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EXISTENCE AND CONTROLLABILITY OF IMPULSIVE FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATION WITH STATE DEPENDENT INFINITE DELAY VIA SECTORIAL OPERATOR

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Abstract. In the article, we handle with the existence and controllability results for fractional impulsive neutral functional integro-differential equation in Banach spaces. We have used advanced phase space definition for infinite delay. State dependent infinite delay is the main motivation using advanced version of phase space. The results are acquired using Schaefer's fixed point theorem. Examples are given to illustrate the theory.

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1. Introduction

The concept of semigroups of bounded linear operators is precisely associated to solving differential and integro-differential equations in Banach spaces. A little while back, this strategy has been utilized to a substantial type of nonlinear differential equations in Banach spaces. For more points of interest on this concept, we refer Pazy [32]. The principle of fractional differential equation (FDE) have selected up extensive vitality because of their use in numerous sciences, including physical science, mechanics and engineering [8, 21]. The notion of fractional derivatives, as is long familiar, has its commencement in an inquiry postured amid a correspondence in the middle of Leibnitz and L'Hö̈spital. The five millennium extremely ancient inquiry has turned into a significant zone of exploration. As of late, it has been demonstrated that the differential designs including derivatives of fractional order emerge in numerous technological innovations and scientific disciplines as the statistical modeling of frameworks and

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procedures in numerous fields like: physical science, chemical industry, aerodynamics of complex medium, etc. For information, such as some uses and latest outcomes, think about the treatise of Miller and Ross [29], Abbas et al. [1], Baleanu et al. [9], Podlubny [33], Diethelm [18], Kilbas et al. [26], Tarasov [47], and the papers [15, 3, 4, 6, 7, 25, 38, 39, 42, 43, 16], and the references cited therein.

The concept of impulsive differential framework has been a target consideration due to the fact of its extensive uses in physics, biology, engineering, medicinal fields, industry and technology. Impulsive differential equations are appropriate models for describing the real processes that deviate their states rapidly at certain moments and cannot be described by using the classical differential equations [14, 35, 37, 40, 41]. Integro-differential equations arise in the mathematical modeling of several neutral phenomena and various investigations led to the exploration of their different aspects, refer [30, 12].

Fractional differential equations (FDE) are recognized as an indispensable tool to compile the dynamical behavior of real-life phenomena in a accurate manner and can be described very successfully by models using mathematical tools from the fractional calculus. Occurrence in purpose, the nonlinear oscillation of earthquake can be effectively displayed with fractional derivatives. This includes fluid flow, rheology, dynamical processes in self-similar and porous structure, dielectric polarization, electrode-electrolyte polarization, electromagnetic wave, electrical networks, traffic model with fractional derivative, control theory of dynamical systems and so on. Many problems in engineering systems can be resolved by incorporating fractional calculus (pl. refer to ([50, 6, 7, 17])). Fractional equation with delay properties arise in several fields such as biological and physical with state dependent delay (SDD) or non-constant delay. Nowadays, existence results of mild solutions for such problems became very attractive and several researchers are working on it. Recently, several papers have been written on the fractional order problems with state dependent delay (SDD) [14] and the sources therein. Currently, in existence and controllability of mild solutions for such problems became very attractive.

State dependent delays are several places in application, such as 3D printing and oil drilling. The formulation of the problem working with a control of nonlinear systems with state dependent delay on the input can be studied by designing “nonlinear predictor feedback” law that compensates the input delay. In [11], the authors introduced the concept of nonlinear predictor feedback starting from nonlinear systems with constant delays all the way through to predictor feedback for nonlinear systems with state dependent delay.

The existence, controllability, and other qualitative and quantitative attributes of differential and fractional differential equations (FDE) are the most advancing area of interest (for instance, see [50, 7]). Recently, several authors investigated the different types of impulsive fractional differential systems in Banach space under different fixed theorems with weak conditions.Controllability plays a significant role in the evolution of modern mathematical control theory.

This is a qualitative property of dynamical control systems and is of appropriate significance in Control theory. It has many significant applications not only in control theory and systems theory, but also in such fields as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospace engineering and recently in quantum systems theory. To have effective illustration one can refer to [48, 27, 45, 34, 36, 5].

