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SPECTRAL DECOMPOSITION FOR HOMEOMORPHISMS ON NON-METRIZABLE TOTALLY DISCONNECTED SPACES

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Abstract. We introduce the notions of symbolic expansivity and symbolic shadowing for homeomorphisms on non-metrizable compact spaces which are generalizations of expansivity and shadowing, respectively, for metric spaces. The main result is to generalize the Smale's spectral decomposition theorem to symbolically expansive homeomorphisms with symbolic shadowing on non-metrizable compact Hausdorff totally disconnected spaces.

1. Introduction

The famous spectral decomposition theorem by Smale [6] says that the non-wandering set $\Omega(f)$ of an Axiom A diffeomorphism f on a compact C^∞ manifold can be decomposed as a finite union of disjoint closed invariant sets on which f is topologically transitive. In the case, we say that f has the spectral decomposition.

Aoki [1] extended the result to homeomorphisms on compact metric spaces as follows: if f is an expansive homeomorphism with shadowing on a compact metric space, then f has the spectral decomposition. Afterwards, there are many works that generalize the Smale's spectral decomposition theorem to more general settings. For example, to multi dynamical systems (e.g. [5]), to homeomorphisms on non compact spaces (e.g. [4]), and to homeomorphisms with measure expansivity and measure shadowing (e.g. [3]).

Recently, Good and Meddaugh [2] introduced a notion of shadowing for homeomorphisms f on compact Hausdorff spaces, and showed that f has the shadowing if and only if the system is conjugate the inverse limit of directed systems satisfying the Mittag-Leffler condition and consisting of shifts of finite type. More precisely, let X be a compact Hausdorff space, and $\mathcal{FO}(X)$ denote the collection of all finite open covering of X. Let f be a homeomorphism on X. For $\beta \in \mathcal{FO}(X)$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a β-pseudo orbit of f if for any $i \in \mathbb{Z}$, $f(x_i), x_{i+1} \in U$ for some $U \in \beta$. We say that f has the shadowing

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property if for any $\alpha \in \mathcal{FO}(X)$, there is $\beta \in \mathcal{FO}(X)$ such that $\beta \succ \alpha$ and any β-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ can be α-shadowed by a point in X, i.e., there is $y \in X$ such that for any $i \in \mathbb{Z}$, $f^i(y), x_i \in U$ for some $U \in \alpha$ (see Definition 5 in [2]).

In this paper, we introduce another types of expansivity (called symbolic expansivity) and shadowing (called symbolic shadowing) for homeomorphisms on non-metrizable compact spaces which are generalizations of usual expansivity and usual shadowing, respectively, for metric spaces. Then we extend the Smale's spectral decomposition theorem to symbolically expansive homeomorphisms with symbolic shadowing as follows.

Main Theorem. *If a homeomorphism f on a non-metrizable compact Hausdorff totally disconnected space X is symbolically expansive and has symbolic shadowing, then f has the spectral decomposition, i.e., the nonwandering set $\Omega(f)$ is decomposed by a disjoint union of finitely many invariant and closed subsets*

$$\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$$

such that f is topologically transitive on each Ω_i for $1 \leq i \leq n$.

2. Symbolic shadowing

In this section, we study the symbolic shadowing of homeomorphisms on compact totally disconnected spaces. Precisely, we prove that if a homeomorphism f has the symbolic shadowing, then its restriction on nonwandering set has the symbolic shadowing. First, we introduce some definitions and notations.

Let X be a non-metrizable compact Hausdorff totally disconnected space X. A finite collection $\boldsymbol{\epsilon} = \{U_1, \ldots, U_n\}$ consisting of open subsets of X is called as an open partition if U_i's are pairwise disjoint and $\bigcup_{i=1}^n U_i = X$. We denote by $\mathcal{P}(X)$ the collection of all finite open partitions of X. We note that $\mathcal{P}(X)$ is nonempty. For any $\alpha, \beta \in \mathcal{P}(X)$, we say that α is a *refinement* of β, (denoted by $\alpha \succ \beta$), if for any $U \in \alpha$ there exists $V \in \beta$ such that $U \subset V$.

First of all, we can check the basic property of refinement from the following lemma.

Lemma 2.1. *Let $f : X \to X$ be a homeomorphism. For any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, there exists $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that $f(\boldsymbol{\delta}(x)) \subset \boldsymbol{\epsilon}(f(x))$ for all $x \in X$.*

Proof. Let $\boldsymbol{\epsilon} = \{A_1, A_2, \ldots, A_n\}$ and $B_{ij} = f^{-1}(A_i) \cap A_j$ for $1 \leq i, j \leq n$. Put $\Lambda = \{(i,j) : B_{ij} \neq \emptyset \text{ for } 1 \leq i,j \leq n\}$, we consider $\boldsymbol{\delta} = \{B_{ij} : (i,j) \in \Lambda\}$. Then $\boldsymbol{\delta}$ satisfies that $\boldsymbol{\delta}$ covers X, and $\boldsymbol{\delta} \succ \boldsymbol{\epsilon}$. We check it as follows.

