J. Korean Math. Soc. **59** (2022), No. 5, pp. 869–890
https://doi.org/10.4134/JKMS.j210650
pISSN: 0304-9914 / eISSN: 2234-3008

DYNAMIC BEHAVIOR OF CRACKED BEAMS AND SHALLOW ARCHES

Semion Gutman, Junhong Ha, and Sudeok Shon

Abstract. We develop a rigorous mathematical framework for studying dynamic behavior of cracked beams and shallow arches. The governing equations are derived from the first principles, and stated in terms of the subdifferentials of the bending and the axial potential energies. The existence and the uniqueness of the solutions is established under various conditions. The corresponding mathematical tools dealing with vector-valued functions are comprehensively developed. The motion of beams and arches is studied under the assumptions of the weak and strong damping. The presence of cracks forces weaker regularity results for the arch motion, as compared to the beam case.

1. Introduction

The main goal of this paper is to investigate the dynamic behavior of cracked beams and arches based on a rigorous mathematical framework. In [8] we developed a variational formulation for such cracked structural elements. Using this approach, the equations of motions were derived in [7]. In this paper we continue the study by establishing the existence and the uniqueness results for such equations. See [7, 8] for a literature review.

A brief review of the main concepts is presented in Section 2, where we describe special Hilbert spaces V, H_0^1, H. These spaces are broad enough to contain continuous functions with discontinuous derivatives at the joint points.

Then we introduce the operator $\mathcal{A} : V \to V'$. The main result in [8] is that the solution u of the equation $\mathcal{A}u = f$ in H satisfies the joint conditions at the crack points, including the slope discontinuities.

Next, in Section 3, we review the equations of motion for cracked beams and arches in a non-dimensional form following [7]. These equations are expressed in terms of the subdifferentials of the potential energies U_b, and U_a due to the

Received October 27, 2021; Revised June 21, 2022; Accepted June 28, 2022.
2010 *Mathematics Subject Classification.* 47J35, 35Q74, 35D30, 70G75.
Key words and phrases. Shallow arch, beam, subdifferential, cracks, equation of motion.
Sudeok Shon is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020 R1I1A1A01065032).

bending and axial force, correspondingly. The Appendix (Section 9) presents a brief review of these concepts, and relevant examples.

In this framework the abstract equation of motion for cracked beams and arches is

$$\ddot{y} + \partial U_b(y) + \partial U_a(y) + c_d \dot{y} = p, \tag{1.1}$$

where $\dot{y}, \ddot{y}$ denote the time derivatives.

The main result in Section 3 is that the "classical" equation for a cracked shallow arch is

$$\ddot{y} + y'''' - \frac{1}{\pi}\left(\beta + \frac{1}{2}\int_0^L |y'(x,t)|^2\,dx\right)\left(y'' + \sum_{i=1}^m \theta_i y''(x_i,t)\delta(x - x_i)\right) + c_d\dot{y} = p,$$

where $\delta = \delta(x)$ is the delta function.

Motion in viscous media results in the additional term $\mu\mathcal{A}\dot{y}$, $\mu > 0$ in the governing equations. Such a case is referred to as the strong damping motion. If the viscous effects are neglected ($\mu = 0$), we have the weak damping case.

In Section 4, we study some properties of functions with values in Hilbert spaces, including the generalization of the Chain Rule for the subdifferentials, used to derive critical a priori estimates.

The motion of a cracked beam in the cases of the weak ($\mu = 0$) and strong ($\mu > 0$) damping is investigated in Section 5. We also consider the case of the vanishing damping, and show that $y^{(\mu)} \to y$ as $\mu \to 0$. This approach can be considered as a "parabolic regularization" method.

While studying the motion of cracked arches, we encounter the following obstacle. The solution y of the problem is sought to be the weak limit of the approximate solutions, but the subdifferential ∂U_a is a *non-linear* operator. Generally speaking, non-linear operators are not weakly continuous. In Section 6 we undertake an additional examination of the weak convergence, and use the special structure of the operator ∂U_a to justify the passage to the limit under these circumstances.

The strong arch damping problem is studied in Section 7. In this case the results for the arches are similar to the ones for the beams. The weak arch damping problem is considered in Section 8. We show that y can also be obtained as a weak limit of the corresponding strong damping solutions $y^{(\mu)}$.

The presence of the nonlinear subdifferential of the axial potential energy ∂U_a results in weaker results for the arches. In particular, in the weak arch damping case the solutions are less regular, than for the beams. Also, their uniqueness could not be obtained.

2. Variational setting for cracked beams and arches

This section contains a brief review of our results from [8], to which we refer for further details.

The transverse motion of a beam or an arch is described by the function $y(x,t)$, $x \in [0,\pi]$, $t \geq 0$, which represents the deformation of the beam/arch measured from the x-axis. For definiteness, the boundary conditions are of the hinged type

$$y(0,t) = y''(0,t) = 0, \quad y(\pi,t) = y''(\pi,t) = 0, \quad t \in (0,T). \tag{2.1}$$

Other types of boundary conditions can be treated similarly.

According to the common practice in the field, see [4], a crack is modeled by a massless rotational spring with the spring flexibility θ. The flexibility θ is equal to 0 if there is no crack, and it increases with the crack depth.

Suppose that there are m cracks along the length of the arch (or a beam), located at $0 < x_1 < \cdots < x_m < \pi$. For convenience, we denote $x_0 = 0$, and $x_{m+1} = \pi$. Consequently, the cracked arch is modeled as a collection of $m+1$ uniform arches over the intervals $l_i = (x_{i-1}, x_i)$, $i = 1, \ldots, m+1$.

Let H be the Hilbert space

$$H = \bigoplus_{i=1}^{m+1} L^2(l_i). \tag{2.2}$$

Let the inner product and the norm in $L^2(l_i)$ be denoted by $(\cdot,\cdot)_i$ and $|\cdot|_i$ correspondingly. The inner product and the norm in H are defined by

$$(u,v)_H = \sum_{i=1}^{m+1} (u,v)_i, \quad |u|_H^2 = \sum_{i=1}^{m+1} |u|_i^2. \tag{2.3}$$

Define the linear space

$$V = \left\{ u \in \bigoplus_{i=1}^{m+1} H^2(l_i) \, : \, u(0) = u(\pi) = 0, \; J[u](x_i) = 0, \; i = 1, \ldots, m \right\}. \tag{2.4}$$

Let the inner product in V be

$$((u,v))_V = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} J[u'](x_i) J[v'](x_i) \quad \text{for any } u, v \in V, \tag{2.5}$$

where $(u'', v'')_i = \int_{l_i} u''(x) v''(x)\, dx$.

The corresponding norm in V is

$$\|u\|_V^2 = \sum_{i=1}^{m+1} |u''|_i^2 + \sum_{i=1}^{m} |J[u'](x_i)|^2 \quad \text{for any } u \in V, \tag{2.6}$$

where $|\cdot|_i$ is the norm in $L^2(l_i)$. It can be shown that V is a Hilbert space.

The Hilbert space $H_0^1 = H_0^1(0,\pi)$ is equipped with the inner product and the norm given by

$$(u,v)_1 = (u', v')_H, \quad \|u\|_1^2 = |u'|_H^2, \quad u, v \in H_0^1. \tag{2.7}$$

The norm in $(H_0^1)'$ will be denoted by $\|\cdot\|_{-1}$. It can be shown that the identity embedding $i : V \to H_0^1$ is linear, continuous, with a dense range in H_0^1. Furthermore, it is compact.

We have

$$V \subset H_0^1 \subset H \subset \left(H_0^1\right)' \subset V', \tag{2.8}$$

with dense embeddings. Furthermore, the embeddings $V \subset H_0^1 \subset H$ are compact.

Now we can introduce the operator $\mathcal{A} : V \to V'$ that "absorbs" the junction boundary conditions. This operator is central to the variational setting of problems for cracked beams and arches. Here $J[u](x) = u(x^+) - u(x^-)$.

Definition 1. Define the operator $\mathcal{A}$ on V by

$$\langle \mathcal{A}u, v\rangle_V = \sum_{i=1}^{m+1} (u'', v'')_i + \sum_{i=1}^{m} \frac{1}{\theta_i} J[u'](x_i) J[v'](x_i) \tag{2.9}$$

for any $u, v \in V$. We will also write $\langle \mathcal{A}u, v\rangle$ for $\langle \mathcal{A}u, v\rangle_V$ if it does not cause a confusion.

Recall that a linear operator $A : V \to V'$ is called coercive if there exists $c > 0$ such that $\langle Au, u\rangle \geq c\|u\|_V^2$ for any $u \in V$. We have:

Lemma 2.1. *Let $\mathcal{A}$ be defined by* (2.9)*. Then $\mathcal{A}$ is a symmetric, continuous, linear, and coercive operator from V onto V'.*

Functions $u = u(x)$ modeling an arch with cracks satisfy the hinged boundary conditions

$$u(0) = u(\pi) = 0, \quad u''(0) = u''(\pi) = 0, \tag{2.10}$$

and

$$J[u](x_i) = 0, \quad J[u''](x_i) = 0, \quad J[u'''](x_i) = 0, \quad J[u'](x_i) = \theta_i u''(x_i^+), \tag{2.11}$$

for $i = 1, \ldots, m$, at the crack (or joint) points.

The next theorem is the main result of this section.

