and hence of Theorem 4.3. □

Remark 4.4. Actually, when $\vec{p}=(p,\dots,p)\in(0,1]^n$, the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ in Theorem 4.3 is just the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [1] in the sense of equivalent quasi-norms. Thus, we point out that Theorem 4.3 covers [2, Corollary 8]. Moreover, if $A=d\,\mathrm{I}_{n\times n}$ for some $d\in\mathbb{R}$ with $|d|\in(1,\infty)$, then the anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$, with $\vec{p}=(p,\dots,p)\in(0,1]^n$, coincides with the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman and Stein [12]. In this case, $\rho_*(x)\sim|x|^n$ for any $x\in\mathbb{R}^n$, and hence (23) is just the classic Hardy–Littlewood inequality as in (2).

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