However, existence results for impulsive fractional neutral integro differential equations (IFNIDE) with state dependent delay (SDD) in phase space ($\mathcal{B}_h$) adages have not yet been completely examined. In addition, Kailasavalli et al. [28] acknowledged the existence and controllability of fractional neutral integro-differential systems with state dependent delay, Banach Contraction and resolvent operator technique as the main reference. Dabas et al. [19] studied the existence, uniqueness and continuous dependence of mild solution for an impulsive neutral fractional order differential equation with infinite time delay, but not the state delay.

Motivated by the above works, we show that a particular class of impulsive fractional neutral integrodifferential equation in Banach space is controllable provided that some sufficient conditions are satisfied. The system considered here is untreated in the literature, which is a main motivation of the current work.

Inspired by the effort of the above stated papers, the primary inspiration driving this manuscript is to research the existence of mild solution for an IFNIDE with SDD of the model

$$ {}^{C}D_t^q\Big[x(t)-\mathscr{G}(t,x_{\varrho(t,x_t)})\Big]=\mathcal{A}x(t)+\mathscr{F}\Big[t,x_{\varrho(t,x_t)},\int_0^t \mathfrak{W}(t,s,x_{\varrho(s,x_s)})ds\Big],\quad t\in\mathscr{I}; \tag{1.1} $$

$$ \Delta x|_{t=t_k}=\mathcal{I}_k(x(t_k));\quad k=1,2,...m; \tag{1.2} $$

$$ x_0=\phi\in\mathcal{B}_h,\quad t\in(-\infty,0]; \tag{1.3} $$

where ${}^{C}D_t^q$ denote the Caputo fractional derivative of order $0<q<1$, $t\in\mathscr{I}=[0,T]$ with the lower limit zero, $\mathcal{A}$ is a fractional sectorial (unbounded) operator on a Banach space $\mathbb{X}$, having its norm recognized as $\|.\|_{\mathbb{X}}$, $\mathscr{G}:\mathscr{I}\times\mathcal{B}_h\to\mathbb{X}$, $\mathscr{F}:\mathscr{I}\times\mathcal{B}_h\times\mathbb{X}\to\mathbb{X}$, $\mathfrak{W}:\mathscr{D}\times\mathcal{B}_h\to\mathbb{X}$, $\varrho:\mathscr{I}\times\mathcal{B}_h\to(-\infty,T]$ are appropriate operators. $\mathcal{B}_h$ is a theoretical phase space adages outlined in preliminaries. Here, $\mathscr{D}=\{(t,s)\in\mathscr{I}\times\mathscr{I}:0\le s\le t\le T\}$. Here, $0=t_0<t_1<t_2<...<t_m<t_{m+1}=T$, $\mathcal{I}_k:\mathbb{X}\to\mathbb{X}$ $(k=1,2,\dots,m)$ are impulsive functions which portray the jump of the solutions at impulse points t_k, $\Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-)$ with $x(t_k^+)$, $x(t_k^-)$ representing the right and left limits of x at the points t_k, respectively. Consider the space

$$ PC:=\big\{x:(-\infty,T]\to\mathbb{X}\ such\ that\ x(t_k^+)\ and\ x(t_k^-)\ exist\ and\ x(t_k)=x(t_k^-) \\ x(t)=\phi(t)\ for\ t\in(-\infty,T],\ x_k\in C(\mathcal{I}_k,\mathbb{X}),\ k=1,2,\dots,m\big\} $$

.

For almost any continuous function x characterized on $(-\infty, T]$ and any $t \geq 0$, we designate by x_t the part of $\mathcal{B}_h$ characterized by $x_t(\theta) = x(t+\theta)$ for $\theta \leq 0$. Now, $x_t(.)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$, likely the current time t.

The purpose of this paper is to analyze this fascinating model (1.1)-(1.3). In Section 2, we recollect some definitions, theorems and notations. In Section 3, the existence results of mild solutions for the model (1.1)-(1.3) is discussed under a suitable fixed point theorem. In Section 4, the controllability results of mild solutions for the model (4.1)-(4.3) is discussed under the suitable conditions. As a final part of the article we illustrate a couple of theoretical results.

2. Preliminaries

In this section, we recall some basic definition, notations and lemmas that are used throughout this paper.

