J. Appl. & Pure Math. Vol. **4***(2022), No. 3 - 4, pp. 123 - 139*
https://doi.org/10.23091/japm.2022.123

DISTRIBUTION OF ZEROS OF THE COSINE-TANGENT AND SINE-TANGENT POLYNOMIALS†

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Abstract. In this paper we give some interesting properties of the cosine tangent polynomials and sine tangent polynomials. In addition, we give some identities for these polynomials and the distribution of zeros of these polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : Cosine-Bernoulli polynomials, sine-Bernoulli polynomials, tangent numbers and polynomials, cosine-tangent polynomials, sine-tangent polynomials.

1. Introduction

Many mathemations studied in the field of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). It is well known that the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x=0, B_n=B_n(0)$ are called the Bernoulli numbers. The tangent polynomials are given by the generating function to be

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x=0, T_n=T_n(0)$ are called the tangent numbers (see [6, 7]).

Received February 17, 2022. Revised April 11, 2022. Accepted May 20, 2022. *Corresponding author.
†This work was supported by 2021 Hannam University Research Fund.

The Bernoulli polynomials $B_n^{(r)}(x)$ of order r are defined by the following generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad (|t|<2\pi). \tag{3}$$

The Frobenius–Euler polynomials of order r, denoted by $H_n^{(r)}(u,x)$, are defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt}=\sum_{n=0}^{\infty} H_n^{(r)}(u,x)\frac{t^n}{n!}. \tag{4}$$

The values at $x=0$ are called Frobenius-Euler numbers of order r; when $r=1$, the polynomials or numbers are called ordinary Frobenius-Euler polynomials or numbers. In [4], we introduced the cosine-Bernoulli, sine-Bernoulli, cosine-Euler, and sine-Euler polynomials. We also obtained some identities for these polynomials. The cosine-Bernoulli polynomials $B_n^{(C)}(x,y)$ and cosine-Euler polynomials $E_n^{(C)}(x,y)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} B_n^{(C)}(x,y)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt}\cos yt, \tag{5}$$

and

$$\sum_{n=0}^{\infty} E_n^{(C)}(x,y)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt}\cos yt, \tag{6}$$

respectively.

In this paper, we introduce some special polynomials which are related to tangent polynomials. In addition, we give some identities for these polynomials. Finally, we investigate the distribution of zeros of these polynomials.

2. Cosine-tangent and sine-tangent polynomials

In this section, we obtain some properties of the cosine-tangent and sine-tangen polynomials. In [6, 7], we introduced tangent numbers and polynomials. After that we investigated some their properties. In [5], Park and Kang defined the cosine-tangent and sine-tangen polynomials. From now on, some results are the same as [5], but we describe them again for better understanding. We consider the tagent polynomials that are given by the generating function to be

$$\sum_{n=0}^{\infty} T_n(x+iy)\frac{t^n}{n!}=\frac{2}{e^{2t}+1}e^{(x+iy)t}. \tag{7}$$

On the other hand, we note that

$$e^{(x+iy)t}=e^{xt}e^{iyt}=e^{xt}(\cos yt+i\sin yt). \tag{8}$$

From (7) and (8), we obtain

$$\sum_{n=0}^{\infty} T_n(x+iy)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}(\cos yt + i\sin yt), \tag{9}$$

and

$$\sum_{n=0}^{\infty} T_n(x-iy)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}(\cos yt - i\sin yt). \tag{10}$$

Hence, by (9) and (10), we obtain

$$\frac{2}{e^{2t}+1}e^{xt}\cos yt = \sum_{n=0}^{\infty}\left(\frac{T_n(x+iy)+T_n(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{11}$$

and

$$\frac{2}{e^{2t}+1}e^{xt}\sin yt = \sum_{n=0}^{\infty}\left(\frac{T_n(x+iy)-T_n(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{12}$$

It follows that Park and Kang defined the following cosine-tangent and sine-tangent polynomials (see [5]).