First, $\boldsymbol{\delta}$ covers X. For any $x \in X$, let $x \in A_j$ and $f(x) \in A_i$ for some $1 \leq i, j \leq n$. That is, $x \in f^{-1}(A_i) \cap A_j = B_{ij} \in \boldsymbol{\delta}$. And it satisfies either $B_{ij} \cap B_{kl} = \emptyset$ or $B_{ij} = B_{kl}$ for some $(k,l) \in \Lambda$. Suppose that $B_{ij} \cap B_{kl} \neq \emptyset$. Then

$$x \in B_{ij} \cap B_{kl} = (f^{-1}(A_i) \cap A_j) \cap (f^{-1}(A_k) \cap A_l).$$

This means that $x \in A_j \cap A_l$ and $x \in f^{-1}(A_i) \cap f^{-1}(A_k)$. So, $f(x) \in A_i \cap A_k$, we can see that $i = k$ and $j = l$. Thus, $B_{ij} = B_{kl}$.

Second, $\boldsymbol{\delta} \in \mathcal{P}(X)$, $\boldsymbol{\delta} \succ \boldsymbol{\epsilon}$. For any $x \in X$, $\boldsymbol{\delta}(x) = B_{ij}$ for some $(i, j) \in \Lambda$. Since $B_{ij} = f^{-1}(A_i) \cap A_j$,

$$f(x) \in f(\boldsymbol{\delta}(x)) \subset A_i \cap f(A_j) \subset A_i = \boldsymbol{\epsilon}(f(x)). \qquad \square$$

We say that a point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{Z} \setminus \{0\}$, denote by $\mathrm{Per}(f)$ the set of all periodic points of f. A point $x \in X$ is called *nonwandering* if for any neighborhood U of x, there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points of f is called the nonwandering set of f, denoted by $\Omega(f)$.

A sequence $\xi = \{x_0, \ldots, x_n\}$ is called an *$\boldsymbol{\epsilon}$-chain of f ($\boldsymbol{\epsilon} \in \mathcal{P}(X)$) from x to y* ($x, y \in X$) if $f(x_i) \in \boldsymbol{\epsilon}(x_{i+1})$ for all $0 \leq i \leq n-1$, and $x_0 = x$ and $x_n = y$. A point $x \in X$ is called *chain recurrent* if for any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, there is an $\boldsymbol{\epsilon}$-chain $\xi = \{x_i\}_{i=0}^n$ from x to itself. The set of all chain recurrent points of f is called the chain recurrent of f, and denote it by $CR(f)$. We see that $CR(f)$ is closed and invariant.

For any $x, y \in CR(f)$ and $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, we say that $x \sim^{\boldsymbol{\epsilon}} y$ if there are an $\boldsymbol{\epsilon}$-chain from x to y and an $\boldsymbol{\epsilon}$-chain from y to x. We define $x \sim y$ whenever $x \sim^{\boldsymbol{\epsilon}} y$ for any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$. It is clear that $\sim$ is an equivalence relation on the set $CR(f)$ and we call an equivalence class of $\sim$ by a *chain component of f*. We see that any chain component of f is closed and invariant.

A sequence $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is called a *$\boldsymbol{\delta}$-pseudo orbit of f* if $f(x_i) \in \boldsymbol{\delta}(x_i)$ for all $i \in \mathbb{Z}$. A $\boldsymbol{\delta}$-pseudo orbit is said to be *$\boldsymbol{\epsilon}$-shadowed by $x \in X$* if $f^i(x) \in \boldsymbol{\epsilon}(x_i)$ for all $i \in \mathbb{Z}$.

Definition. We say that f has the *symbolic shadowing* if for all $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ there exists $\delta \in \mathcal{P}(X)$ with $\boldsymbol{\delta} \succ \boldsymbol{\epsilon}$ such that every δ-pseudo orbit $\xi = \{x_i\}$ is $\boldsymbol{\epsilon}$-shadowed by a point in X.

We observe that the notion of symbolic shadowing on totally disconnected space is equivalent to one introduced by Good and Meddaugh, see Lemma 15 in [2].

Now we consider the symbolic shadowing on $\Omega(f)$ and $CR(f)$.

Lemma 2.2. *Let f be a homeomorphism on X. Then $\Omega(f) \subset CR(f)$. Moreover, if f has the symbolic shadowing, then $\Omega(f) = CR(f)$.*

Proof. We first show that $\Omega(f) \subset CR(f)$. Let $x \in \Omega(f)$ and $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ be arbitrary. Choose $\boldsymbol{\delta} \in \mathcal{P}(X)$ with $\boldsymbol{\delta} \succ \boldsymbol{\epsilon}$ such that $f(\boldsymbol{\delta}(x)) \subset \boldsymbol{\epsilon}(f(x))$ for all $x \in X$. Since $x \in \Omega(f)$ there exists $n \in \mathbb{N}$ such that $f^n(\boldsymbol{\delta}(x)) \cap \boldsymbol{\delta}(x) \neq \emptyset$. That is, there is $y \in \boldsymbol{\delta}(x)$ satisfying $f^n(y) \in \boldsymbol{\delta}(x)$. Then we can see an $\boldsymbol{\epsilon}$-chain

$$\xi = \{x_0 = x, f(y), f^2(y), \ldots, f^{n-1}(y), x_n = x\}$$

from x to x. It satisfies that $f(x_i) \in \boldsymbol{\epsilon}(x_{i+1})$ for $i = 0, 1, 2, \ldots, n-1$. More precisely, we can check that $f(x) \in \boldsymbol{\epsilon}(x_1) = \boldsymbol{\epsilon}(f(y))$. Since $y \in \boldsymbol{\delta}(x)$ and

$x \in \boldsymbol{\delta}(y) = \boldsymbol{\delta}(f(x))$, $f(x) \in f(\boldsymbol{\delta}(y)) \subset \boldsymbol{\epsilon}(f(y))$. Also, $f(x_{n-1}) = f^n(y) \in \boldsymbol{\delta}(x) \subset \boldsymbol{\epsilon}(x)$. Therefore, $x \in CR(f)$.