Let $\mathcal{L}(\mathbb{X})$ symbolize the Banach space of all bounded linear operator from $\mathbb{X}$ into $\mathbb{X}$, having its norm recognized as $\|.\|_{\mathcal{L}(\mathbb{X})}$.

Let $C(\mathscr{I}, \mathbb{X})$ symbolize the space of all continuous functions from $\mathscr{I}$ into $\mathbb{X}$, having the norm recognized as $\|.\|_{C(\mathscr{I},\mathbb{X})}$.

We assume that $\mathcal{A} : \mathscr{D}(\mathcal{A}) \to \mathbb{X}$ be the infinitesimal generator of an analytic semigroup $\{\mathcal{R}_q(t)\}_{t\geq 0}$. Without loss of simplification, we expect that $0 \in \rho(\mathcal{A})$. Then it is possible to determine the fractional power $\mathcal{A}^\alpha$ for $0 < \alpha \leq 1$, as closed linear operator on its domain $\mathscr{D}(\mathcal{A}^\alpha)$ being dense in $\mathbb{X}$. For $0 < \beta \leq \alpha \leq 1$, $\mathbb{X}_\alpha \to \mathbb{X}_\beta$ and the imbedding is compact whenever the resolvent operator of $\mathcal{A}$ is compact. Also for every $0 < \alpha \leq 1$, there exists $\mathcal{M}_\alpha > 0$ such that

$$\|\mathcal{A}^\alpha \mathcal{R}_q(t)\| \leq \frac{\mathcal{M}_\alpha}{t^\alpha}.$$

With this discussion, we recall fundamental properties of fractional power $\mathcal{A}^\alpha$ from Pazy [32].

It needs to be outlined that, once the delay is infinite, we need to talk about the theoretical phase space $\mathcal{B}_h$ in a beneficial way. In this manuscript, we deliberate phase space $\mathcal{B}_h$ which are same as described in [19]. So, we bypass the details.

Now we define the abstract phase space $\mathcal{B}_h$. Assume that $\mathfrak{W} : (-\infty, 0] \to (0, \infty)$ be a continuous function with $l = \int_{-\infty}^{0} \mathfrak{W}(s)ds < \infty$. For any $a > 0$, we define,

$$\mathcal{B} = \Big\{\phi : [-a, 0] \to \mathbb{X} \ \text{ such that } \phi(l) \text{ is bounded and measurable}\Big\},$$

and equip the space $\mathcal{B}$ with the norm

$$\big\|\phi\big\|_{[-a,0]} = \sup\nolimits_{\mathrm{s}\in[-\mathrm{a},0]} \Big\|\phi(s)\Big\|, \ \forall \ \phi \in \mathcal{B}.$$

Let us define

$$\mathcal{B}_h = \Big\{\phi : (-\infty, 0] \to \mathbb{X} \ \text{ such that, for any } c > 0,\ \phi|_{[-c,0]} \in \mathcal{B}$$

$$\text{and } \int_{-\infty}^{0} \mathfrak{W}(s)\ \|\phi\|_{[s,o]} ds < \infty \Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} \mathfrak{W}(s)\ \|\phi\|_{[s,0]} ds,\ \forall\ \phi \in \mathcal{B}_h,$$

then it is clear that $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.
Now we consider the space

$$\mathcal{B}'_h = \Big\{ x : (-\infty, T] \to \mathbb{X} \text{ such that } x|_{\mathscr{I}} \in C(\mathscr{I}, \mathbb{X}),\ x_0 = \phi \in \mathcal{B}_h \Big\}.$$

Set $\|.\|_b$ be a seminorm in $\mathcal{B}'_h$ defined by

$$\|x\|_T = \|\phi\|_{\mathcal{B}_h} + \sup\{\|x(s)\| : s \in [0,T]\},\ x \in \mathcal{B}'_h.$$

We assume that the phase space $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and fulfilling the following elementary axioms as a result of Hino et al. (see [23]).