Theorem 2.2. *Let the domain of $\mathcal{A}$ be $D(\mathcal{A}) = \{v \in V : \mathcal{A}v \in H\}$.*

(i) *If $u \in D(\mathcal{A})$, then $u|_{l_i} \in H^4(l_i)$, $\mathcal{A}u = u''''$ a.e. on l_i, $i = 1, \ldots, m+1$, and u satisfies conditions* (2.10)–(2.11)*.*
(ii) *If $f \in H$, then equation $\mathcal{A}u = f$ in V' has a unique solution $u \in D(\mathcal{A})$.*

Furthermore, we have:

Lemma 2.3. *Let $\mathcal{A}$ be the operator defined in* (2.9)*. Then*

(i) *There exists an increasing sequence of its real positive eigenvalues $\lambda_1^4, \lambda_2^4, \ldots$, with $\lim_{k\to\infty} \lambda_k^4 = \infty$.*
(ii) *The corresponding eigenfunctions $\varphi_k \in D(\mathcal{A}) \subset V$, $k \geq 1$, and they satisfy the junction conditions* (2.10)–(2.11)*.*
(iii) *The eigenfunctions φ_k satisfy $\mathcal{A}\varphi_k = \lambda_k^4 \varphi_k$ in H, $k \geq 1$. That is, $\varphi_k''''(x) = \lambda_k^4 \varphi_k(x)$ a.e. on every interval l_i, $i = 1, \ldots, m+1$.*
(iv) *The set $\{\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal basis in H.*

An efficient method for a computational determination of the eigenvalues and the eigenfunctions of $\mathcal{A}$ (Modified Shifrin's method) is discussed in [8].

3. Beam and arch equations of motion

In this section we review our results from [7]. A brief review of convex functions ϕ, and their subdifferentials $\partial\phi$ is presented in the Appendix. This review also treats the potential energy $U_b(u)$ due to bending, the potential energy $U_a(u)$ due to the axial force, and their subdifferentials. It is shown that both functions are convex, lower-semicontinuous function on V and H_0^1 correspondingly.

The variational formulation of the previous section allows us to apply the Extended Hamilton's Principle to derive the following abstract equation of motion for beams and aches

$$\ddot{y} + \partial U_b(y) + \partial U_a(y) + c_d\dot{y} = p. \tag{3.1}$$

The equation is satisfied in V', a.e. for $t \in [0, T]$. Note: $\partial U_b : V \to V'$, and $\partial U_a : H_0^1 \to (H_0^1)'$.

Beam equations. In the classical Euler-Bernoulli beam theory the influence of the axial force is disregarded, so we let $U_a = 0$. This results in the following abstract equation for the beam with cracks

$$\ddot{y} + \mathcal{A}y + c_d\dot{y} = p, \tag{3.2}$$

which is satisfied in V', a.e. for $t \in [0, T]$.

If $u \in D(\mathcal{A})$, then u satisfies the boundary conditions of the problem, i.e., (2.10) and (2.11), as well as $\mathcal{A}u = u''''$ a.e. on every interval l_i, $i = 1, \ldots, m+1$.

Then equation (3.2) can be written as

$$\ddot{y} + y'''' + c_d\dot{y} = p \tag{3.3}$$

on every subinterval l_i, $i = 1, \ldots, m + 1$. We can call it the classical Beam equation for cracked beams.

Strong damping. Viscous effects on the beam and arch motion are discussed in [1, 5]. Considerations based on the Voigt model for viscoelasticity result in the additional term $\mu\mathcal{A}\dot{y}$ in the governing equations. Here $\mu > 0$ is a non-dimensional normalized dynamic viscosity coefficient.

If such a term is present, we refer to the model as having the *strong damping*. Otherwise, if $\mu = 0$, the model is for the *weak damping*. In particular, equations (3.2) and (3.3) describe the weak beam damping motion case. The corresponding non-dimensional abstract and classical equations in the presence of the strong damping $\mu > 0$ are

$$\ddot{y} + \mathcal{A}y + \mu\mathcal{A}\dot{y} + c_d\dot{y} = p, \tag{3.4}$$

and

$$\ddot{y} + y'''' + \mu\dot{y}'''' + c_d\dot{y} = p. \tag{3.5}$$

Arch with cracks. The axial potential energy $U_a(y)$ of the arch has the expression

$$U_a(y) = \frac{1}{2\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)^2, \tag{3.6}$$

and its subdifferential $\partial U_a(u)$ is computed in Example 9.3 as

$$\partial\psi(u) = \mathcal{B}u = -\sum_{i=1}^{m} J[u'](x_i)\delta(x-x_i) - u'', \quad u \in V. \tag{3.7}$$

Then equation (3.1) becomes

$$\ddot{y} + \mathcal{A}y - \frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\left(\sum_{i=1}^{m} J[y'](x_i)\delta(x-x_i) - y''\right) + c_d\dot{y} = p, \tag{3.8}$$

which is the abstract equation for a shallow arch with cracks. It is satisfied in V', a.e. $t \in [0,T]$.

Then, assuming that the function y is smooth, as discussed in Example 9.3, we can use (9.9) for the subdifferential $\partial U_a(u)$, and $\partial U_b(u) = \mathcal{A}u = u''''$. This results in

$$\ddot{y} + y'''' - \frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\left(\sum_{i=1}^{m} \theta_i y''(x_i,t)\delta(x-x_i) - y''\right) + c_d\dot{y} = p, \tag{3.9}$$

which can be called the "classical" form of the shallow arch equation with cracks. These equations are also referred to as describing the weak arch damping motion. The strong damping equations ($\mu > 0$) are obtained from (3.8) and (3.9) by adding to them the terms $\mu\mathcal{A}\dot{y}$ and $\mu\dot{y}''''$ correspondingly.

4. Vector functions I

In this section we develop some mathematical tools needed to study the beam motion. First, introduce the Hilbert space of vector-valued functions

$$W[0,T] = \{y \,:\, y \in L^2(0,T;V),\ \dot{y} \in L^2(0,T;H),\ \ddot{y} \in L^2(0,T;V')\}, \tag{4.1}$$

where the functions y, and their time derivatives $\dot{y}$, $\ddot{y}$ are understood in the sense of distributions with the values in V, H and V', correspondingly, see [9]. The inner product in $W[0,T]$ is set to be the sum of the inner products in the constituent spaces.

Similarly to the definition of $W[0,T]$, we let

$$W_r[0,T] = \{y \,:\, y \in L^2(0,T;V),\ \dot{y} \in L^2(0,T;V),\ \ddot{y} \in L^2(0,T;V')\}. \tag{4.2}$$

For functions y in $W_r[0,T]$, we have $\dot{y} \in L^2(0,T;V)$, resulting in more regular functions than the ones in $W[0,T]$.

Our main tool is the following crucial result established in [10, Lemma 2.4.1]:

Lemma 4.1. *Let $A : V \to V'$ be a linear, continuous and symmetric operator, coercive on V. Suppose that $y \in L^2(0,T;V)$, $\dot{y} \in L^2(0,T;H)$, and $\ddot{y} + Ay \in L^2(0,T;H)$. Then, after a modification on a set of measure zero, $y \in C([0,T];V)$, $\dot{y} \in C([0,T];H)$ and, in the sense of distributions on $(0,T)$, one has*

$$(\ddot{y} + Ay, \dot{y}) = \frac{1}{2}\frac{d}{dt}\left(|\dot{y}|^2 + \langle Ay, y\rangle\right). \tag{4.3}$$

Sometimes, instead of Lemma 4.1, we can use:

Lemma 4.2. (i) *Let $y \in L^2(0,T;V)$, $\dot{y} \in L^2(0,T;V')$. Then $y \in C([0,T];H)$, and*

$$\frac{d}{dt}|y|_H^2 = 2\langle \dot{y}, y\rangle. \tag{4.4}$$

(ii) *Let $y \in W_r[0,T]$. Then $y \in C([0,T];V)$, $\dot{y} \in C([0,T];H)$, and*

$$\frac{d}{dt}|\dot{y}|_H^2 = 2\langle \ddot{y}, \dot{y}\rangle. \tag{4.5}$$

(iii) *Let $y, \dot{y} \in L^2(0,T;V)$. Let an operator A satisfy the conditions of Theorem 9.1, and let $\phi(u) = \frac{1}{2}\langle Au, u\rangle$, $u \in V$. Then*

$$\frac{d}{dt}\phi(y) = \langle Ay, \dot{y}\rangle = \langle \partial\phi(y), \dot{y}\rangle. \tag{4.6}$$

Proof. (i) This is [10, Lemma 2.3.2].

(ii) Replace y with $\dot{y}$ in (i). The continuity $y \in C([0,T];V)$ follows from $y(t) - y(s) = \int_s^t \dot{y}(\tau)\, d\tau$ in V.

(iii) We have

$$\lim_{s\to t} \frac{y(s) - y(t)}{s - t} = \dot{y}(t),$$

a.e. for $t \in (0,T)$. By Theorem 9.1, function ϕ is Fréchet differentiable on V (and $\partial\phi = A$). Furthermore,

$$|\phi(y(s)) - \phi(y(t)) - \langle Ay(t), y(s) - y(t)\rangle| \le \frac{C}{2}\|y(s) - y(t)\|_V^2 \tag{4.7}$$

for any $s, t \in [0,T]$. Then, since $y \in C([0,T];V)$,

$$\begin{aligned} &\lim_{s\to t}\left|\frac{\phi(y(s)) - \phi(y(t))}{s - t} - \left\langle Ay(t), \frac{y(s) - y(t)}{s - t}\right\rangle\right| \\ &\le \frac{C}{2}\lim_{s\to t}\left\|\frac{y(s) - y(t)}{s - t}\right\|_V \|y(s) - y(t)\|_V = 0, \end{aligned} \tag{4.8}$$

a.e. $t \in (0,T)$, and (4.6) follows. □

For completeness, we mention the following related result established in [2, Lemma 4.4], [3, Lemma 3.3]. It shows that the subdifferential is the proper concept to generalize the Chain Rule for the differentiation of vector-valued functions.