Definition 2.1. The cosine-tangent polynomials $T_n^{(C)}(x,y)$ and sine-tangent polynomials $T_n^{(S)}(x,y)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}\cos yt, \tag{13}$$

and

$$\sum_{n=0}^{\infty} T_n^{(S)}(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}\sin yt, \tag{14}$$

respectively.

Note that $T_n^{(C)}(x,0) = T_n(x), T_n^{(S)}(x,0) = 0, (n \geq 0)$.

By (11)-(14), we have

$$T_n^{(C)}(x,y) = \frac{T_n(x+iy)+T_n(x-iy)}{2},$$
$$T_n^{(S)}(x,y) = \frac{T_n(x+iy)-T_n(x-iy)}{2i}.$$

Clearly, we obtain the following explicit representations of $T_n(x+iy)$

$$T_n(x+iy) = \sum_{l=0}^{n}\binom{n}{l}T_l(x+iy)^{n-l},$$
$$T_n(x+iy) = \sum_{l=0}^{n}\binom{n}{l}T_l(x)(iy)^{n-l}.$$

Let

$$e^{xt}\cos yt=\sum_{l=0}^{\infty}C_l(x,y)\frac{t^l}{l!},\qquad e^{xt}\sin yt=\sum_{l=0}^{\infty}S_l(x,y)\frac{t^l}{l!}. \tag{15}$$

Then, by Taylor expansions of $e^{xt}\cos yt$ and $e^{xt}\sin yt$, we get

$$e^{xt}\cos yt=\sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m x^{l-2m}y^{2m}\right)\frac{t^l}{l!} \tag{16}$$

and

$$e^{xt}\sin yt=\sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l-1}{2}]}\binom{l}{2m+1}(-1)^m x^{l-2m-1}y^{2m+1}\right)\frac{t^l}{l!}, \tag{17}$$

where '[ ]' denotes taking the integer part (see [4]).

By (15), (16) and (17), we get

$$C_l(x,y)=\sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m x^{l-2m}y^{2m},$$

and

$$S_l(x,y)=\sum_{m=0}^{[\frac{l-1}{2}]}\binom{l}{2m+1}(-1)^m x^{l-2m-1}y^{2m+1}, (l\geq 0).$$

Now, we observe that

$$\begin{aligned}\frac{2}{e^{2t}+1}e^{xt}\cos yt&=\left(\sum_{l=0}^{\infty}T_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}C_m(x,y)\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_lC_{n-l}(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we obtain the following theorem (see [5]).

Theorem 2.2. *For $n\geq 0$, we have*

$$T_n^{(C)}(x,y)=\sum_{l=0}^{n}\binom{n}{l}T_lC_{n-l}(x,y)$$

and

$$T_n^{(S)}(x,y)=\sum_{l=0}^{n}\binom{n}{l}T_lS_{n-l}(x,y).$$

From (13), we have

$$\begin{aligned}2e^{xt}\cos yt &= \left(\sum_{l=0}^{\infty} T_l^{(C)}(x,y)\frac{t^l}{l!}\right)\left(e^{2t}+1\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_l^{(C)}(x,y)2^{n-l}+T_n^{(C)}(x,y)\right)\frac{t^n}{n!}.\end{aligned} \tag{18}$$

By (15) and (18), we get

$$C_n(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}T_l^{(C)}(x,y)2^{n-l}+T_n^{(C)}(x,y)\right).$$

Therefore, we obtain the following theorem(see [5]).

Theorem 2.3. *For $n \geq 0$, we have*

$$C_n(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}T_l^{(C)}(x,y)2^{n-l}+T_n^{(C)}(x,y)\right),$$

and

$$S_n(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}T_l^{(S)}(x,y)2^{n-l}+T_n^{(S)}(x,y)\right).$$

From (15), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} T_n^{(C)}(2-x,y)\frac{t^n}{n!} &= \frac{2}{e^{2t}+1}e^{(2-x)t}\cos yt\\ &= \frac{2}{e^{-2t}+1}e^{-xt}\cos(-yt)\\ &= \left(\sum_{m=0}^{\infty}(-1)^m T_m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}(-1)^m C_{m,}(x,y)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left((-1)^n\sum_{l=0}^{n}\binom{n}{l}T_l C_{n-l}(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we have the following theorem (see [5]).