(A) If $x : (\infty, T] \to \mathbb{X},\ T > 0$ is continuous on $\mathscr{I}$ and $x_0 \in \mathcal{B}_h$, then for every $t \in \mathscr{I}$, the following conditions hold:

- (P_1) x_t is in $\mathcal{B}_h$;
- (P_2) $\|x(t)\|_{\mathbb{X}} \le H\|x_t\|_{\mathcal{B}_h}$;
- (P_3) $\|x_t\|_{\mathcal{B}_h} \le \mathscr{D}_1(t)\sup\{\|x(s)\|_{\mathbb{X}} : 0 \le s \le t\} + \mathscr{D}_2(t)\|x_0\|_{\mathcal{B}_h}$, where $H > 0$ is a constant and $\mathscr{D}_1(.) : [0,\infty) \to [0,\infty)$ is continuous, $\mathscr{D}_2(.) : [0,\infty) \to [0,\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $x(.)$.
- (P_4) The function $t \to \phi_t$ is well described and continuous from the set:

  $$\mathcal{R}(\varrho^-) = \{\varrho(s,\psi) : (s,\psi) \in \mathscr{I} \times \mathcal{B}_h\}$$

  into $\mathcal{B}_h$, and there is a continuous and bounded function $\mathscr{I}^{\phi} : \mathcal{R}(\varrho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}_h} \le \mathscr{I}^{\phi(t)}\|\phi\|_{\mathcal{B}_h}$ for every $t \in \mathcal{R}(\varrho^-)$.
- $(A1)$ For the function $x(.)$ in (A), x_t is a $\mathcal{B}_h$-valued continuous function on $[0,T)$.
- $(A2)$ The space $\mathcal{B}_h$ is complete.

Here, we consider some examples of phase spaces.

Example 1. *(The phase space $C_r \times L^p(g, \mathbb{X})$) Let $r \ge 0, 1 \le p < \infty$ and let $g : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the following conditions:*

$(g-5)$

$$\int_{\chi}^{0} g(\theta)d\theta < \infty,\ forall\ \chi \in (-\infty, 0)$$

$(g-6)$ *There is a nonnegative function G, which is a locally bounded in $(-\infty, 0]$ such that $g(\chi+\theta) \le G(\chi)g(\theta),\ \forall\ \chi \le 0$ and $\theta \in (-\infty, -r) \setminus N_{\chi}$, where $N_{\chi} \subset (-\infty, -r)$ with Lebesgue measure zero.*

The space $\mathcal{B}_h = C_r \times L^p(g, \mathbb{X})$ consists of all classes of Lebesgue-measurable functions $\phi : (-\infty, 0] \to \mathbb{X}$ such that ϕ is continuous on $[-r, 0]$ and $g\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in this space is defined by

$$\|\phi\|_{\mathcal{B}_h} := \sup\Big\{\|\phi(\theta)\| : -r \le \theta \le 0\Big\} + \bigg(\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p d\theta\bigg)^{\frac{1}{p}}.$$

Proceeding as in the proof of [[23], Theorem 1.3.8], it follows that $\mathcal{B}_h$ is a space that satisfies axioms $(A), (A1), (B)$. Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $\mathscr{D}_1(t) = G(-t)^{\frac{1}{2}}$ and $\mathscr{D}_2(t) = 1 + \Big(\int_{-t}^{0} g(\theta)d\theta\Big)^{\frac{1}{2}}$ for $t \ge 1$.

For additional details concerning phase space, we refer the reader to [23].

Lemma 2.1. *([20]) Let $x : (-\infty, T] \to \mathbb{X}$ be a function in a way that $x_0 = \phi$, $x|_{\mathscr{I}_k} \in \mathcal{C}(\mathscr{I}_k, \mathbb{X})$, and (P_4) holds. Then*

$$\|x_s\|_{\mathcal{B}_h} \le (\mathscr{D}_2^* + J^{\phi})\|\phi\|_{\mathcal{B}_h} + \mathscr{D}_1^* \sup\{\|x(\theta)\|_{\mathbb{X}} : \theta \in [0, \max\{0, s\}]\},\ s \in \mathcal{R}(\rho^-) \cup \mathscr{I},$$

where $J^{\phi} = \sup_{t \in \mathcal{R}(\rho^-)} J^{\phi}(t)$, $\quad \mathscr{D}_1^* = \sup_{s \in [0,T]} \mathscr{D}_1(s)$, $\quad \mathscr{D}_2^* = \sup_{s \in [0,T]} \mathscr{D}_2(s)$.