Lemma 4.3. *Let X be a Hilbert space, ϕ be a proper, convex, lower semicontinuous function on X, and $\partial\phi \subset X \times X$ be its subdifferential. Suppose that $y, \dot{y} \in L^2(0,T;X)$, and $g \in L^2(0,T;X)$ is such that $g(t) \in \partial\phi(y(t))$ a.e. for $t \in (0,T)$.*

Then the function $t \to \phi(y(t))$ is absolutely continuous on $[0,T]$, and

$$\frac{d}{dt}\phi(y(t)) = (g(t), \dot{y}(t))_X\,, \tag{4.9}$$

a.e. for $t \in (0,T)$.

We conclude this section with some eigenfunction expansion results for the operator $\mathcal{A}$, introduced in Section 2. The existence of its eigenvalues λ_k^4, and the eigenfunctions φ_k, $k \geq 1$ was shown in Lemma 2.3.

By Lemma 2.1, the operator $\mathcal{A}$ is linear, bounded, symmetric, and coercive on V. Therefore it defines an equivalent inner product and the norm on V by

$$((u,v))_{\mathcal{A}} = \langle \mathcal{A}u, v\rangle_V\,, \quad \|u\|^2_{\mathcal{A}} = \langle \mathcal{A}u, u\rangle_V\,, \quad u, v \in V. \tag{4.10}$$

This space is denoted by $V_{\mathcal{A}}$.

Lemma 4.4. (i) *System $\{\varphi_k\}_{k=1}^\infty$ is an orthonormal basis in H.*
(ii) *System $\{\frac{1}{\lambda_k^2}\varphi_k\}_{k=1}^\infty$ is an orthonormal basis in $V_{\mathcal{A}}$.*

Proof. Part (i) was stated in Lemma 2.3. For (ii) we have

$$\left(\left(\frac{1}{\lambda_j^2}\varphi_j, \frac{1}{\lambda_k^2}\varphi_k\right)\right)_{\mathcal{A}} = \left\langle \frac{1}{\lambda_j^2}\mathcal{A}\varphi_j, \frac{1}{\lambda_k^2}\varphi_k\right\rangle_V = \left(\lambda_j^2\varphi_j, \frac{1}{\lambda_k^2}\varphi_k\right)_H = 0 \tag{4.11}$$

for any $j \neq k \geq 1$. If $j = k$, then (4.11) shows that $\|\varphi_k\|^2_{\mathcal{A}} = \langle \mathcal{A}\varphi_k, \varphi_k\rangle_V = \lambda_k^4|\varphi_k|^2_H = \lambda_k^4$, $k \geq 1$.

To see that this system is a basis in $V_{\mathcal{A}}$, suppose that there exists $w \in V_{\mathcal{A}}$, such that $((\varphi_k, w))_{\mathcal{A}} = 0$ for any $k \geq 1$. Then $((\varphi_k, w))_{\mathcal{A}} = \lambda_k^4(\varphi_k, w)_H = 0$. But $\{\varphi_k\}_{k=1}^\infty$ is a basis in H, thus $w = 0$. $\square$

Lemma 4.5. *Let $m \geq 1$, and $V_m = span\{\varphi_1, \ldots, \varphi_m\} \subset V$. Define operators $P_m : H \to V_m$, and $P_m^* : V' \to V'$ by*

$$P_m h = \sum_{k=1}^m (h, \varphi_k)_H \varphi_k, \quad h \in H, \tag{4.12}$$

and

$$(P_m^* g, v) = \langle g, P_m v\rangle_{\mathcal{A}}, \quad g \in V',\ v \in V. \tag{4.13}$$

Then

(i) *$P_m : H \to V_m$ is an orthogonal projection in H, and $|P_m h| \leq |h|$ for any $h \in H$. Also $|h - P_m h| \to 0$ as $m \to \infty$.*
(ii) *$P_m : V_{\mathcal{A}} \to V_m$ is an orthogonal projection in $V_{\mathcal{A}}$, and $\|P_m v\|_{\mathcal{A}} \leq \|v\|_A$ for any $v \in V$. Also $\|v - P_m v\|_{\mathcal{A}} \to 0$ as $m \to \infty$.*

(iii) *Operator* $P_m^*: V' \to V'$ *satisfies* $\|P_m^* g\|_{V'} \le \|g\|_{V'}$, *and* $P_m g \rightharpoonup g$ *weakly in* V' *as* $m \to \infty$ *for any* $g \in V'$.

Proof. Part (i) follows from Lemma 4.4(i). For (ii), let $v \in V_{\mathcal{A}}$, and $\hat{P}_m v = \sum_{k=1}^m \left(\left(v, \frac{1}{\lambda_k^2}\varphi_k\right)\right)_{\mathcal{A}} \frac{1}{\lambda_k^2}\varphi_k$. Then we can verify directly that $\hat{P}_m = P_m$ on V. Lemma 4.4(ii) implies the other assertions in (ii). For part (iii), we have $|(P_m^* g, v)| = |\langle g, P_m v\rangle_{\mathcal{A}}| \le \|g\|_{V'}\,\|v\|_{\mathcal{A}}$, and $(P_m^* g, v) = \langle g, P_m v\rangle_{\mathcal{A}} \to \langle g, v\rangle_{\mathcal{A}}$ as $m \to \infty$ for any $v \in V$. $\square$

5. Beam motion

In this section we establish the uniqueness and the existence of motion for a cracked beam under the assumption of the weak and strong damping $\mu > 0$. Since the proofs are standard and straightforward, they will be omitted. Our approach closely follows the one in [10, Section 2.4.1]. Extended proofs are provided for the arch motion in Sections 7 and 8.

We also show that $y^{(\mu)} \to y$ as $\mu \to 0$, where $y^{(\mu)}$ and y are the solutions of the corresponding strong and weak damping problems. This can be considered to be a regularization method having some similarities with the "parabolic regularization" in [9, Section 3.8.5].

Definition 2. Let $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$. A function $y \in W[0,T]$ is called a solution of the weak beam damping problem if it satisfies $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$, $y(0) = u_0$, $\dot{y}(0) = v_0$, and

$$\ddot{y} + \mathcal{A}y + c_d\dot{y} = f, \tag{5.1}$$

in V', a.e. for $t \in [0,T]$. Here the operator $\mathcal{A}: V \to V'$ is defined by (2.9). We will write $y = y(u_0, v_0, f)$ to emphasize the dependence of the solution on the data.

Lemma 5.1. *Let* y *be a solution of the weak beam damping problem* (5.1). *Then* $y \in W[0,T] \cap C([0,T];V)$, $\dot{y} \in C([0,T];H)$, *and*

$$|\dot{y}(t)|^2 + \|y(t)\|_{\mathcal{A}}^2 \le c\left(|v_0|^2 + \|u_0\|_{\mathcal{A}}^2 + \|f\|_{L^2(0,T;H)}^2\right) \tag{5.2}$$

for any $t \in [0,T]$. *Furthermore, the solution is unique.*

Now we establish the existence of the solution y by taking the limit of approximate solutions.

Given $m \ge 1$, a function y_m is called an approximate solution of the weak beam damping problem if it satisfies the same conditions as in Definition 2, except that $y_m(0) = P_m u_0$, $\dot{y}_m(0) = P_m v_0$, and

$$\ddot{y}_m + \mathcal{A}y_m + c_d\dot{y}_m = P_m f, \tag{5.3}$$

in V', a.e. for $t \in [0,T]$.

Theorem 5.2. *Let $u_0 \in V$, $v_0 \in H$, $T > 0$, and $f \in L^2(0,T;H)$. Then there exists a unique solution y of the weak beam damping problem* (5.1)*, and it satisfies $y \in W[0,T] \cap C([0,T];V)$, $\dot{y} \in C([0,T];H)$. Furthermore,*

$$\begin{aligned} (5.4) \qquad & |\dot{y}(t) - \dot{y}_m(t)|^2 + \|y(t) - y_m(t)\|_{\mathcal{A}}^2 \\ & \leq c\left(|v_0 - P_m v_0|^2 + \|u_0 - P_m u_0\|_{\mathcal{A}}^2 + \|f - P_m f\|_{L^2(0,T;H)}^2\right) \end{aligned}$$

for any approximate solution y_m, $m \geq 1$, $t \in [0,T]$.

Note that inequality (5.4) implies that $y_m \to y$ strongly in $C([0,T];V)$, and $\dot{y}_m \to \dot{y}$ strongly in $C([0,T];H)$ as $m \to \infty$.

Definition 3. Let $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$. A function $y \in W_r[0,T]$ is called a solution of the strong beam damping problem with $\mu > 0$ if it satisfies $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$, $y(0) = u_0$, $\dot{y}(0) = v_0$, and

$$(5.5) \qquad \ddot{y} + \mathcal{A}y + \mu\mathcal{A}\dot{y} + c_d\dot{y} = f,$$

in V', a.e. for $t \in [0,T]$. The operator $\mathcal{A} : V \to V'$ is defined by (2.9). We will write $y = y(u_0, v_0, f) = y^{(\mu)}(u_0, v_0, f)$ to emphasize the dependence of the solution on the data.