Theorem 2.4. *For $n \geq 0$, we have*

$$\begin{aligned}T_n^{(C)}(2-x,y) &= (-1)^n T_n^{(C)}(x,y)\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}T_l C_{n-l}(x,y),\end{aligned}$$

and

$$\begin{aligned}T_n^{(S)}(2-x,y) &= (-1)^{n+1} T_n^{(S)}(x,y)\\ &= (-1)^{n+1}\sum_{l=0}^{n}\binom{n}{l}T_l S_{n-l}(x,y).\end{aligned}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} T_n^{(C)}(x+2,y)\frac{t^n}{n!} &= \frac{2}{e^{2t}+1}e^{(x+2)t}\cos yt \\ &= \frac{2}{e^{2t}+1}e^{xt}(e^{2t}-1+1)\cos yt \\ &= 2e^{xt}\cos yt - \frac{2}{e^{2t}+1}e^{xt}\cos yt\end{aligned}$$

Hence we have

$$\sum_{n=0}^{\infty}\left(T_n^{(C)}(x+2,y)+T_n^{(C)}(x,y)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(2C_n(x,y)\right)\frac{t^n}{n!}.$$

By comparing the coefficients on the both sides, we get the following theorem (see [5]).

Theorem 2.5. *For $n \geq 1$, we have*

$$T_n^{(C)}(x+2,y)+T_n^{(C)}(x,y) = 2C_n(x,y),$$

and

$$T_n^{(S)}(x+2,y)+T_n^{(S)}(x,y) = 2S_n(x,y).$$

From (15), we have

$$\sum_{m=0}^{\infty} C_m(0,y)\frac{t^m}{m!} = \sum_{m=0}^{\infty}(-1)^m y^{2m}\frac{t^{2m}}{(2m)!}. \tag{19}$$

Therefore, by Theorem 2.5 and (19), we obtain the following corollary.

Corollary 2.6. *For $n \geq 0$, we have*

$$T_{2n}^{(C)}(2,y)+T_{2n}^{(C)}(0,y) = 2(-1)^n y^{2n},$$

and

$$T_{2n+1}^{(S)}(2,y)+T_{2n+1}^{(S)}(0,y) = 2(-1)^n y^{2n+1}.$$

By (15), we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_n^{(C)}(x+r,y)\frac{t^n}{n!} &= \left(\frac{2}{e^{2t}+1}e^{xt}\cos yt\right)e^{rt} \\ &= \left(\sum_{l=0}^{\infty} T_l^{(C)}(x,y)\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty} r^k\frac{t^k}{k!}\right) \qquad (20) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}T_k^{(C)}(x,y)r^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.7. *For $n \geq 0, r \in \mathbb{N}$, we have*

$$T_n^{(C)}(x+r,y) = \sum_{k=0}^{n} \binom{n}{k} T_k^{(C)}(x,y) r^{n-k},$$

and

$$T_n^{(S)}(x+r,y) = \sum_{k=0}^{n} \binom{n}{k} T_k^{(S)}(x,y) r^{n-k}.$$

Taking $r = 2$ in Theorem 2.7, by Theorem 2.5, we obtain the following corollary.

Corollary 2.8. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} T_k^{(C)}(x,y) 2^{n-k} = 2C_n(x,y) - T_n^{(C)}(x,y),$$

and

$$\sum_{k=0}^{n} \binom{n}{k} T_k^{(S)}(x,y) 2^{n-k} = 2S_n(x,y) - T_n^{(S)}(x,y).$$

From Corollary 2.8, we note that

$$T_n^{(C)}(0,y) + \sum_{k=0}^{n} \binom{n}{k} T_k^{(C)}(0,y) = \begin{cases} 0 & \text{if } n = 2m+1 \\ 2(-1)^m y^{2m} & \text{if } n = 2m, \end{cases}$$

and

$$T_n^{(S)}(0,y) + \sum_{k=0}^{n} \binom{n}{k} T_k^{(S)}(0,y) = \begin{cases} 2(-1)^m y^{2m+1} & \text{if } n = 2m+1 \\ 0 & \text{if } n = 2m. \end{cases}$$