Moreover, suppose that f has the symbolic shadowing on X. For any $x \in CR(f)$ and any neighborhood U of x, choose $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ such that $\boldsymbol{\epsilon}(x) \subset U$. Since f has the symbolic shadowing, there is $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that every $\boldsymbol{\delta}$-chain can be $\boldsymbol{\epsilon}$-shadowed. Since $x \in CR(f)$ and given $\boldsymbol{\delta} \in \mathcal{P}(X)$, there is a $\boldsymbol{\delta}$-chain

$$\xi = \{x = x_0, x_1, \ldots, x_{n-1}, x_n = x\}$$

from x to itself. Then there is $y \in X$ such that $f^i(y) \in \boldsymbol{\epsilon}(x_i)$ for all $i = 0, 1, \ldots, n-1$. Since $y \in \boldsymbol{\epsilon}(x) \subset U$ and $f^n(y) \in \boldsymbol{\epsilon}(x) \subset U$, these imply that $f^n(y) \in U \cap f^n(U) \neq \emptyset$. Therefore, $x \in \Omega(f)$. □

Now let $C_{\boldsymbol{\epsilon}}(x) = \{y \in CR(f) : y \sim^{\boldsymbol{\epsilon}} x\}$ be the $\boldsymbol{\epsilon}$-chain component for f. Then we can see that it is f-invariant following lemma.

Lemma 2.3. *For any $x \in CR(f)$, any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, $C_{\boldsymbol{\epsilon}}(x)$ is f-invariant.*

Proof. Let $y \in C_{\boldsymbol{\epsilon}}(x)$. First, we will show that $f(y) \in C_{\boldsymbol{\epsilon}}(x)$ and only if $f(y) \sim^{\boldsymbol{\epsilon}} x$. For given $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ since $y \in C_{\boldsymbol{\epsilon}}(x)$, there exist $\boldsymbol{\epsilon}$-chains

$$\{x = x_0, x_1, x_2, \ldots, x_n = y\} : \text{from } x \text{ to } y,$$
$$\{y = y_0, y_1, y_2, \ldots, y_m = x\} : \text{from } y \text{ to } x.$$

Since $f(y_1) \in \boldsymbol{\epsilon}(y_2)$, $\boldsymbol{\epsilon}(f(y_1)) = \boldsymbol{\epsilon}(y_2)$ and $f(f(x)) \in \boldsymbol{\epsilon}(f(y_1)) = \boldsymbol{\epsilon}(y_2)$. Hence it is easy to see that two $\boldsymbol{\epsilon}$-chains

$$\{x, x_1, \ldots, x_m = y, f(y)\} : \text{from } x \text{ to } f(y),$$
$$\{f(y), y_2, y_3, \ldots, y_m = x\} : \text{from } f(y) \text{ to } x.$$

So, $f(y) \sim^{\boldsymbol{\epsilon}} x$.

Now, let $\bar{C}_{\boldsymbol{\epsilon}}(x)$ be the $\boldsymbol{\epsilon}$-chain component of f^{-1}. By the first fact of that, $f^{-1}(\bar{C}_{\boldsymbol{\epsilon}}(x)) \subset \bar{C}_{\boldsymbol{\epsilon}}(x)$. We will show that $\bar{C}_{\boldsymbol{\epsilon}}(x) = C_{\boldsymbol{\epsilon}}(x)$.

At first, $C_{\boldsymbol{\epsilon}}(x) \subset \bar{C}_{\boldsymbol{\epsilon}}(x)$, i.e., if $y \in C_{\boldsymbol{\epsilon}}(x)$, then $y \in \bar{C}_{\boldsymbol{\epsilon}}(x)$. For any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ since $y \in C_{\boldsymbol{\epsilon}}(x)$, there exist $\boldsymbol{\epsilon}$-chains

$$\{y = y_0, \ldots, y_n = x\} \text{ satisfying } f(y_i) \in \boldsymbol{\epsilon}(y_{i+1}),$$
$$\{x = z_0, \ldots, z_m = y\} \text{ satisfying } f(z_i) \in \boldsymbol{\epsilon}(z_{i+1}).$$

Then we have that $y_i \in \boldsymbol{\epsilon}(f^{-1}(y_{i+1}))$ and $z_i \in \boldsymbol{\epsilon}(f^{-1}(z_{i+1}))$. Thus, we can get that $\{x = y_n, y_{n-1}, y_{n-2}, \ldots, y_2, y_1, y_0 = y\}$ is an $\boldsymbol{\epsilon}$-chain of f^{-1} from x to y.