Now, we provide some fundamental definitions and results of the fractional calculus [33, 26] theory that are used further as an aspect of this manuscript.

Definition 2.2. *The fractional integral of order γ with the lower limit zero for a function f is defined by*

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds,\ t > 0,\ \gamma > 0,$$

the right part is point wise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.3. *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f \in \mathcal{L}^1(\mathscr{I}, \mathbb{X})$ is defined by*

$$D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}} ds,\ t > 0,\ n-1 < \gamma < n.$$

Definition 2.4. *The Caputo derivative of order γ for a function $f \in \mathcal{L}^1(\mathscr{I}, \mathbb{X})$ is defined by*

$${}^{C}D_t^{\gamma} f(t) = D_t^{\gamma}(f(t) - f(0)),\ t > 0,\ 0 < \gamma < 1.$$

Remark 2.1. *To be able to determine mild solution of the model (1.1)-(1.3), we require mild solution of the subsequent Cauchy problem:*

$$\begin{cases} {}^{C}D_t^{q} x(t) = \mathcal{A}x(t) + f(t),\ t \in \mathscr{I}, \\ x(0) = x_0 \in \mathbb{X}. \end{cases} \tag{2.1}$$

The mild solution [13, 46] of the above Cauchy problem can be described by

$$x(t) = \mathcal{R}_q(t)x_0 + \int_0^t \mathcal{S}_q(t-s)f(s)ds,$$

where

$$\mathcal{R}_q(t) = \frac{1}{2\pi i}\int_\Gamma e^{\tau t}\tau^{q-1}R(\tau^q,\mathcal{A})d\tau, \quad \mathcal{S}_q(t) = \frac{1}{2\pi i}\int_\Gamma e^{\tau t}R(\tau^q,\mathcal{A})d\tau,$$

for suitable path Γ, *and* $f : \mathscr{I} \to \mathbb{X}$ *is continuous.*

Definition 2.5. *(*[24]*) An operator* $\mathcal{A}$ *is called a sectorial operator if there are constants* $\eta \in [\frac{\pi}{2},\pi]$, $\rho \in \mathbb{R}$ *and* $\mathcal{M} > 0$, *such that the following conditions are satisfied:*

(1) $\nu(\mathcal{A}) \subset \sum_{\eta,\rho}$, *and*
(2) $\|R(\lambda,\mathcal{A})\|_{L(X)} \leq \frac{\mathcal{M}}{|\lambda-\rho|}, \mathcal{M} > 0, \lambda \in \sum_{\eta,\rho}$,

where $\sum_{\eta,\rho} = \{\lambda \in \mathbb{C} : \lambda \neq \rho, |arg(\lambda-\rho)| < \eta\}$.

Sectorial operators are well studied in the literature. Since the spectrum of a sectorial operator integral is unbounded, one has to integrate along infinite lines (means the boundary of a sector). As a matter of fact, this is only possible for a restricted collection of functions. Dealing with these functions requires some sectors in the space. For a recent reference including several examples and properties, we refer the reader to [24]. In this work, we will assume that the operator $\mathcal{A}$ is sectorial of type ρ with $0 \leq \rho_0 < \eta_0(0,\frac{\pi}{2}]$. In this case $\mathcal{A}$ is a generator of a solution operator given by

$$\mathcal{R}_q(t) = \frac{1}{2\pi i}\int_\Gamma e^{\tau t}\tau^{q-1}R(\tau^q,\mathcal{A})d\tau,$$
$$\mathcal{S}_q(t) = \frac{1}{2\pi i}\int_\Gamma e^{\tau t}R(\tau^q,\mathcal{A})d\tau;$$

where Γ is a suitable path lying on $\sum_{\eta,\rho}$.

Definition 2.6. [2] *A family* $\{\mathcal{R}_q(t)\}_{t\geq 0}$ *is called a solution operator of the Cauchy problem (2.1) if the following conditions are satisfied:*

(i) $\mathcal{R}_q(t)$ *is strongly continuous for* $t \geq 0$, *and* $\mathcal{R}_q(0) = I$, *where* I *is the identity operator.*
(ii) $\mathcal{R}_q(t)\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ *and* $\mathcal{A}\mathcal{R}_q(t)x = \mathcal{R}_q(t)\mathcal{A}x$ *for all* $x \in \mathcal{D}(\mathcal{A})$ *and* $t \geq 0$.
(iii) $\mathcal{R}_q(t)x$ *is a solution of (2.1) for all* $x \in \mathcal{D}(\mathcal{A})$ *and* $t \geq 0$.