By (13), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial x} T_n^{(C)}(x,y) \frac{t^n}{n!} &= \frac{\partial}{\partial x} \left(\frac{2}{e^{2t}+1} e^{xt} \cos yt \right) \\ &= \frac{2t}{e^{2t}+1} e^{xt} \cos yt \\ &= \sum_{n=1}^{\infty} \left(n T_{n-1}^{(C)}(x,y) \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

Comparing the coefficients on the both sides of (21), we have

$$\frac{\partial}{\partial x} T_n^{(C)}(x,y) = n T_{n-1}^{(C)}(x,y).$$

Similarly, for $n \geq 1$, we have

$$\frac{\partial}{\partial x} T_n^{(S)}(x,y) = nT_{n-1}^{(S)}(x,y),$$
$$\frac{\partial}{\partial y} T_n^{(C)}(x,y) = -nT_{n-1}^{(S)}(x,y),$$
$$\frac{\partial}{\partial y} T_n^{(S)}(x,y) = nT_{n-1}^{(C)}(x,y).$$

Now, we define the new type polynomials that are given by the generating functions to be

$$\frac{2}{e^{2t}+1}\cos yt = \sum_{n=0}^{\infty} T_n^{(C)}(y)\frac{t^n}{n!}, \tag{22}$$

and

$$\frac{2}{e^{2t}+1}\sin yt = \sum_{n=0}^{\infty} T_n^{(S)}(y)\frac{t^n}{n!}, \tag{23}$$

respectively.

Note that $T_n^{(C)}(0) = T_n$, $T_n^{(S)}(0) = 0$. The new type polynomials can be determined explicitly. A few of them are

$$T_0^{(C)}(y) = 1, \quad T_1^{(C)}(x,y) = -1,$$
$$T_2^{(C)}(x,y) = -y^2, \quad T_3^{(C)}(y) = 2 + 3y^2,$$
$$T_4^{(C)}(y) = y^4, \quad T_5^{(C)}(y) = -16 - 20y^2 - 5y^4,$$
$$T_6^{(C)}(y) = -y^6,$$

and

$$T_0^{(S)}(x,y) = 0, \quad T_1^{(S)}(x,y) = y,$$
$$T_2^{(S)}(x,y) = -2y, \quad T_3^{(S)}(x,y) = -y^3,$$
$$T_4^{(S)}(x,y) = 8y + 4y^3, \quad T_5^{(S)}(x,y) = y^5,$$
$$T_6^{(S)}(x,y) = -96y - 40y^3 - 6y^5.$$

From (22) and (23), we derive the following equations:

$$\frac{2}{e^{2t}+1}\cos yt = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}\binom{k}{2m}(-1)^m T_{k-2m}y^{2m}\right)\frac{t^k}{k!} \tag{24}$$

and

$$\frac{2}{e^{2t}+1}\sin yt = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m T_{k-2m-1}y^{2m+1}\right)\frac{t^k}{k!}. \tag{25}$$

By (22), (23), (24) and (25), we get

$$T_n^{(C)}(y)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m y^{2m}T_{n-2m},$$

and

$$T_n^{(S)}(y)=\sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}(-1)^m y^{2m+1}T_{n-2m-1}.$$

From (13), (14), (22) and (23), we derive the following theorem:

Theorem 2.9. *For $n\geq 0$, we have*

$$T_n^{(C)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}T_k^{(C)}(y)x^{n-k},$$

and

$$T_n^{(S)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(S)}(y)x^{n-k}.$$

We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and of the second kind $S_2(n,k)$ are defined by the relations(see [10])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k \text{ and } x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k, \tag{26}$$

respectively. Here, $(x)_n=x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$(e^t-1)^m=m!\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}. \tag{27}$$