Lemma 5.3. *Let $y = y(u_0, v_0, f)$ be a solution of the strong beam damping problem* (5.5)*.*

(i) *Then*

$$(5.6) \quad |\dot{y}(t)|_H^2 + \|y(t)\|_{\mathcal{A}}^2 + \mu\|\dot{y}\|_{L^2(0,t;V_{\mathcal{A}})}^2 \leq c\left(|v_0|_H^2 + \|u_0\|_{\mathcal{A}}^2 + |f|_{L^2(0,T;H)}^2\right).$$

(ii) *Let $y_1 = y(u_{0,1}, v_{0,1}, f_1)$ and $y_2 = y(u_{0,2}, v_{0,2}, f_2)$ be two solutions of* (5.5)*. Then their difference $z = y(u_{0,1}, v_{0,1}, f_1) - y(u_{0,2}, v_{0,2}, f_2)$ satisfies*

$$\begin{aligned} (5.7) \qquad & |\dot{z}(t)|_H^2 + \|z(t)\|_{\mathcal{A}}^2 + \mu\|\dot{z}\|_{L^2(0,t;V_{\mathcal{A}})}^2 \\ & \leq C\left(|v_{0,1} - v_{0,2}|_H^2 + \|u_{0,1} - u_{0,2}\|_{\mathcal{A}}^2 + |f_1 - f_2|_{L^2(0,T;H)}^2\right). \end{aligned}$$

(iii) *The solution $y(u_0, v_0, f)$ is unique.*

Given $m \geq 1$, a function y_m is called an approximate solution of the strong beam damping problem if it satisfies the same conditions as in Definition 3, except that $y_m(0) = P_m u_0$, $\dot{y}_m(0) = P_m v_0$, and

$$(5.8) \qquad \ddot{y}_m + \mathcal{A}y_m + \mu\mathcal{A}\dot{y}_m + c_d\dot{y}_m = P_m f,$$

in V', a.e. for $t \in [0,T]$.

Theorem 5.4. *Given $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$, there exists a unique solution $y = y^{(\mu)}$ of the strong beam damping problem* (5.5)*. The solution satisfies $y \in C([0,T];V)$, and $\dot{y} \in C([0,T];H)$. Furthermore,*

$$\begin{aligned} (5.9) \qquad & |\dot{y}(t) - \dot{y}_m(t)|^2 + \|y(t) - y_m(t)\|_{\mathcal{A}}^2 + \mu\|\dot{y} - \dot{y}_m\|_{L^2(0,t;V_{\mathcal{A}})}^2 \\ & \leq c\left(|v_0 - P_m v_0|^2 + \|u_0 - P_m u_0\|_{\mathcal{A}}^2 + \|f - P_m f\|_{L^2(0,T;H)}^2\right) \end{aligned}$$

for any approximate solution y_m, $m \geq 1$, $t \in [0,T]$.

Note that inequality (5.9) implies that $y_m \to y$ strongly in $C([0,T];V)$, $\dot{y}_m \to \dot{y}$ strongly in $C([0,T];H)$ and in $L^2(0,T;V)$ as $m \to \infty$.

Finally in this section, we show that the vanishing damping $\mu \to 0$ causes the strong damping solutions to converge to the weak damping solution.

Theorem 5.5. *Let* $\mu > 0$, *and* $y^{(\mu)} = y^{(\mu)}(u_0, v_0, f)$ *be the solution of the strong beam damping problem* (5.5) *for some* $u_0 \in V$, $v_0 \in H$, $f \in L^2(0,T;H)$. *Let* $y = y(u_0, v_0, f)$ *be the solution of the weak beam damping problem* (5.1). *If* $\mu \to 0$, *then* $y^{(\mu)} \to y$ *in* $C([0,T];V)$, *and* $\dot{y}^{(\mu)} \to \dot{y}$ *in* $C([0,T];H)$.

Proof. Let $y_m^{(\mu)}$ and y_m be the approximate solutions for the strong and the weak beam damping problems. For any $t \in [0,T]$, we have

$$\|y(t)-y^{(\mu)}(t)\| \leq \|y(t)-y_m(t)\|+\|y_m(t)-y_m^{(\mu)}(t)\|+\|y_m^{(\mu)}(t)-y^{(\mu)}(t)\|. \tag{5.10}$$

Let $\epsilon > 0$. By estimates (5.4) and (5.9), we can make the first and the third terms in the right side of (5.10) to be less than ϵ, by choosing a sufficiently large m.

Now let $z_m = y_m - y_m^{(\mu)}$. It satisfies equation

$$\ddot{z}_m + \mathcal{A} z_m + c_d \dot{z}_m = \mu \mathcal{A} \dot{y}_m^{(\mu)}, \tag{5.11}$$

with $z_m(0) = 0$, and $\dot{z}_m(0) = 0$. By the properties of the approximate solutions, $z_m, \dot{z}_m \in C([0,T];V_m)$. Multiply both sides of (5.11) by $\dot{z}_m$, and use Lemma 4.2 to obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\dot{z}_m|_H^2 + \|z_m\|_{\mathcal{A}}^2\right) + c_d|\dot{z}_m|_H^2 = \mu\left(\mathcal{A}\dot{y}_m^{(\mu)}, \dot{z}_m\right). \tag{5.12}$$

Integrate it on $[0,t]$, and get

$$|\dot{z}_m(t)|_H^2 + \|z_m(t)\|^2 \leq c\mu \|\mathcal{A}\dot{y}_m^{(\mu)}\|_{L^2(0,T;V')} \|\dot{z}_m\|_{L^2(0,T;V)}, \tag{5.13}$$

where the constant $c > 0$ is independent of $m \in \mathbb{N}$ and $\mu > 0$.

Recall that any two norms on a finite-dimensional space are equivalent. In particular, the norms $|\cdot|_H$ and $\|\cdot\|_V$ are equivalent on V_m. This is also seen directly from $\|\varphi_k\|_{\mathcal{A}}^2 = \lambda_k^4 |\varphi_k|_H^2 = \lambda_k^4$, $k \geq 1$, see Lemma 4.4. Thus, there exists a constant C_m, such that $\|u\|_V \leq C_m |u|_H$ for any $u \in V_m$. In particular, $\|\dot{z}_m\|_{L^2(0,T;V)} \leq cC_m |\dot{z}_m|_{L^2(0,T;H)}$.

Using (5.2) and (5.6), which are also valid for y_m and $y_m^{(\mu)}$ correspondingly, we conclude that $|\dot{z}_m|_{L^2(0,T;H)} \leq c$. Therefore (5.13) becomes

$$|\dot{z}_m(t)|_H^2 + \|z_m(t)\|^2 \leq cC_m \mu \|\mathcal{A}\dot{y}_m^{(\mu)}\|_{L^2(0,T;V')}. \tag{5.14}$$

Since (5.6) is valid for $\dot{y}_m^{(\mu)}$ for any $m \geq 1$, we conclude that the set $\{\sqrt{\mu}\dot{y}_m^{(\mu)}\}_{\mu>0}$ is bounded in $L^2(0,T;V)$. Then the boundedness of $\mathcal{A}$ on V implies that $\mu\|\mathcal{A}\dot{y}_m^{(\mu)}\|_{L^2(0,T;V')} \to 0$ as $\mu \to 0$.

Therefore, the left side in (5.14) approaches zero as $\mu \to 0$. Thus we can choose a sufficiently small $\mu > 0$, such that $\|y_m(t) - y_m^{(\mu)}(t)\| = \|z_m\| < \epsilon$, and

(5.10) becomes $\|y(t) - y^{(\mu)}(t)\| \leq 3\epsilon$. This implies that $y^{(\mu)} \to y$ as $\mu \to 0$, in $C([0,T];V)$, and the first claim of the theorem is established. The second claim is proved using the same arguments applied to $|\dot{y}(t) - \dot{y}^{(\mu)}(t)|_H$. □

6. Vector functions II

In preparation for investigating shallow arch dynamics, we have to develop some additional results on vector functions. Some results on functions with values in Hilbert spaces have already been discussed in Section 4. The Hilbert spaces $W[0,T]$ and $W_r[0,T]$ were defined in (4.1) and (4.2). The space $V_{\mathcal{A}}$ was introduced in (4.10).

In attempting to use the same approach for arches as for the beams, we encounter the problem of passing to the weak limit in the *non-linear* term ∂U_a. This requires us to study the weak convergence in more detail.

Let X be a Hilbert space. A function $y : [0,T] \to X$ is called *weakly continuous* with values in X if scalar functions $t \to \langle y(t), w\rangle_X$ are continuous on $[0,T]$ for every $w \in X'$.

The next lemma deals with weakly continuous functions in Hilbert spaces X and Y satisfying $X \subset Y$. Since the condition $X \subset Y$ implies that $Y' \subset X'$, Lemma 6.1 is a non-trivial result.

Lemma 6.1. *Let X and Y be two Hilbert spaces such that $X \subset Y$ is a continuous and dense embedding.*

(i) *Let a function $g : [0,T] \to X$ satisfy $g \in L^\infty(0,T;X)$. If g is weakly continuous with values in Y, then g is weakly continuous with values in X.*

(ii) *Let a sequence $\{x_n\}_{n\geq 1}$ be bounded in X. If $x_n \rightharpoonup y_0$ weakly in Y, then $y_0 \in X$, and $x_n \rightharpoonup y_0$ weakly in X as $n \to \infty$.*

Proof. Part (i) is [10, Lemma 2.3.3].