Lemma 2.7. *If* $\mathcal{A} \in \mathcal{A}^q(\eta_0,\rho_0)$ *for some* $\eta_0 \in (0,\frac{\pi}{2}]$, *and* $\rho_0 \in \mathbb{R}$, *then*

$$\|\mathcal{R}_q(t)\|_{\mathcal{L}(X)} \leq \hat{\mathcal{M}}_1, \qquad \|\mathcal{S}_q(t)\|_{\mathcal{L}(X)} \leq \hat{\mathcal{M}}_2 t^{q-1},$$

where

$$\hat{\mathcal{M}}_1 = \sup_{t\in\mathscr{I}}\|\mathcal{R}_q(t)\|_{L(X)}, \qquad \hat{\mathcal{M}}_2 = \sup_{t\in\mathscr{I}}\mathcal{M}e^{\rho t}(1+t^{1-q}),$$

where $\mathcal{M} = \mathcal{M}(\eta,\rho)$ *is a constant.*

Definition 2.8. *A function* $x : (-\infty,T] \to \mathbb{X}$ *is a mild solution of the model (1.1)-(1.3) if :* $x_0 = \phi \in \mathcal{B}_h$ *on* $(-\infty,0]$; $\Delta x|_{t=t_k} = \mathcal{I}_k(x(t_k))$, $k = 1,2,...,m$, *the constraint of* $x(.)$ *to be interval* $\mathcal{I}_k, k = 0,1,2,...,m$, *is continuous equation* *and there exists* $x(.) \in \mathcal{L}^1(\mathcal{I}_k, \mathbb{X})$*, such that* $x(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)})$ *a.e.* $t \in \mathscr{I}$ *and* x *fulfills the subsequent integral equation:*

$$x(t) = \begin{cases} \mathcal{R}_q(t)\Big[\phi(0) - \mathscr{G}(0,\phi)\Big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ + \int_0^t (t-s)^{q-1} \mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ + \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\mathscr{F}\Big(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big)ds \\ + \sum_{k=1}^m \mathcal{R}_q(t-t_k)\mathcal{I}_k(x(t_k)), \quad t \in \mathscr{I}; \end{cases} \tag{2.2}$$

is satisfied, where $\mathcal{R}_q(.)$ *and* $\mathcal{S}_q(.)$ *are called characteristic solution operators and given by*

$$\mathcal{R}_q(t) = \int_0^\infty \psi_q(\theta)\mathcal{R}_q(t^q\theta)d\theta,$$
$$\mathcal{S}_q(t) = q\int_0^\infty \theta\psi_q(\theta)\mathcal{R}_q(t^q\theta)d\theta,$$

and for $\theta \in (0,\infty)$*,*

$$\psi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\nu(\theta^{-\frac{1}{q}}) \geq 0,$$
$$\nu(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}sin(n\pi q).$$

Here, ψ_q *is a probability density function defined on* $(0,\infty)$*, that is,*

$$\psi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \ \ and \ \ \int_0^\infty \psi_q(\theta)d\theta = 1.$$

The following results of $\mathcal{R}_q(.)$ *and* $\mathcal{G}_q(.)$ *are used throughout this manuscript.*

Remark 2.2. *(*[52]*) It is not difficult to verify that for* $\zeta \in [0,1]$*,*

$$\int_0^\infty \theta^\zeta\psi_q(\theta)d\theta = \int_0^\infty \theta^{-q\zeta}\nu(\theta)d\theta = \frac{\Gamma(1+\zeta)}{\Gamma(1+q\zeta)}.$$

Lemma 2.9. *(*[52, 51]*) The operators* $\mathcal{R}_q(t)$ *and* $\mathcal{S}_q(t)$ *have the following properties:*