By (13), (27) and by using Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty}T_n^{(C)}(x,y)\frac{t^n}{n!}&=\left(\frac{2}{e^{2t}+1}\right)(1-(1-e^{-t}))^{-x}\cos yt\\&=\left(\frac{2}{e^{2t}+1}\right)\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-e^{-t})^l\cos yt\\&=\sum_{l=0}^{\infty}<x>_l\frac{(e^t-1)^l}{l!}\left(\frac{2}{e^{2t}+1}\right)e^{-lt}\cos yt\\&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_n^{(C)}(-l,y)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(C)}(-l,y)<x>_l\right)\frac{t^n}{n!},\end{aligned} \tag{28}$$

where $< x >_l = x(x+1)\cdots(x+l-1)(l \geq 1)$ denotes the rising factorial polynomial of order l and $< x >_0 = 1$. By comparing the coefficients on both sides of (28), we have the following theorem:

Theorem 2.10. *For $n \in \mathbb{Z}_+$, we have*

$$T_n^{(C)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(C)}(-l,y) < x >_l,$$
$$T_n^{(S)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(S)}(-l,y) < x >_l .$$

By (13), (14), (26), (27) and by using Cauchy product, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} &= \left(\frac{2}{e^{2t}+1}\right)((e^t-1)+1)^x \cos yt \\
&= \frac{2}{e^{2t}+1}\cos yt \sum_{l=0}^{\infty}\binom{x}{l}(e^t-1)^l \\
&= \sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!}\left(\frac{2}{e^{2t}+1}\cos yt\right) \\
&= \sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}T_n^{(C)}(y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)T_{n-i}^{(C)}(y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{29}$$

By comparing the coefficients on both sides of (29), we have the following theorem:

Theorem 2.11. *For $n \geq 0$, we have*

$$T_n^{(C)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)T_{n-i}^{(C)}(y),$$
$$T_n^{(S)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)T_{n-i}^{(S)}(y).$$

By (3), (13), (22) and by using Cauchy product, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} = \left(\frac{2}{e^{2t}+1}\right) e^{xt}\cos yt \\
&= \frac{(e^t-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt}\sum_{n=0}^{\infty} T_n^{(C)}(y)\frac{t^n}{n!} \\
&= \frac{(e^t-1)^r}{r!}\left(\sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n^{(C)}(y)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i} B_i^{(r)}(x) T_{n-l-i}^{(C)}(y)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 2.12. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned}
T_n^{(C)}(x,y) &= \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i} B_i^{(r)}(x) T_{n-l-i}^{(C)}(y), \\
T_n^{(S)}(x,y) &= \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i} B_i^{(r)}(x) T_{n-l-i}^{(S)}(y).
\end{aligned}$$

By (4), (13) and by using the Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} = \left(\frac{2}{e^{2t}+1}\right) e^{xt}\cos yt \\
&= \frac{(e^t-u)^r}{(1-u)^r}\left(\frac{1-u}{e^t-u}\right)^r e^{xt}\left(\frac{2}{e^{2t}+1}\right)\cos yt \\
&= \sum_{n=0}^{\infty} H_n^{(r)}(u,x)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i} e^{it}(-u)^{r-i}\frac{1}{(1-u)^r}\left(\frac{2}{e^{2t}+1}\right)\cos yt \\
&= \frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty} H_n^{(r)}(u,x)\frac{t^n}{n!}\sum_{n=0}^{\infty} T_n^{(C)}(i,y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l} H_l^{(r)}(u,x) T_{n-l}^{(C)}(i,y)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 2.13. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$T_n^{(C)}(x,y) = \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} T_{n-l}^{(C)}(i,y) H_l^{(r)}(u,x),$$

$$T_n^{(S)}(x,y) = \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} T_{n-l}^{(S)}(i,y) H_l^{(r)}(u,x).$$

By Theorem 2.11, Theorem 2.12 and Therem 2.13, we have the following corollary.