For part (ii), suppose that $\|x_n\|_X \leq M$ for some $M > 0$, and any $n \geq 1$. Identify Y with its dual Y'. Then $X \subset Y \subset X'$ with continuous and dense embeddings, and the duality pairing $\langle\cdot,\cdot\rangle_X$ extends the inner product $(\cdot,\cdot)$ in Y. In particular, there exists $c > 0$ such that $\|w\|_{X'} \leq c|w|_Y$ for any $w \in Y$.

First, fix $w \in Y$. Then

$$|(x_n, w)| = |\langle x_n, w\rangle_X| \leq M\|w\|_{X'} \leq Mc|w|_Y.$$

It is assumed that $x_n \rightharpoonup y_0$ weakly in Y. Then passing to the limit as $n \to \infty$, gives

$$|(y_0, w)| = |\langle y_0, w\rangle_X| \leq M\|w\|_{X'}. \tag{6.1}$$

Since the embedding $Y \subset X'$ is dense, we conclude that inequality $|\langle y_0, w\rangle_X| \leq M\|w\|_{X'}$ is satisfied for any $w \in X'$. Therefore $y_0 \in X$, and $\|y_0\|_X \leq M$.

Now we want to show that $x_n \rightharpoonup y_0$ weakly in X as $n \to \infty$. So, let $w \in X'$. Because of the density of the embedding $Y \subset X'$, given $\epsilon > 0$, there exists

$w_\epsilon \in Y$, such that $\|w - w_\epsilon\|_{X'} < \epsilon$. Since $x_n \rightharpoonup y_0$ weakly in Y as $n \to \infty$, we have

$$|\langle x_n - y_0, w\rangle_X| \le |\langle x_n - y_0, w - w_\epsilon\rangle_X| + |(x_n - y_0, w_\epsilon)| \le 2M\epsilon + \epsilon$$

for sufficiently large n. Since $\epsilon > 0$ is arbitrary, we get $x_n \rightharpoonup y_0$ weakly in X as $n \to \infty$, as claimed. □

The next lemma is needed for the weak limit passage in nonlinear operators.

Lemma 6.2. (i) *Let X be a Hilbert space. Suppose that functions $y_n \rightharpoonup y$ weakly in $L^2(0,T;X)$, and $y_n \rightharpoonup y^*$ in the w^*-topology of $L^\infty(0,T;X)$ as $n \to \infty$. Then $y = y^*$ a.e. on $[0,T]$.*

(ii) *Suppose that functions $\{y_n\}_{n\ge1}$ and their derivatives $\{\dot y_n\}_{n\ge1}$ belong to bounded sets in $L^\infty(0,T;V)$, and $L^\infty(0,T;H)$ correspondingly. Also assume that $y_n \rightharpoonup y$ weakly in $L^2(0,T;V)$ as $n \to \infty$. Then $y_n(t) \rightharpoonup y(t)$ weakly in V as $n \to \infty$, for any $t \in [0,T]$.*

Proof. (i) The weak convergence $y_n \rightharpoonup y$ in $L^2(0,T;X)$ means that

$$\int_0^T (y_n(s), v(s))\, ds \to \int_0^T (y(s), v(s))\, ds \tag{6.2}$$

for any $v \in L^2(0,T;X)$. The *weak** convergence $y_n \rightharpoonup y^*$ means that

$$\int_0^T (y_n(s), v(s))\, ds \to \int_0^T (y^*(s), v(s))\, ds \tag{6.3}$$

for any $v \in L^1(0,T;X)$. In particular, (6.3) is satisfied for any $v \in L^2(0,T;X)$. Therefore $\int_0^T (y - y^*, v)\, ds = 0$ for any such v, which implies that $y = y^*$ in $L^2(0,T;X)$, and a.e. on $[0,T]$.

(ii) We have

$$|y_n(t) - y_n(s)|_H \le \int_s^t |\dot y_n(\tau)|_H\, d\tau$$

for any $0 \le s \le t \le T$. Therefore all the functions $\{y_n\}_{n\ge1}$ are equicontinuous and equibounded in $C([0,T];H)$.

By assumption $\|y_n(t)\|_V \le M$ a.e. on $[0,T]$ for some $M \ge 0$. Since the embedding of V into H is compact, functions $\{y_n\}_{n\ge1}$ have all their values in the same compact set $K \subset H$. By the Arzela-Ascoli Theorem the set $\{y_n\}_{n\ge1}$ is precompact in $C([0,T];H)$, therefore in $L^2(0,T;H)$.

The embedding $L^2(0,T;V) \to L^2(0,T;H)$ is linear and continuous. Therefore $y_n \rightharpoonup y$ weakly in $L^2(0,T;V)$, implies $y_n \rightharpoonup y$ weakly in $L^2(0,T;H)$. Since the functions are in a compact set in $C([0,T];H)$, we conclude that $y_n \to y$ strongly in $C([0,T];H)$ as $n \to \infty$.

In particular, $y_n(t) \to y(t)$ strongly (and weakly) in H for any $t \in [0,T]$. By Lemma 6.1(ii), $y_n(t) \rightharpoonup y(t)$ weakly in V as $n \to \infty$, as claimed. □

Let $H_0^1 = H_0^1(0,\pi)$ be the Hilbert space defined in (2.7), and $\langle\cdot,\cdot\rangle_1$ be the duality pairing between H_0^1 and $(H_0^1)'$, which is consistent with the inner product $(\cdot,\cdot)$ in H.

Following [7], the non-dimensional axial potential energy is given by

$$U_a(u) = \frac{1}{2\pi}\left(\beta + \frac{1}{2}|u'|_H^2\right)^2, \quad u \in H_0^1, \tag{6.4}$$

and the subdifferential $\partial U_a : H_0^1 \to (H_0^1)'$ of U_a, at $u \in H_0^1$ is given by

$$\partial U_a(u) = \frac{1}{\pi}\left(\beta + \frac{1}{2}|u'|_H^2\right)\partial\left(\frac{1}{2}|u'|_H^2\right) = \frac{1}{\pi}\left(\beta + \frac{1}{2}|u'|_H^2\right)\mathcal{B}u. \tag{6.5}$$

Here $\mathcal{B} : H_0^1 \to (H_0^1)'$ is the linear continuous, symmetric, positive and coercive operator on H_0^1 defined by

$$\langle \mathcal{B}u, v\rangle_1 = (u', v')_H, \quad u, v \in H_0^1, \tag{6.6}$$

see Example 9.3. Note that $|\langle \mathcal{B}u, v\rangle_1| \le \|u\|_1\,\|v\|_1$, where the norm is in H_0^1. Therefore $\|\mathcal{B}\| \le 1$.

By Lemma 4.2(iii) used with $A = \mathcal{B}$, we have

$$\frac{d}{dt}U_a(y) = \frac{1}{\pi}\left(\beta + \frac{1}{2}|y'|_H^2\right)\langle \mathcal{B}y, \dot{y}\rangle_1 = \langle \partial U_a(y), \dot{y}\rangle_1. \tag{6.7}$$

Lemma 6.3. *The subdifferential $\partial U_a : H_0^1 \to (H_0^1)'$ is a continuous nonlinear operator, which is Lipschitz continuous on bounded subsets of H_0^1. More precisely, if $\|u\|_1, \|\bar{u}\|_1 \le M$, where $M \ge 1$, then*

$$\|\partial U_a(u) - \partial U_a(\bar{u})\|_{-1} \le cM^2\|u - \bar{u}\|_1. \tag{6.8}$$

Furthermore, ∂U_a maps weakly convergent sequences in V into strongly convergent ones in $(H_0^1)'$.

Proof. Let $u, \bar{u} \in H_0^1$, with $\|u\|_1, \|\bar{u}\|_1 \le M$. Using $\|\mathcal{B}\| \le 1$, we get

$$\begin{aligned}
&\|\partial U_a(u) - \partial U_a(\bar{u})\|_{-1} \\
&\le \frac{1}{2\pi}\left||u'|_H^2 - |\bar{u}'|_H^2\right|\,\|\mathcal{B}u\|_{-1} + \frac{1}{\pi}\left|\beta + \frac{1}{2}|\bar{u}'|_H^2\right|\,\|\mathcal{B}u - \mathcal{B}\bar{u}\|_{-1} \\
&\le \frac{M}{\pi}\left||u'|_H - |\bar{u}'|_H\right|\|u\|_1 + \frac{1}{\pi}\left|\beta + \frac{1}{2}M^2\right|\,\|u - \bar{u}\|_1.
\end{aligned}$$

Since $\left||u'|_H - |\bar{u}'|_H\right| \le |u' - \bar{u}'|_H = \|u - \bar{u}\|_1$, inequality (6.8) follows.

Now, let $u_n \rightharpoonup u$ weakly in V as $n \to \infty$. The embedding of V into H_0^1 is compact. Therefore $u_n \to u$ strongly in H_0^1 as $n \to \infty$, and the second assertion follows from the Lipschitz continuity of ∂U_a. $\square$

7. Arch motion with strong damping $\mu > 0$

The abstract equation for the strong damping arch motion is (3.8) with the addition of the term $\mu\mathcal{A}\dot{y}$. The difficulty for the arch motion is that we cannot claim that the eigenfunctions φ_k of $\mathcal{A}$ are also the eigenfunctions of the operator $\mathcal{B}$. Thus $\mathcal{B}y_m \notin V_m$.