(i) *For any fixed* $t \geq 0$*,* $\mathcal{R}_q(t)$ *and* $\mathcal{S}_q(t)$ *are linear and bounded operators, that is, for any* $x \in \mathbb{X}$*,*

$$\big\|\mathcal{R}_q(t)x\big\| \leq \hat{\mathcal{M}}_1\|x\| \ \ and \ \ \big\|\mathcal{S}_q(t)x\big\| \leq \frac{q\hat{\mathcal{M}}_2}{\Gamma(1+q)}\|x\|.$$

(ii) $\{\mathcal{R}_q(t), t \geq 0\}$ *and* $\{\mathcal{S}_q(t), t \geq 0\}$ *are strongly continuous.*

(iii) *For* $t \in \mathscr{I}$ *and any bounded subsets* $\mathcal{D} \subset X$*,* $t \to \{\mathcal{R}_q(t)x : x \in \mathcal{D}\}$ *and* $t \to \{\mathcal{S}_q(t)x : x \in \mathcal{D}\}$ *are equicontinuous if* $\|\mathcal{R}_q(t_2^q(\theta))x - \mathcal{R}_q(t_1^q(\theta))x\| \to 0$ *with respect to* $x \in \mathcal{D}$ *as* $t_2 \to t_1$ *for each fixed* $\theta \in [0,\infty)$*.*

(iv) *For any* $x \in \mathbb{X}$, $\alpha, \beta \in (0,1)$, *we have*

$$\mathcal{A}\mathcal{R}_q(t)x = \mathcal{A}^{1-\beta}\mathcal{R}_q(t)\mathcal{A}^{\beta}x, \quad t \in \mathscr{I},$$

$$\|\mathcal{A}^{\alpha}\mathcal{R}_q(t)\| \le \frac{q\hat{M}_{1\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} t^{-\alpha q}, \quad 0 < t \le T.$$

Theorem 2.10. *(Hölder's inequality) Assume that* $p, q \ge 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $l \in \mathcal{L}^p(\mathscr{I}, R)$, $m \in \mathcal{L}^q(\mathscr{I}, R)$, *then for* $1 \le q \le \infty$ *and* $lm \in \mathcal{L}^1(\mathscr{I}, R)$ *and*

$$\|lm\|_{\mathcal{L}^1(\mathscr{I})} \le \|l\|_{\mathcal{L}^p(\mathscr{I})}\|m\|_{\mathcal{L}^q(\mathscr{I})}.$$

Theorem 2.11. *(*[49]*) (PC-Ascoli-Arzela Theorem) Let* $\mathbb{X}$ *be a Banach space and* $\mathcal{A} \subset PC(\mathscr{I}, \mathbb{X})$. *If the following conditions are satisfied:*

(i) $\mathcal{A}$ *is uniformly bounded subset of* $PC(\mathscr{I}, \mathbb{X})$*;*

(ii) $\mathcal{A}$ *is equicontinuous in* (t_k, t_{k+1}), $k = 0, 1, 2, ...m$*; where* $t_0 = 0$, $t_{m+1} = T$*;*

(iii) $\mathcal{A}(t) \equiv \{x(t) \mid x \in \mathcal{A}; \ t \in J\backslash\{t_1...t_m\}\}$, $\mathcal{A}(t_k^+) = \{x(t_k^+)|x \in \mathcal{A}\}$ *and* $\mathcal{A}(t_k^-) = \{x(t_k^-)|x \in \mathcal{A}\}$ *is a relatively compact subsets of* $\mathbb{X}$.

Then $\mathcal{A}$ *is a relatively compact subset of* $PC(\mathscr{I}, \mathbb{X})$.

Theorem 2.12. *(Schaefer's fixed point theorem) Let* $\mathbb{X}$ *be a Banach space and* $F : \mathbb{X} \to \mathbb{X}$ *be a completely continuous operator. If the set* $E = \{y \in \mathbb{X} : y = \lambda F(y), 0 < \lambda < 1\}$ *is bounded, then* F *has at least a fixed point in* $\mathbb{X}$.