And second, $\bar{C}_{\boldsymbol{\epsilon}}(x) \subset C_{\boldsymbol{\epsilon}}(x)$, i.e., if $y \in \bar{C}_{\boldsymbol{\epsilon}}(x)$, then $y \in C_{\boldsymbol{\epsilon}}(x)$. For any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ since $y \in \bar{C}_{\boldsymbol{\epsilon}}(x)$, there exist $\boldsymbol{\epsilon}$-chains

$$\{y = y_0, \ldots, y_n = x\} \text{ satisfying } f^{-1}(y_i) \in \boldsymbol{\epsilon}(y_{i+1}),$$
$$\{x = z_0, \ldots, z_m = y\} \text{ satisfying } f^{-1}(z_i) \in \boldsymbol{\epsilon}(z_{i+1}).$$

Then $y_i \in f(\boldsymbol{\epsilon}(y_{i+1})) = \boldsymbol{\epsilon}(f(y_{i+1}))$, we have that $\{y_n = x, y_{n-1}, \ldots, y_1, y_0 = y\}$ is an $\boldsymbol{\epsilon}$-chain from y to x. Similarly, $\{z_m = y, \ldots, z_0 = x\}$ is an $\boldsymbol{\epsilon}$-chain from y to x. So, $y \sim^{\boldsymbol{\epsilon}} x$ under f. Therefore, $C_{\boldsymbol{\epsilon}}(x)$ is f-invariant. □

Lemma 2.4. *For any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, there exists $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that $CR(f) \cap \boldsymbol{\delta}(x) \subset C_{\boldsymbol{\epsilon}}(x)$ for all $x \in CR(f)$.*

Proof. Let $y \in CR(f) \cap \boldsymbol{\delta}(x)$ and $\boldsymbol{\delta}$ be as before. We show that $x \sim^{\boldsymbol{\epsilon}} y$.

Since $y \in CR(f)$, there is a $\boldsymbol{\delta}$-chain $\{y = y_0, y_1, \ldots, y_m = y\}$ from y to itself. Since $x \in \boldsymbol{\delta}(x)$, we have

$$f(x) \in f(\boldsymbol{\delta}(x)) \subset \boldsymbol{\epsilon}(f(x)) = \boldsymbol{\epsilon}(y_1).$$

Then we get an $\boldsymbol{\epsilon}$-chain $\{x, y_1, y_2, \ldots, y_m = y\}$ from x to y.

Similarly, since $x \in CR(f)$, there is a $\boldsymbol{\delta}$-chain $\{x = x_0, x_1, \ldots, x_n = x\}$. Because $f(y) \in \boldsymbol{\epsilon}(x_1)$, we can construct an $\boldsymbol{\epsilon}$-chain $\{y, x_1, \ldots, x_n = x\}$ from y to x. Since

$$f(y) \in f(\boldsymbol{\delta}(x)) \subset \boldsymbol{\epsilon}(f(x)) = \boldsymbol{\epsilon}(x_1),$$

$y \in C_{\boldsymbol{\epsilon}}(x)$. □

Lemma 2.5. *For any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, there exists $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that any $\boldsymbol{\delta}$-chain in $CR(f)$ is contained in a common $\boldsymbol{\epsilon}$-chain component of f.*

Proof. For $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, we take $\boldsymbol{\delta} \in \mathcal{P}(X)$ corresponding to $\boldsymbol{\epsilon}$ by Lemma 2.4. Let $x \in CR(f)$ and $C_{\boldsymbol{\epsilon}}(x)$ be the $\boldsymbol{\epsilon}$-chain component containing x, and

$$U_{\boldsymbol{\delta}} = \bigcup_{y \in C_{\boldsymbol{\epsilon}}(x)} (CR(f) \cap \boldsymbol{\delta}(y)).$$

By Lemma 2.4, we see that $U_{\boldsymbol{\delta}} = C_{\boldsymbol{\epsilon}}(x)$, and so $C_{\boldsymbol{\epsilon}}(x)$ is open in $CR(f)$.

Moreover, let $\xi = \{x_0, x_1, \ldots, x_n\}$ be a $\boldsymbol{\delta}$-chain in $CR(f)$. Since $C_{\boldsymbol{\epsilon}}(x_0)$ is f-invariant, we have $f(x_0) \in C_{\boldsymbol{\epsilon}}(x_0)$. Moreover, since ξ is a $\boldsymbol{\delta}$-chain of f, we see that

$$x_1 \in \boldsymbol{\delta}(f(x_0)) \cap CR(f) \subset U_{\boldsymbol{\delta}} = C_{\boldsymbol{\epsilon}}(x_0).$$

Then $C_{\boldsymbol{\epsilon}}(x_0) = C_{\boldsymbol{\epsilon}}(x_1)$. Continue the process, we derive that $\xi \subset C_{\boldsymbol{\epsilon}}(x_0)$. □

We say that $f : X \to X$ has the *finite symbolic shadowing* if for given $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, there exists $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that if for a set $\{x_0, x_1, \ldots, x_m\} \subset Y$ satisfying that $f(x_k) \in \boldsymbol{\delta}(x_{k+1}),\ 0 \le k \le m-1$, then there is $x \in X$ such that $f^k(x) \in \boldsymbol{\epsilon}(x_k),\ 0 \le k \le m-1$.