Corollary 2.14. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{i=l}^{n} \binom{n}{i} (x)_l S_2(i,l) T_{n-i}^{(C)}(y) \\
&= \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} H_l^{(r)}(u,x) T_{n-l}^{(C)}(i,y) \\
&= \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} T_{n-l-i}^{(C)}(y) B_i^{(r)}(x).
\end{aligned}$$

3. Distribution of zeros of the cosine-tangent and sine-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the cosine-tangent polynomials $T_n^{(C)}(x,y)$ and sine-tangent polynomials $T_n^{(S)}(x,y)$. The sine-tangent polynomials $T_n^{(S)}(x,y)$ can be determined explicitly. A few of them are

$$\begin{aligned}
T_0^{(S)}(x,y) &= 0, \\
T_1^{(S)}(x,y) &= y, \\
T_2^{(S)}(x,y) &= -2y + 2xy \\
T_3^{(S)}(x,y) &= -6xy + 3x^2y - y^3, \\
T_4^{(S)}(x,y) &= 8y - 12x^2y + 4x^3y + 4y^3 - 4xy^3, \\
T_5^{(S)}(x,y) &= 40xy - 20x^3y + 5x^4y + 20xy^3 - 10x^2y^3 + y^5, \\
T_6^{(S)}(x,y) &= -96y + 120x^2y - 30x^4y + 6x^5y - 40y^3 + 60x^2y^3 - 20x^3y^3 - 6y^5 \\
&\quad + 6xy^5.
\end{aligned}$$

We investigate the beautiful zeros of the sine-tangent polynomials $T_n^{(S)}(x,y)$ by using a computer. We plot the zeros of the poly-tangent polynomials $T_n^{(S)}(x,y)$ for $n = 50, x = 2,4,6,8$ and $y \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose

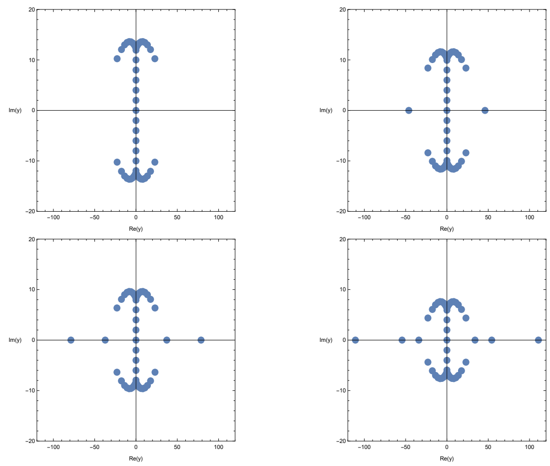

FIGURE 1. Zeros of $T_n^{(S)}(x,y)$

$n = 50$ and $x = 2$. In Figure 1(top-right), we choose $n = 50$ and $x = 4$. In Figure 1(bottom-left), we choose $n = 50$ and $x = 6$. In Figure 1(bottom-right), we choose $n = 50$ and $x = 8$.

Stacks of zeros of $T_n^{(S)}(x,y)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 2). In Figure 2(top-left), we choose $x = 2$. In Figure 2(top-right),

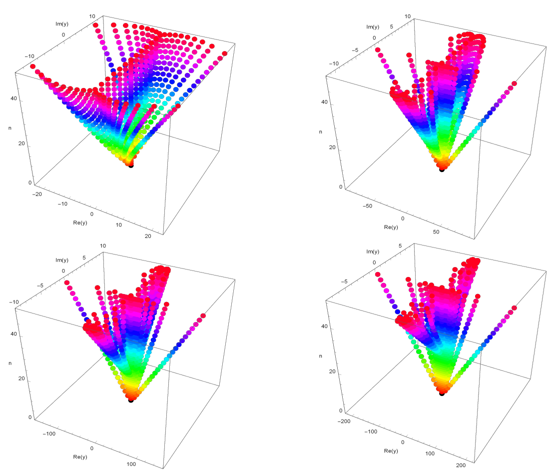

FIGURE 2. Stacks of zeros of $T_n^{(S)}(x,y)$ for $1 \leq n \leq 50$

we choose $x = 4$. In Figure 2(bottom-left), we choose $x = 6$. In Figure 2(bottom-right), we choose $x = 8$.