Definition 4. Let $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$. A function $y \in W_r[0,T]$ is called a solution of the strong arch damping problem with $\mu > 0$ if it satisfies $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$, $y(0) = u_0$, $\dot{y}(0) = v_0$, and

$$\ddot{y} + \mathcal{A}y + \partial U_a(y) + \mu\mathcal{A}\dot{y} + c_d\dot{y} = f, \tag{7.1}$$

in V', a.e. for $t \in [0,T]$. Here the operator $\mathcal{A} : V \to V'$ is defined by (2.9), and the axial potential energy U_a by (6.4). We write $y = y^{(\mu)} = y^{(\mu)}(u_0, v_0, f)$ to emphasize the dependence of the solution on the data.

Lemma 7.1. *Let $y = y(u_0, v_0, f)$ be a solution of the strong arch damping problem* (7.1) *with $\mu > 0$, and $t \in [0,T]$.*

(i) *Then*

$$\begin{aligned} &|\dot{y}(t)|_H^2 + \|y(t)\|_{\mathcal{A}}^2 + \mu\|\dot{y}\|_{L^2(0,t;V_{\mathcal{A}})}^2 \\ &\quad\le c\left(|v_0|_H^2 + \|u_0\|_{\mathcal{A}}^2 + U_a(u_0) + |f|_{L^2(0,T;H)}^2\right). \end{aligned} \tag{7.2}$$

(ii) *Let $y_1 = y(u_{0,1}, v_{0,1}, f_1)$ and $y_2 = y(u_{0,2}, v_{0,2}, f_2)$ be two solutions of* (7.1). *Then their difference $z = y_1 - y_2$ satisfies*

$$\begin{aligned} &|\dot{z}(t)|_H^2 + \|z(t)\|_{\mathcal{A}}^2 + \mu\|\dot{z}\|_{L^2(0,t;V_{\mathcal{A}})}^2 \\ &\quad\le \frac{C}{\mu}\left(|v_{0,1} - v_{0,2}|_H^2 + \|u_{0,1} - u_{0,2}\|_{\mathcal{A}}^2 + \|f_1 - f_2\|_{L^2(0,T;H)}^2\right), \end{aligned} \tag{7.3}$$

where the constant C depends only on the bounds of the initial conditions, and the loads f_1 and f_2.

(iii) *The solution $y^{(\mu)} = y(u_0, v_0, f)$ is unique.*

Proof. (i) Multiply (7.1) by $\dot{y}$, and then use Lemma 4.2 and (6.7) to obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\dot{y}|_H^2 + \|y\|_{\mathcal{A}}^2\right) + \frac{d}{dt}U_a(y) + \mu\langle\mathcal{A}\dot{y}, \dot{y}\rangle + c_d|\dot{y}|_H^2 = (f, \dot{y})_H. \tag{7.4}$$

Integrate both sides of (7.4) from 0 to t. Note that all the terms in the left side are non-negative. Thus

$$\begin{aligned} &|\dot{y}(t)|_H^2 + \|y(t)\|_{\mathcal{A}}^2 + 2\mu\|\dot{y}\|_{L^2(0,t;V_{\mathcal{A}})}^2 \\ &\quad\le |v_0|_H^2 + \|u_0\|_{\mathcal{A}}^2 + 2U_a(u_0) + 2\int_0^t (f(s), \dot{y}(s))\,ds. \end{aligned}$$

Since $f \in L^2(0,T;H)$, we have

$$2\left|\int_0^t (f(s), \dot{y}(s))\,ds\right| \le |f|_{L^2(0,T;H)}^2 + |\dot{y}|_{L^2(0,T;H)}^2, \tag{7.5}$$

and estimate (7.2) follows by Gronwall's inequality.

(ii) The difference $z = y_1 - y_2$ satisfies

$$\ddot{z} + \mathcal{A}z + \mu\mathcal{A}\dot{z} + c_d\dot{z} = \partial U_a(y_2) - \partial U_a(y_1) + f_1 - f_2.$$

Multiply this equality by $\dot{z} \in L^2(0,T;V)$, and use Lemma 4.2 to obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\dot{z}|_H^2 + \|z\|_{\mathcal{A}}^2\right) + \mu\langle\mathcal{A}\dot{z},\dot{z}\rangle \le |\langle\partial U_a(y_1) - \partial U_a(y_2),\dot{z}\rangle_1| + |(f_1 - f_2,\dot{z})|.$$

The pairing $\langle\cdot,\cdot\rangle_1$ refers to the space H_0^1. By (7.2), the values of y_1 and y_2 remain in a bounded set in H_0^1, that depends only on the initial data and the loads. The subdifferential ∂U_a is Lipschitz continuous on bounded sets according to Lemma 6.3. Therefore

$$\begin{aligned}(7.6)\qquad & |\dot{z}(t)|_H^2 + \|z(t)\|_{\mathcal{A}}^2 + 2\mu\|\dot{z}\|_{L^2(0,t;V_{\mathcal{A}})}^2 \\ &\le C\bigg(|\dot{z}(0)|_H^2 + \|z(0)\|_{\mathcal{A}}^2 + \int_0^T \|z(s)\|_1\|\dot{z}(s)\|_1\,ds \\ &\qquad + \int_0^T |f_1(s) - f_2(s)|_H\,|\dot{z}(s)|_H\,ds\bigg).\end{aligned}$$

The last term is estimated as in (7.5), and for the previous term we have

$$(7.7)\qquad \int_0^t \|z(s)\|_{\mathcal{A}}\|\dot{z}(s)\|_{\mathcal{A}}\,ds \le \frac{C}{\mu}\|z\|_{L^2(0,t;V_{\mathcal{A}})}^2 + \frac{\mu}{C}\|\dot{z}\|_{L^2(0,t;V_{\mathcal{A}})}^2.$$

The Gronwall's inequality gives (7.3).

(iii) The uniqueness follows from (7.3). □

Given $m \ge 1$, a function $y_m = y_m^{(\mu)}$ is called an approximate solution of the strong arch damping problem if it satisfies the same conditions as in Definition 4, except that $y_m(0) = P_m u_0$, $\dot{y}_m(0) = P_m v_0$, and

$$(7.8)\qquad \ddot{y}_m + \mathcal{A}y_m + P_m^*\partial U_a(y_m) + \mu\mathcal{A}\dot{y}_m + c_d\dot{y}_m = P_m f,$$

in V', a.e. for $t \in [0,T]$. The operators P_m and P_m^* were defined in Lemma 4.5.

Theorem 7.2. *Given $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$, there exists a unique solution y of the strong damping problem* (7.1). *The solution satisfies $y \in C([0,T];V)$, and $\dot{y} \in C([0,T];H)$.*

Proof. Let $y_m(t) = \sum_{k=1}^m g_{k,m}(t)\varphi_k$, where the functions $g_{k,m}$, $k = 1,\dots,m$ satisfy of the following system of m ordinary differential equations

$$\begin{aligned}(7.9)\qquad & \langle\ddot{y}_m + \mathcal{A}y_m + P_m^*\partial U_a(y_m) + \mu\mathcal{A}\dot{y}_m + c_d\dot{y}_m,\varphi_k\rangle = (P_m f,\varphi_k)_H, \\ & ((y_m(0),\varphi_k))_{\mathcal{A}} = ((P_m u_0,\varphi_k))_{\mathcal{A}}, \quad (\dot{y}_m(0),\varphi_k) = (P_m v_0,\varphi_k),\end{aligned}$$

where $k = 1,\dots,m$.

Using Lemma 6.3, all the coefficients of equation (7.9) are Lipschitz continuous on bounded sets. Therefore the system has unique solutions $g_{k,m}$ satisfying $g_{k,m}$, $\dot{g}_{k,m} \in C[0,T]$, $\ddot{g}_{k,m} \in L^2(0,T)$. Thus y_m, $\dot{y}_m \in C([0,T];V_m)$, and $\ddot{y}_m \in L^2(0,T;V_m)$.

Note that

$$\langle P_m^* \partial U_a(y_m), \varphi_k\rangle = \langle \partial U_a(y_m), \varphi_k\rangle \text{ for } 1 \le k \le m, \text{ and}$$
$$\langle P_m^* \partial U_a(y_m), \varphi_k\rangle = 0 \text{ for } k > m.$$

Therefore, in fact, equation (7.9) are satisfied for any $k \ge 1$. This implies that y_m satisfies (7.8), that is, y_m is an approximate solution of the strong arch damping problem.

Notice that (6.7) is still applicable to y_m, i.e., $\frac{d}{dt} U_a(y_m) = \langle \partial U_a(y_m), \dot{y}_m\rangle$. Therefore, the estimates in Lemma 7.1 are valid for y_m as well. They show that all the approximate solutions y_m, $m \ge 1$ remain within the same bounded ball in $L^\infty(0, T; V)$. Their derivatives $\dot{y}_n$ remain in a bounded ball in $L^\infty(0, T; H)$, as well as in $L^2(0, T; V)$, since $\mu > 0$ is fixed.

Furthermore, let us move all the terms in equation (7.1) to its right side, except $\ddot{y}$. The estimates for y and $\dot{y}$ give an estimate for $\ddot{y}$ in $L^2(0, T; V')$. Clearly, the same estimate is valid for any approximate solution y_m. Thus $\ddot{y}_m$, $m \ge 1$ remain within the same bounded ball in $L^2(0, T; V')$.