Lemma 2.6. *A homeomorphism $f : X \to X$ has the finite symbolic shadowing if and only if it has the symbolic shadowing.*

Proof. It is clear that if f has the symbolic shadowing, then it has the finite symbolic shadowing. Suppose that f has the finite symbolic shadowing. It is enough to show that f has the symbolic shadowing.

Let $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ and take $\boldsymbol{\delta} \in \mathcal{P}(X)$ corresponding to $\boldsymbol{\epsilon}$ by the finite symbolic shadowing of f. Let $\xi = \{x_k : k \in \mathbb{Z}\}$ be a $\boldsymbol{\delta}$-pseudo orbit of f. Fix $m > 0$ and set ${x_k}' = x_{k-m}$ for all $k \in \mathbb{Z}$. We consider

$$\{x'_0, x'_1, \ldots, x'_{2m}\} = \{x_{-m}, x_{-m+1}, \ldots, x_{-m+k}, \ldots, x_{-m+2m}\}.$$

Since f has the finite symbolic shadowing, there exists $y_m \in X$ such that $f^k(y_m) \in \boldsymbol{\epsilon}(x'_k),\ 0 \le k \le 2m$. Put $z_m = f^m(y_m)$ then $f^k(z_m) \in \boldsymbol{\epsilon}(x_k),\ -m \le k \le m$. Let $z_m \to z$ then we can see that

$$f^k(z) \in \boldsymbol{\epsilon}(x_k),\ \forall k \in \mathbb{Z}.$$

This means that ξ is $\boldsymbol{\epsilon}$-shadowed by z. Therefore, f has the symbolic shadowing. □

Theorem 2.7. *If $f : X \to X$ has the symbolic shadowing, then $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ has the symbolic shadowing.*

Proof. By Lemma 2.6, it is enough to show that if f has the symbolic shadowing, then its restriction $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ has the finite symbolic shadowing.

For any $\boldsymbol{\epsilon} \in \mathcal{P}(X)$, choose $\boldsymbol{\gamma} \in \mathcal{P}(X)$ such that every $\boldsymbol{\gamma}$-chain in X is $\boldsymbol{\epsilon}$-shadowed by a point in X. By Lemma 2.2, we have $CR(f) = \Omega(f)$. For $\boldsymbol{\gamma} \in \mathcal{P}(X)$ with $\boldsymbol{\gamma} \succ \boldsymbol{\delta}$, take $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that if $\xi = \{x_0, x_1, x_2, \ldots, x_n\}$ is a finite $\boldsymbol{\delta}$-chain in $\Omega(f)$, then it is contained in one $\boldsymbol{\gamma}$-chain component by Lemma 2.5. That is, $\xi \subset C_{\boldsymbol{\gamma}}(x_0)$.

Let $\xi = \{x_0, x_1, \ldots, x_k\}$ be a $\boldsymbol{\delta}$-chain in $\Omega(f)$. Since $x_k, x_0 \in C_\gamma(x_0)$, there exists a γ-chain $\{x_k, x_{k+1}, \ldots, x_{n-1}, x_n = x_0\}$ from x_k to x_0. Consider periodic $\boldsymbol{\gamma}$-chain

$$\{x_0, x_1, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_n = x_0, x_1, \ldots, x_n, \ldots\} = \{y_i\}_{i \in \mathbb{Z}}.$$

Since f has the symbolic shadowing, there exists $y \in X$ such that $f^i(y) \in \boldsymbol{\epsilon}(y_{i+1})$ for all $i \in \mathbb{Z}$. Then we have

$$f^n(y) \in \boldsymbol{\epsilon}(x_0), f^{2n}(y) \in \boldsymbol{\epsilon}(x_0), \ldots, f^{in}(y) \in \boldsymbol{\epsilon}(x_0)$$

for all $i \in \mathbb{Z}$.

Since $\boldsymbol{\epsilon}(x_0)$ is compact, there is a convergent subsequence of $\{f^{in}(y) : i \in \mathbb{N}\}$. We assume that $f^{n_j}(y) \to z \in \boldsymbol{\epsilon}(x_0)$. Note that $z \in \Omega(f)$.

Now we show that $z \in \Omega(f)$ is an $\boldsymbol{\epsilon}$-shadowing point of $\{x_0, x_1, \ldots, x_k\}$. In fact, since $\boldsymbol{\epsilon}(x_i)$ is closed in X for all $0 \le i \le k$, we have

$$\begin{aligned} f^{n_j+1}(y) &= f(f^{n_j}(y)) \longrightarrow f(z) \in \boldsymbol{\epsilon}(x_1), \\ &\vdots \\ f^{n_j+k}(y) &= f^k(f^{n_j}(y)) \longrightarrow f^k(z) \in \boldsymbol{\epsilon}(x_k). \end{aligned}$$

Then we get that

$$f(z) \in \boldsymbol{\epsilon}(x_1), f^2(z) \in \boldsymbol{\epsilon}(x_2), \ldots, f^k(z) \in \boldsymbol{\epsilon}(x_k).$$

This means that z is an $\boldsymbol{\epsilon}$-shadowing point of $\{x_0, x_1, \ldots, x_k\}$. Therefore, f has the finite symbolic shadowing on $\Omega(f)$. □

3. Proof of Main Theorem

In this section, we prove the spectral decomposition theorem for symbolic expansive homeomorphisms with symbolic shadowing on compact totally disconnected spaces. More precisely, we show that if a homeomorphism f on a compact totally disconnected space is symbolically expansive and has the symbolic shadowing, then it has the spectral decomposition, i.e., the nonwandering $\Omega(f)$ can be decomposed as a finite union of disjoint closed invariant sets on which f is topologically transitive. For this, we first introduce the notion of symbolic expansivity for f.