The plot of real zeros of $T_n^{(S)}(x,y)$ for $1 \leq n \leq 50$ structure are presented(Figure 3). In Figure 3(top-left), we choose $x = 2$. In Figure 3(top-right), we choose

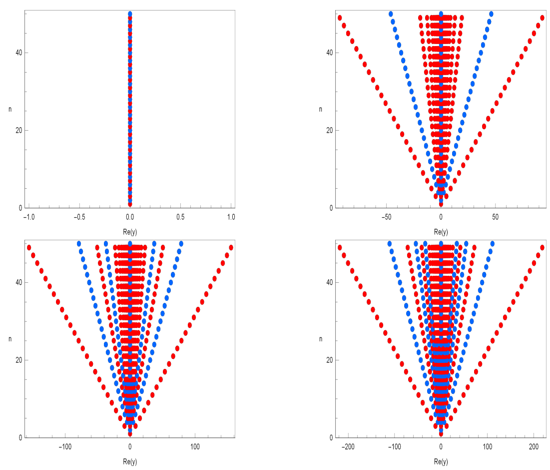

FIGURE 3. Stacks of zeros of $T_n^{(S)}(x,y)$ for $1 \leq n \leq 50$

$x = 4$. In Figure 3(bottom-left), we choose $x = 6$. In Figure 3(bottom-right), we choose $x = 8$.

Next, we calculated an approximate solution satisfying sine-tangent polynomials $T_n^{(S)}(x,y) = 0$ for $y \in \mathbb{R}$. The results are given in Table 1.

Table 1. Approximate solutions of $T_n^{(S)}(2,y) = 0$

degree n	y
1	0
2	0
3	-4.8990 0, 4.8990
4	-2.4495, 0, 2.4495
5	-8.8288, -1.4327, 0, 1.4327, 8.8288
6	-4.3778, -0.91370, 0, 0.91370, 4.3778
7	-12.692, -2.4474, -0.96366, 0, 0.96366, 2.4474, 12.692
8	-6.2659, 0, 6.2659

The cosine-tangent polynomials $T_n^{(C)}(x,y)$ can be determined explicitly. A few of them are

$$T_0^{(C)}(x,y) = 1, \quad T_1^{(C)}(x,y) = -1 + x, \quad T_2^{(C)}(x,y) = -2x + x^2 - y^2$$
$$T_3^{(C)}(x,y) = 2 - 3x^2 + x^3 + 3y^2 - 3xy^2,$$
$$T_4^{(C)}(x,y) = 8x - 4x^3 + x^4 + 12xy^2 - 6x^2y^2 + y^4,$$
$$T_5^{(C)}(x,y) = -16 + 20x^2 - 5x^4 + x^5 - 20y^2 + 30x^2y^2 - 10x^3y^2 - 5y^4 + 5xy^4,$$
$$T_6^{(C)}(x,y) = -96x + 40x^3 - 6x^5 + x^6 - 120xy^2 + 60x^3y^2 - 15x^4y^2 - 30xy^4 + 15x^2y^4 - y^6,$$
$$T_7^{(C)}(x,y) = 272 - 336x^2 + 70x^4 - 7x^6 + x^7 + 336y^2 - 420x^2y^2 + 105x^4y^2 - 21x^5y^2 + 70y^4 - 105x^2y^4 + 35x^3y^4 + 7y^6 - 7xy^6.$$

We investigate the beautiful zeros of the cosine-tangent polynomials $T_n^{(C)}(x,y)$ by using a computer. We plot the zeros of the poly-tangent polynomials $T_n^{(C)}(x,y)$ for $n = 50, y = 2, 4, 6, 8$ and $x \in \mathbb{C}$(Figure 4). In Figure 4(top-left), we choose