Since the Hilbert space $W_r[0, T]$ defined in (4.2) is reflexive, we can find a subsequence of $\{y_m\}_{m\ge1}$ (still denoted by y_m) such that functions y_m, $\dot{y}_m$, $\ddot{y}_m$, $m \ge 1$ are weakly convergent in the corresponding spaces. Since the derivatives are taken in the distributional sense, it follows that there exists $y \in W_r[0, T]$, such that

$$y_m \rightharpoonup y, \quad \dot{y}_m \rightharpoonup \dot{y}, \quad \ddot{y}_m \rightharpoonup \ddot{y}, \tag{7.10}$$

weakly as $m \to \infty$. Furthermore, using Lemma 6.2, we can as well assume that $y_m \rightharpoonup y$, and $\dot{y}_m \rightharpoonup \dot{y}$ in the *weak** topologies of the spaces $L^\infty(0, T; V)$ and $L^\infty(0, T; H)$ correspondingly.

Now we show that equation (7.1) is satisfied for the constructed function y. Indeed, the weak convergence implies the distributional convergence. Since each y_m satisfies (7.9), we can certainly pass to the limit as $m \to \infty$ in V' in all the linear terms of this equation. As for the nonlinear operator ∂U_a, we use Lemma 6.2 to conclude that $y_m(t) \rightharpoonup y(t)$ weakly in V for any $t \in [0, T]$. Then Lemma 6.3 shows that $\partial U_a(y_m(t)) \rightharpoonup \partial U_a(y(t))$ weakly in V' for any $t \in [0, T]$.

This argument also shows that $u_{0,m} = y_m(0) \rightharpoonup y(0)$ weakly in V. Thus $y(0) = u_0$. Using standard methods we can also conclude that $\dot{y}(0) = v_0$, [9, Section 3.8.2]. The conclusion is that y is the solution of the strong damping problem (7.1). The required continuity of y follows from Lemma 4.2. $\square$

8. Arch motion with weak damping $\mu = 0$

In this section we show the existence of a solution for the weak arch damping problem. Unlike the previous sections, we do not establish the uniqueness of the solution. Also, we show only that $y \in C([0, T]; H_0^1)$, and $\dot{y} : [0, T] \to H$ is weakly continuous. Stronger results are available for uniform (no cracks) arches, see [6].

The abstract equation for the arch motion is (3.8), which was derived from (3.1). The solution is defined as follows.

Definition 5. Let $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$. A function $y \in W[0,T]$ is called a solution of the weak arch damping problem if it satisfies $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$, $y(0) = u_0$, $\dot{y}(0) = v_0$, and

$$\ddot{y} + \mathcal{A}y + \partial U_a(y) + c_d \dot{y} = f, \tag{8.1}$$

in V', a.e. for $t \in [0,T]$. Here the operator $\mathcal{A} : V \to V'$ is defined by (2.9). We will write $y = y(u_0, v_0, f)$ to emphasize the dependence of the solution on the data.

Lemma 8.1. *Let y be a solution of the weak arch damping problem. Then $y \in C([0,T]; H_0^1)$, and $\dot{y} : [0,T] \to H$ is weakly continuous in H.*

Proof. Since $\dot{y} \in L^2(0,T;H)$, we conclude that $y \in C([0,T];H)$. Thus it is also weakly continuous in H. Since $y \in L^\infty(0,T;V)$, Lemma 6.1 shows that y is weakly continuous from $[0,T]$ to V. The embedding $V \subset H_0^1$ is compact. Therefore y is strongly continuous in H_0^1.

Since $\ddot{y} \in L^2(0,T;V')$, we get $\dot{y} \in C([0,T];V')$. Thus $\dot{y}$ is weakly continuous from $[0,T]$ to V'. Also $\dot{y} \in L^\infty(0,T;H)$. Then Lemma 6.1 shows that $\dot{y}$ is weakly continuous from $[0,T]$ to H. $\square$

We establish the existence of the solution by two methods: by taking the weak limit of the approximate solutions, and by the "parabolic regularization". That is, by taking the weak limit of the strong damping solutions $y^{(\mu)}$ as $\mu \to 0$.

Given $m \geq 1$, a function $y_m \in W_r[0,T] \cap C([0,T];V_m)$ is called an approximate solution of the weak arch damping problem, if it satisfies the same conditions as in Definition 5, except that $y_m(0) = P_m u_0$, $\dot{y}_m(0) = P_m v_0$, and

$$\ddot{y}_m + \mathcal{A}y_m + P_m^* \partial U_a(y_m) + c_d \dot{y}_m = P_m f, \tag{8.2}$$

in V', a.e. for $t \in [0,T]$. The operators P_m and P_m^* were defined in Lemma 4.5.

Theorem 8.2. *Given $u_0 \in V$, $v_0 \in H$, and $f \in L^2(0,T;H)$, there exists a solution y of the weak arch damping problem* (8.1)*. The solution satisfies $y \in C([0,T]; H_0^1)$, and $\dot{y}$ is weakly continuous from $[0,T]$ to H.*

Proof. The required continuity properties of the solution y are established in Lemma 8.1. The existence of the solution is shown by two methods.

Limit of the approximate solutions. Arguing as in Theorem 7.2, the functions $y_m(t) = \sum_{k=1}^m g_{k,m}(t)\varphi_k$, are approximate solutions provided that the coefficient functions $g_{k,m}$, $k = 1, \ldots, m$ satisfy of the following system of m ordinary differential equations

$$\begin{aligned} &\langle \ddot{y}_m + \mathcal{A}y_m + P_m^* \partial U_a(y_m) + c_d \dot{y}_m, \varphi_k \rangle = (P_m f, \varphi_k)_H, \\ &((y_m(0), \varphi_k))_{\mathcal{A}} = ((P_m u_0, \varphi_k))_{\mathcal{A}}, \quad (\dot{y}_m(0), \varphi_k) = (P_m v_0, \varphi_k), \end{aligned} \tag{8.3}$$

where $k = 1, \ldots, m$. The solution of this system is unique, and we conclude that so defined functions y_m satisfy y_m, $\dot{y}_m \in C([0,T];V_m)$, and $\ddot{y}_m \in L^2(0,T;V_m)$.

Multiply (8.2) by $\dot{y}_m$. Then use Lemma 4.2 and (6.7) to obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\dot{y}_m|_H^2 + \|y_m\|_{\mathcal{A}}^2\right) + \frac{d}{dt}U_a(y_m) + c_d|\dot{y}_m|_H^2 = (f, \dot{y}_m)_H. \tag{8.4}$$

Integrate both sides of (8.4) from 0 to t. Note that all the terms in the left side are non-negative. Thus

$$|\dot{y}_m(t)|_H^2 + \|y_m(t)\|_{\mathcal{A}}^2 \leq |v_0|_H^2 + \|u_0\|_{\mathcal{A}}^2 + 2U_a(u_0) + 2\int_0^t (f(s), \dot{y}_m(s))\, ds.$$

Since $f \in L^2(0, T; H)$, we have

$$2\left|\int_0^t (f(s), \dot{y}_m(s))\, ds\right| \leq |f|_{L^2(0,T;H)}^2 + |\dot{y}_m|_{L^2(0,T;H)}^2, \tag{8.5}$$

and estimate

$$\begin{aligned} &|\dot{y}_m(t)|_H^2 + \|y_m(t)\|_{\mathcal{A}}^2 \\ &\leq c\left(1 + |v_0|_H^2 + \|u_0\|_{\mathcal{A}}^2 + \|u_0\|_{\mathcal{A}}^4 + |f|_{L^2(0,T;H)}^2\right) \end{aligned} \tag{8.6}$$

follows by Gronwall's inequality. Here we used the fact that $\|u\|_1 \leq c\|u\|_{\mathcal{A}}$ for any $u \in V$.

The boundedness estimate (8.6) allows us to select a subsequence of the approximate solutions (still denoted by $\{y_m\}_{m\geq 1}$) that converges weakly in $W[0, T]$, as well as weak* in $L^\infty(0, T; V)$ for y_m, and weak* in $L^\infty(0, T; H)$ for the derivatives $\dot{y}_m$ as $m \to \infty$. Let the limit of the sequence be denoted by y.

Arguing as in Theorem 7.2, we conclude that we can pass to the limit in (8.2) as $m \to \infty$, and obtain (8.1). Thus y is a solution of the weak arch damping problem.

Limit of the strong damping solutions. Given $\mu > 0$, let $y^{(\mu)} = y^{(\mu)}(u_0, v_0, f)$ be the solution of the strong damping problem (7.1). Its existence and the uniqueness was proved in Theorem 7.2. Note that $y^{(\mu)}$ satisfies estimate (7.2), where the constant c is independent of μ.

We conclude from (7.2) that all the solutions $y^{(\mu)}$, $\mu > 0$ are bounded in $L^2(0, T; V)$, and $L^\infty(0, T; V)$, and the derivatives $\dot{y}^{(\mu)}$, $\mu > 0$ are bounded in $L^2(0, T; H)$, and $L^\infty(0, T; H)$. Also, the set $\{\mu\|\dot{y}^{(\mu)}\|_{L^2(0,T;V)}^2\}_{\mu>0}$ is bounded in $\mathbb{R}$. In other words, the set $\{\sqrt{\mu}\dot{y}^{(\mu)}\}_{\mu>0}$ is bounded in $L^2(0, T; V)$. In addition, moving all the terms, except $\ddot{y}^{(\mu)}$, to the right side of (7.2), we conclude that the set $\{\ddot{y}^{(\mu)}\}_{\mu>0}$ is bounded in $L^2(0, T; V')$.