Definition. We say that f is *symbolically expansive* if there is $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ such that $\Gamma_{\boldsymbol{\epsilon}}(x) = \{x\}$ for all $x \in X$, where

$$\Gamma_{\boldsymbol{\epsilon}}(x) = \{y \in X : f^i(y) \in \boldsymbol{\epsilon}(f^i(x)) \text{ for all } i \in \mathbb{Z}\}.$$

Here, $\boldsymbol{\epsilon}$ is called an *expansive partition.*

To prove Main Theorem, we need several lemmas.

Lemma 3.1. *Let f be a symbolically expansive homeomorphism on X with an expansive partition $\boldsymbol{\epsilon}$. If*

$$f^n(y) \in \boldsymbol{\epsilon}(f^n(x)) \ \forall n \geq 0, \ \ [\textit{resp.} \ \forall n \leq 0]$$

for some $x, y \in X$, then for any $\boldsymbol{\delta} \succ \boldsymbol{\epsilon}$, there exists $N > 0$ such that $f^n(y) \in \boldsymbol{\delta}(f^n(x))$ for $\forall n \geq N$ [resp. $n \geq -N$].

Proof. Suppose by contradiction that there exist $x \neq y \in X$ and $\boldsymbol{\delta} \succ \boldsymbol{\epsilon}$ such that

(i) $\qquad f^n(y) \in \boldsymbol{\epsilon}(f^n(x)), \ \forall n \geq N,$

(ii) $\qquad \forall k \in \mathbb{N}, \ \exists n_k \in \mathbb{N}$ such that $f^{n_k}(y) \notin \boldsymbol{\delta}(f^{n_k}(x)).$

Let $f^{n_k}(y) \to y_0 \in X$ and $f^{n_k}(x) \to x_0 \in X$ as $k \to \infty$.

Assume that $y_0 \in \boldsymbol{\delta}(x_0)$ is a neighborhood of y_0 and a neighborhood of x_0. Then there is n_k such that $f^{n_k}(y) \in \boldsymbol{\delta}(x_0) = \boldsymbol{\delta}(f^{n_k}(x))$ and $f^{n_k}(x) \in \boldsymbol{\delta}(x_0) = \boldsymbol{\delta}(f^{n_k}(x))$. This is a contradiction by (ii). So, we have $y_0 \notin \boldsymbol{\delta}(x_0)$, i.e., $x_0 \neq y_0$. On the other hand, for all $n \in \mathbb{Z}$,

$$\begin{aligned} f^n(y_0) = f^n(\lim_{k\to\infty} f^{n_k}(y)) = \lim_{k\to\infty}(f^{n+n_k}(y)) &\in \boldsymbol{\epsilon}(f^{n+n_k}(x)) \text{ if } n_k > n \text{ by (i)} \\ &= \boldsymbol{\epsilon}(f^n(f^{n_k}(x))) \\ &= \boldsymbol{\epsilon}(f^n(x_0)). \end{aligned}$$

So, $f^n(y_0) \in \boldsymbol{\epsilon}(f^n(x_0))$. Since f is symbolically expansive, $x_0 = y_0$. This contradiction completes the proof of the lemma. □

Lemma 3.2. *If f is symbolically expansive and has the symbolic shadowing, then $CR(f) = \overline{Per(f)}$.*

Proof. Let $x \in CR(f)$, $\boldsymbol{\epsilon}$ be an expansive partition of f. For any neighborhood U of x, there exists a refinement $\boldsymbol{\gamma} \in \mathcal{P}(X)$ of $\boldsymbol{\epsilon}$ such that $\boldsymbol{\gamma}(x) \subset U$. In fact, we can consider $\boldsymbol{\gamma}$ as following

$$\begin{aligned}\boldsymbol{\gamma} =& \{U \cap E : E \in \boldsymbol{\epsilon} \text{ and } U \cap E \neq \emptyset\} \\ & \cup \{E - U : E \in \boldsymbol{\epsilon} \text{ and } U \cap E \neq \emptyset\} \\ & \cup \ \{E : E \in \boldsymbol{\epsilon} \text{ and } U \cap E = \emptyset\}.\end{aligned}$$

Since f has the symbolic shadowing, there is a partition $\boldsymbol{\delta} \in \mathcal{P}(X)$ such that every $\boldsymbol{\delta}$-chain $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is $\boldsymbol{\gamma}$-shadowed by a point in X. As $x \in CR(f)$, there is a $\boldsymbol{\delta}$-chain $\xi = \{x_0 = x, x_1, x_2, \ldots, x_n = x\}$ of f. We extend it to a $\boldsymbol{\delta}$-pseudo orbit through x by letting $x_i = x_{i+kn}$ for all $k \in \mathbb{Z}$ and $0 \leq i \leq n-1$. By the symbolic shadowing of f, there is $y \in X$ such that $f^i(y) \in \boldsymbol{\gamma}(x_i)$ for all $i \in \mathbb{Z}$. Then we see that

$$f^{i+n}(y) \in \boldsymbol{\gamma}(x_{i+n}) = \boldsymbol{\gamma}(x_i) = \boldsymbol{\gamma}(f^i(y)), \ \forall i \in \mathbb{Z},$$

and so $f^n(y) \in \Gamma_{\boldsymbol{\gamma}}(y)$. Since f is symbolically expansive, we get $f^n(y) = y$. It implies that y is periodic and $y \in \boldsymbol{\gamma}(x) \subset U$. Therefore, $x \in \overline{\mathrm{Per}(f)}$. □

End of proof of Main Theorem.