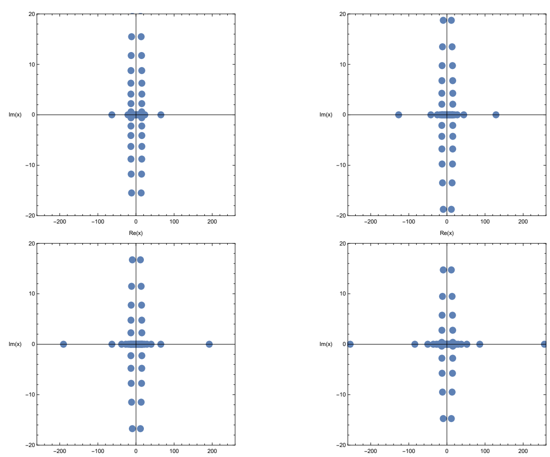

FIGURE 4. Zeros of $T_n^{(S)}(x,y)$

$n = 50$ and $y = 2$. In Figure 4(top-right), we choose $n = 50$ and $y = 4$. In Figure 4(bottom-left), we choose $n = 50$ and $y = 6$. In Figure 4(bottom-right), we choose $n = 50$ and $y = 8$.

Stacks of zeros of $T_n^{(C)}(x,y)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 5). In Figure 5(top-left), we choose $y = 2$. In Figure 5(top-right), we choose $y = 4$. In Figure 5(bottom-left), we choose $y = 6$. In Figure 5(bottom-right), we choose $y = 8$.

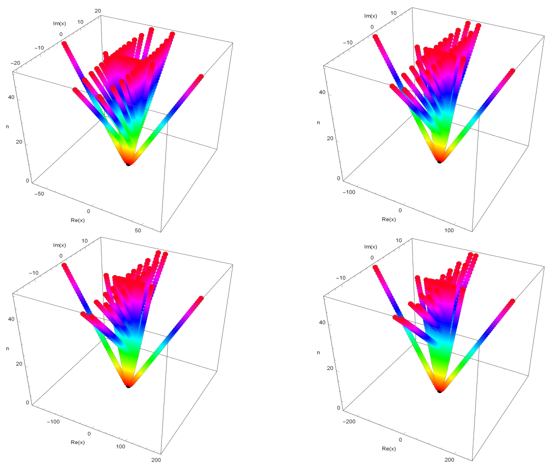

FIGURE 5. Stacks of zeros of $T_n^{(C)}(x, y)$ for $1 \leq n \leq 50$

The plot of real zeros of $T_n^{(C)}(x, y)$ for $1 \leq n \leq 50$ structure are presented(Figure 6). In Figure 6(top-left), we choose $y = 2$. In Figure 6(top-

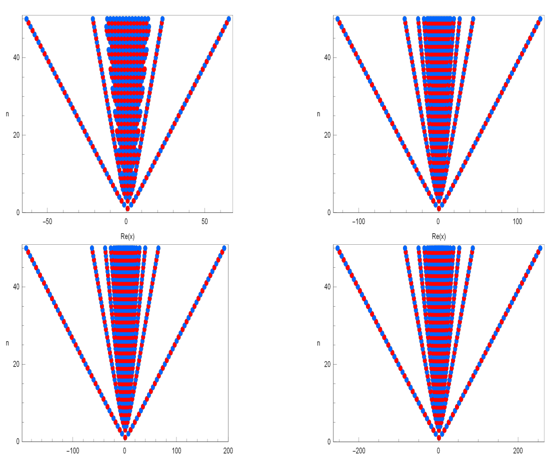

FIGURE 6. Stacks of zeros of $T_n^{(C)}(x, y)$ for $1 \leq n \leq 50$

right), we choose $y = 4$. In Figure 6(bottom-left), we choose $y = 6$. In Figure 6(bottom-right), we choose $y = 8$.

Next, we calculated an approximate solution satisfying cosine-tangent polynomials $T_n^{(C)}(x,y)=0$ for $y=2, x\in\mathbb{R}$.

Table 2. Approximate solutions of $T_n^{(C)}(x,2)=0$

degree n	x
1	1.0000
2	-1.2361, 3.2361
3	-2.8730, 1.0000, 4.8730
4	-4.3307, -0.25841, 2.2584, 6.3307
5	-5.7082, -1.2361, 1.0000, 3.2361, 7.7082
6	-7.0456, -2.0242, -0.059850, 2.0598, 4.0242, 9.0456

Acknowledgements : The graph of the distribution of roots in this paper was assisted by Professor Ryoo. We would like to express our gratitude to Professor Ryoo for his help.

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