Now we can choose a subsequence of $y^{(\mu)}$, $\mu > 0$ (still denoted by $y^{(\mu)}$), such that

$$y^{(\mu)} \rightharpoonup y, \quad \dot{y}^{(\mu)} \rightharpoonup \dot{y}, \quad \ddot{y}^{(\mu)} \rightharpoonup \ddot{y}, \tag{8.7}$$

weakly as $\mu \to 0$ for some $y \in W[0, T]$, in the corresponding spaces. Furthermore, using Lemma 6.2, we can as well assume that $y^{(\mu)} \rightharpoonup y$, and $\dot{y}^{(\mu)} \rightharpoonup \dot{y}$ in the *weak** topologies of the spaces $L^\infty(0, T; V)$ and $L^\infty(0, T; H)$ correspondingly.

Arguing as in Theorem 7.2, we conclude that we can pass to the limit as $\mu \to 0$ in (7.1). Note that $\mu \mathcal{A} \dot{y}^{(\mu)} \to 0$ as $\mu \to 0$, since the set $\{\sqrt{\mu}\dot{y}^{(\mu)}\}_{\mu>0}$ is bounded in $L^2(0,T;V)$, and $\mathcal{A}$ is bounded on V. Thus $y \in W[0,T]$ satisfies equation (8.1) with $y \in L^\infty(0,T;V)$, $\dot{y} \in L^\infty(0,T;H)$. That is, y is a solution of the weak arch damping problem. □

9. Appendix. Convex functions and subdifferentials

Subdifferentials provide the proper mathematical framework for the abstract formulation of equations of motion. See [7] on how this concept is used for the derivation of such equations. Here we restrict ourselves to essential examples.

Let X be a Hilbert space. A function $\phi : X \to (-\infty, +\infty]$ is called *proper and convex* on X if ϕ is not identically $+\infty$, and $\phi((1-\lambda)x + \lambda y) \le (1-\lambda)\phi(x) + \lambda\phi(y)$ for any $x, y \in X$, and $\lambda \in [0,1]$. The function ϕ is called *lower-semicontinuous* on X if every level set $\{x \in X : \phi(x) \le c\}$, $c > -\infty$, is closed in X.

Given a proper, convex, lower-semicontinuous function ϕ on X, the *subdifferential* $\partial\phi : X \to X'$ is defined by

$$\partial\phi(x) = \{x^* \in X' : \phi(y) \ge \phi(x) + \langle x^*, y - x\rangle\} \tag{9.1}$$

for any $y \in X$. Thus $\partial\phi \subset X \times X'$.

Theorem 9.1. *Let X be a Hilbert space, and $A : X \to X'$ be a linear, continuous, and symmetric operator, such that $D(A) = X$, and $\langle Au, u\rangle \ge 0$ for any $u \in X$. Then function $\phi : X \to \mathbb{R}$ defined by*

$$\phi(u) = \frac{1}{2}\langle Au, u\rangle, \quad u \in X, \tag{9.2}$$

is convex, proper, and lower-semicontinuous on X. Moreover, it is Fréchet differentiable on X with $\nabla\phi(u) = \partial\phi(u) = Au$ for any $u \in X$, and $D(\phi) = D(\partial\phi) = X$.

Example 9.2. Let V be the Hilbert space defined in (2.4), and $\mathcal{A}$ be the linear operator defined in (2.9). Let

$$\varphi(u) = \frac{1}{2}\langle \mathcal{A}u, u\rangle, \quad u \in V. \tag{9.3}$$

By Theorem 9.1 with $X = V$, function $\varphi : V \to \mathbb{R}$ is proper, convex, and lower-semicontinuos on V. Furthermore, $D(\partial\varphi) = V$, and $\partial\varphi(u) = \mathcal{A}u$ for any $u \in V$.

For a general $u \in V$, the expression for $\partial\varphi(u) \in V'$ is complicated. However, if we assume that u is somewhat more regular, then we can get a simpler expression for it.

Suppose that $u \in D(\mathcal{A}) \subset V$. Then, by Theorem 2.2, we have $\mathcal{A}u = u''''$ a.e. on every interval l_i, $i = 1, \ldots, m+1$. Thus, we can say that $\partial\varphi(u) = u''''$ a.e. on every such interval.

Suppose further, that $u \in H_0^1(0,\pi) \cap H^4(0,\pi) \subset D(\mathcal{A})$. Then $u'''' \in L^2(0,\pi)$, and u' is smooth. Thus $J[u'](x_i) = 0$ for any $i = 1, \ldots, m$, and we have

$$\langle \partial\varphi(u), v\rangle = \langle \mathcal{A}u, v\rangle = \int_0^\pi u''(x)v''(x)\,dx = \int_0^\pi u''''(x)v(x)\,dx \tag{9.4}$$

for any $v \in V$. Therefore, in this case $\partial\varphi(u) = u''''$ a.e. on $[0,\pi]$.

Example 9.3. Let $H_0^1 = H_0^1(0,\pi)$ be the Hilbert space defined in (2.7), and $\langle\cdot,\cdot\rangle_1$ be the duality pairing between H_0^1 and $(H_0^1)'$. Let $\mathcal{B}$ be the linear operator defined by

$$\langle \mathcal{B}u, v\rangle_1 = (u', v')_H, \quad u, v \in H_0^1. \tag{9.5}$$

Then $\mathcal{B} : H_0^1 \to (H_0^1)'$ is continuous, symmetric and coercive on H_0^1. In particular, $\mathcal{B}$ is positive, and its range is $(H_0^1)'$.

Let

$$\psi(u) = \frac{1}{2}\langle \mathcal{B}u, u\rangle, \quad u \in H_0^1. \tag{9.6}$$

Theorem 9.1 is applicable with $X = H_0^1$, and $A = \mathcal{B}$. We conclude that the function $\psi : H_0^1 \to \mathbb{R}$ is proper, convex, and lower-semicontinuos on H_0^1. Furthermore, $D(\partial\psi) = H_0^1$, and $\partial\psi(u) = \mathcal{B}u \in (H_0^1)'$ for any $u \in H_0^1$.

As in Example 9.2, a simpler expression for the subdifferential $\partial\psi(u)$ can be obtained assuming an additional regularity of $u \in H_0^1$.

Suppose that $u \in V \subset H_0^1$. Then we have

$$\begin{aligned}\langle \mathcal{B}u, v\rangle_1 = (u', v')_H &= \int_0^\pi u'(x)v'(x)\,dx \\ &= -\sum_{i=1}^{m} J[u'(x_i)]v(x_i) - \sum_{i=1}^{m+1}(u'', v)_i\end{aligned} \tag{9.7}$$

for any $v \in H_0^1$. Therefore, in this case,

$$\partial\psi(u) = \mathcal{B}u = -\sum_{i=1}^{m} J[u'](x_i)\delta(x - x_i) - u'', \tag{9.8}$$

where $\delta(x-a)$, $a \in [0,\pi]$ is the element of $(H_0^1)'$, defined by $\langle\delta(x-a), v\rangle_1 = v(a)$ for any $v \in H_0^1$.

Suppose further, that $u \in D(\mathcal{A}) \subset V$. Then, by Theorem 2.2, u satisfies conditions (2.10)–(2.11). In particular, $J[u'](x_i) = \theta_i u''(x_i)$. Thus, with this additional assumption on u, we have

$$\partial\psi(u) = \mathcal{B}u = -\sum_{i=1}^{m} \theta_i u''(x_i)\delta(x - x_i) - u'', \tag{9.9}$$

which is still an element of $(H_0^1)'$.

References

[1] J. M. Ball, *Stability theory for an extensible beam*, J. Differential Equations **14** (1973), 399–418. https://doi.org/10.1016/0022-0396(73)90056-9

[2] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Monographs in Mathematics, Springer, New York, 2010. https://doi.org/10.1007/978-1-4419-5542-5

[3] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, No. 5, Notas de Matemática, No. 50, North-Holland Publishing Co., Amsterdam, 1973.

[4] M. N. Cerri and G. C. Ruta, *Detection of localised damage in plane circular arches by frequency data*, J. Sound Vibration **270** (2004), no. 1, 39–59.

[5] E. Emmrich and M. Thalhammer, *A class of integro-differential equations incorporating nonlinear and nonlocal damping with applications in nonlinear elastodynamics: existence via time discretization*, Nonlinearity **24** (2011), no. 9, 2523–2546. https://doi.org/10.1088/0951-7715/24/9/008

[6] S. Gutman, J. Ha, and S. Lee, *Parameter identification for weakly damped shallow arches*, J. Math. Anal. Appl. **403** (2013), no. 1, 297–313. https://doi.org/10.1016/j.jmaa.2013.02.047

[7] S. Gutman, J. Ha, and S. Shon, *Equations of motion for cracked beams and shallow arches*, Nonlinear Functional Analysis and Applications **27** (2022), no. 2, 405–432.

[8] S. Gutman, J. Ha, and S. Shon, *Variational setting for cracked beams and shallow arches*, Arch. Appl. Mech. **92** (2022), 2225–2236. https://doi.org/10.1007/s00419-022-02174-6

[9] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York, 1972.

[10] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, second edition, Applied Mathematical Sciences, 68, Springer-Verlag, New York, 1997. https://doi.org/10.1007/978-1-4612-0645-3

Semion Gutman
Department of Mathematics
University of Oklahoma
Norman, Oklahoma 73019, USA
Email address: sgutman@ou.edu

Junhong Ha
School of Liberal Arts
Korea University of Technology and Education
Cheonan 31253, Korea
Email address: hjh@koreatech.ac.kr

Sudeok Shon
School of Architectural Engineering
Korea University of Technology and Education
Cheonan 31253, Korea
Email address: sdshon@koreatech.ac.kr