Step 1. If f is symbolically expansive and symbolic shadowing, then every chain component of f is open in $\Omega(f)$.

Proof. Let $\boldsymbol{C} \in \mathcal{P}(X)$ be an expansive partition of f. Take $\boldsymbol{\delta} \in \mathcal{P}(X)$ with $\boldsymbol{\delta} \succ \boldsymbol{C}$ by the definition of the symbolic shadowing property. Since $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ has the symbolic shadowing, it follows that $\Omega(f)$ splits into the equivalence classes of $\Omega(f)$ under the relation "$\sim$" as $\Omega(f) = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$.

For each $\lambda \in \Lambda$, let

$$U_\lambda = \bigcup_{x \in \Omega_\lambda} (\boldsymbol{\delta}(x) \cap \Omega(f)).$$

Then U_λ is open in $\Omega(f)$, that is, $\boldsymbol{\delta}(x) \cap \Omega(f)$ is open in $\Omega(f)$.

To show that Ω_λ is open in $\Omega(f)$, it is enough to show that $U_\lambda = \Omega_\lambda$. Clearly $\Omega_\lambda \subset U_\lambda$, we will prove that $U_\lambda \subset \Omega_\lambda$ by the following two items.

(i) $U_\lambda \cap \mathrm{Per}(f) \subset \Omega_\lambda$, i.e., for all $y \in U_\lambda \cap \mathrm{Per}(f)$ then there exists $x \in \Omega_\lambda$ such that $y \in \boldsymbol{\delta}(x)$. We will prove that for all $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ $(\boldsymbol{\epsilon} \succ \boldsymbol{\delta})$, $x \sim^{\boldsymbol{\epsilon}} y$.

Since $x \in \Omega_\lambda \subset \overline{\mathrm{Per}(f)}$, there exists a sequence $\{p_n\} \in \mathrm{Per}(f)$ such that $p_n \to x$. And there is $p \in \mathrm{Per}(f)$ such that

$$\begin{cases} p \in \boldsymbol{\epsilon}(x) \subset \boldsymbol{\delta}(x) = \boldsymbol{\delta}(y), \\ f(p) \in \boldsymbol{\epsilon}(f(x)). \end{cases}$$

Then $\{x, f(p), f^2(p), \ldots, f^{\pi(p)}(p) = p\}$ is an $\boldsymbol{\epsilon}$-chain x to p, where $\pi(p)$ is a period of p. Also, $\{p, f(p), f^2(p), \ldots, f^{\pi(p)-1}(p), x\}$ is an $\boldsymbol{\epsilon}$-chain p to x. Hence $p \sim^{\boldsymbol{\epsilon}} x$.

For any $\boldsymbol{\epsilon} \succ \boldsymbol{\delta} \succ \boldsymbol{C}$, construct a $\boldsymbol{\delta}$-chain in $\Omega(f)$ through p and y. Define

$$x_n = \begin{cases} f^n(y), & \text{if } n \geq 0, \\ f^n(p), & \text{if } n < 0. \end{cases}$$

Then $\{x_n\}_{n\in\mathbb{Z}}$ is a $\boldsymbol{\delta}$-chain, it satisfies that $p = f(x_{-1}) \in \boldsymbol{\delta}(x_0) = \boldsymbol{\delta}(y)$. By symbolic shadowing, there exists $z \in \Omega(f)$ such that $f^i(z) \in \boldsymbol{C}(x_i)$, $\forall i \in \mathbb{Z}$.

For any $\boldsymbol{\epsilon} \succ \boldsymbol{C}$, by Lemma 3.1,

$$\exists\, N_1 > 0 \text{ such that if } n \geq N_1, \text{ then } f^n(z) \in \boldsymbol{\epsilon}(f^n(y)),$$
$$\exists\, N_2 < 0 \text{ such that if } n \leq N_2, \text{ then } f^n(z) \in \boldsymbol{\epsilon}(f^n(p)).$$

Put $k = \max\{|N_1|, |N_2|\} + 1$,

$$\xi = \{p = f^{-k\pi(p)}(p), f^{-k\pi(p)+1}(z), f^{-k\pi(p)+2}(z), \ldots, f^{\pi(y)-1}(y), y\}$$

is an $\boldsymbol{\epsilon}$-chain from p to y. And $f^{-k\pi(p)+1}(p) \in \boldsymbol{\epsilon}(f^{-k\pi(p)+1}(z))$.

Similarly, we can construct a $\boldsymbol{\delta}$-chain in $\Omega(f)$. Define

$$\tilde{x}_m = \begin{cases} f^m(p), & \text{if } m \geq 0, \\ f^m(y), & \text{if } m < 0. \end{cases}$$

Then $\{\tilde{x}_m\}_{m\in\mathbb{Z}}$ is a $\boldsymbol{\delta}$-chain, it satisfies that $y = f(\tilde{x}_{-1}) \in \boldsymbol{\delta}(\tilde{x}_0) = \boldsymbol{\delta}(p)$. By the symbolic shadowing, there is $\bar{z} \in \Omega(f)$ such that $f^j(\bar{z}) \in \boldsymbol{C}(\tilde{x}_j)$, $\forall j \in \mathbb{Z}$.

For any $\boldsymbol{\epsilon} \succ \boldsymbol{C}$, by Lemma 3.1,

$$\exists\, M_1 > 0 \text{ such that if } m \geq M_1, \text{ then } f^m(\bar{z}) \in \boldsymbol{\epsilon}(f^m(p)),$$
$$\exists\, M_2 < 0 \text{ such that if } m \leq M_2, \text{ then } f^m(\bar{z}) \in \boldsymbol{\epsilon}(f^m(y)).$$

Put $l = \max\{|M_1|, |M_2|\} + 1$,

$$\bar{\xi} = \{y = f^{-l\pi(y)}(y), f^{-l\pi(y)+1}(\bar{z}), f^{-l\pi(y)+2}(\bar{z}), \ldots, f^{\pi(p)-1}(p), p\}$$

is an $\boldsymbol{\epsilon}$-chain from y to p. And $f^{-l\pi(y)+1}(y) \in \boldsymbol{\epsilon}(f^{-l\pi(y)+1}(\bar{z}))$. So, $\bar{x}_i$ is an $\boldsymbol{\epsilon}$-chain from y to p. Therefore, we can see that $x \sim^{\boldsymbol{\epsilon}} p \sim^{\boldsymbol{\epsilon}} y$.

(ii) We can easily see that $U_\lambda = U_\lambda \cap \overline{\mathrm{Per}(f)}$ by the definition of U_λ. Let $y \in U_\lambda \cap \overline{\mathrm{Per}(f)}$ then there exists a sequence $\{p_n\} \in \mathrm{Per}(f)$ such that p_n converges to $y \in \Omega_\lambda$ for large n and $\{p_n\} \in U_\lambda$. So,

$$p_n \to y \in U_\lambda = U_\lambda \cap \overline{\mathrm{Per}(f)} \subset \overline{U_\lambda \cap \mathrm{Per}(f)} \subset \overline{\Omega_\lambda} = \Omega_\lambda.$$

Therefore, we conclude that $U_\lambda = \Omega_\lambda$, that is, Ω_λ is open in $\Omega(f)$. □

Step 2. If $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ has the symbolic shadowing, then $\Omega(f)$ splits into the equivalence classes Ω_λ $(\forall\lambda \in \Lambda)$ under the relation $\sim$. Since X is compact, Ω_λ's are finite,

$$\Omega(f) = \Omega_1 \,\dot{\cup}\, \Omega_2 \,\dot{\cup} \cdots \dot{\cup}\, \Omega_n.$$

That is, Ω_λ is open and f-invariant, and $\Omega(f) = \bigcup_{\lambda\in\Lambda} \Omega_\lambda$.

Finally, we ready to complete the proof of Main Theorem, it is enough to check (iii) f is topologically transitive on each Ω_i.

(iii) Let $\boldsymbol{\epsilon} \in \mathcal{P}(X)$ be an expansive partition. We are going to show that for any nonempty open sets $U, V \subset \Omega_i$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Choose $x \in U$, $y \in V$. Take a partition $\boldsymbol{\gamma} \in \mathcal{P}(X)$ and $\boldsymbol{\gamma} \succ \boldsymbol{\epsilon}$ such that

$$\forall\, W \in \boldsymbol{\gamma} \text{ with } W \cap U \neq \emptyset,\ W \subset U,$$
$$\forall\, W \in \boldsymbol{\gamma} \text{ with } W \cap V \neq \emptyset,\ W \subset V.$$

By the symbolic shadowing, there exists $\boldsymbol{\delta} \in \mathcal{P}(X)$ ($\boldsymbol{\delta} \succ \boldsymbol{\gamma}$) such that every $\boldsymbol{\delta}$-chain in $\Omega_i \subset \Omega(f)$ is $\boldsymbol{\gamma}$-shadowed by a point z in $\Omega(f)$ by Lemma 2.7.

Since $x, y \in \Omega_i$, there is a $\boldsymbol{\delta}$-chain $\xi = \{x = x_0, x_1, \ldots, x_n = y\}$ such that

- $\boldsymbol{\delta}(x_0) \cap U \neq \emptyset \Rightarrow \boldsymbol{\delta}(x_0) \subset U$,
- $z \in \boldsymbol{\delta}(x_0) \subset U \Rightarrow \boldsymbol{\delta}(x_0) \cap U \neq \emptyset \Rightarrow z \in U \Rightarrow f^n(z) \in f^n(U)$,
- $f^n(z) \in \boldsymbol{\delta}(y) \cap V \neq \emptyset \Rightarrow \boldsymbol{\delta}(y) \subset V \Rightarrow f^n(z) \in V$.

So, we can see that $f^n(z) \in f^n(U) \cap V \neq \emptyset$. Therefore, f is topologically transitive.

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