3. Existence Results

We introduce the following hypotheses:

(H1): The function $\mathscr{G} : \mathscr{I} \times \mathcal{B}_h \to \mathbb{X}$ is continuous and there exist constants $\mathcal{W}_{\mathscr{G}} > 0, 0 < \alpha < 1$ such that $\mathscr{G}$ is $\mathbb{X}_\alpha$ valued and satisfies the following conditions:

$$\|\mathcal{A}^{\beta}\mathscr{G}(t,x) - \mathcal{A}^{\beta}\mathscr{G}(t,y)\| \le \mathcal{W}_{\mathscr{G}}\|x-y\|, \quad t \in \mathscr{I}, x, y \in \mathcal{B}_h,$$

$$\|\mathcal{A}^{\beta}\mathscr{G}(t,x)\| \le \mathcal{W}_{\mathscr{G}}(1+\|x\|), \quad t \in \mathscr{I}, x \in \mathcal{B}_h.$$

(H2): The function $\mathscr{F} : \mathscr{I} \times \mathcal{B}_h \times \mathbb{X} \to \mathbb{X}$ satisfies the following properties:

(i) For each $t \in \mathscr{I}$, the function $\mathscr{F}(t, ., .) : \mathcal{B}_h \times \mathbb{X} \to \mathbb{X}$ is continuous.

(ii) For each $(x, \phi) \in \mathcal{B}_h \times \mathbb{X}$, the function $\mathscr{F}(., x, \phi) : \mathscr{I} \to \mathbb{X}$ is strongly measurable.

(iii) There exists a positive integrable function $c_{\mathscr{F}} \in \mathcal{L}^1(\mathscr{I})$ and a continuous non decreasing function $\aleph_{\mathscr{F}} : [0, \infty) \to (0, \infty)$ such that for all $(t, x, \phi) \in \mathscr{I} \times \mathcal{B}_h \times \mathbb{X}$, we have

$$\|\mathscr{F}(t, x, \phi)\| \le c_{\mathscr{F}}(t)\aleph_{\mathscr{F}}(\|x\|_{\mathcal{B}_h} + \|\phi\|_{\mathbb{X}}),$$

$$\lim \inf_{r\to\infty} \frac{\aleph_{\mathscr{F}}(r)}{r} = \Lambda < \infty.$$

(H3): The function $\mathfrak{W} : \mathscr{D} \times \mathcal{B}_h \to \mathbb{X}$, where $\mathscr{D} = \{(t, s) \in \mathscr{I} \times \mathscr{I}; 0 \le s \le t \le T\}$, satisfies followings:

(i) For each $(t,s) \in \mathcal{D}$, the function $\mathfrak{W}(t,s,.) : \mathcal{B}_h \to \mathbb{X}$ is continuous, and for each $x \in \mathcal{B}_h$, the function $\mathfrak{W}(.,.,x) : \mathcal{D} \to \mathbb{X}$ is strongly measurable.

(ii) There exists constants $N_1 > 0$ such that, for all $t, s \in \mathscr{I}$ and $x \in \mathcal{B}_h$, we have

$$\|\mathfrak{W}(t,s,x)\| \leq N_1(1 + \|x\|_{\mathcal{B}_h})$$

(H4): The functions $\mathcal{I}_k : \mathcal{B}_h \to \mathbb{X} \quad k = 1, 2, ..., m$, are continuous, and there exist nondecreasing continuous functions $\mathcal{W}_{\mathcal{I}_k} : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x \in \mathcal{B}_h$, we have

$$\|\mathcal{I}_k(x)\| \leq \mathcal{W}_{\mathcal{I}_k}(\|x\|), \quad \lim \inf_{r\to\infty} \frac{\mathcal{W}_{\mathcal{I}_k}(r)}{r} = \sigma_k < \infty.$$

(H5): $\mathcal{I}_k \in C(\mathbb{X}, \mathbb{X})$ and we can find $L_k \in C[\mathscr{I}, \mathbb{R}^+]$ such that

$$\|\mathcal{I}_k(x)\|_{\mathbb{X}} \leq L_k(t)\|x\|_{\mathbb{X}}, \quad x \in \mathbb{X}, \quad t \in \mathscr{I}.$$

(H6): The functions $\mathcal{I}_k : \mathcal{B}_h \to \mathbb{X}$ are continuous and there are positive constants $\delta_k, \ \ k = 1, 2, ..., m$, such that

$$\|\mathcal{I}_k(x) - \mathcal{I}_k(y)\| \leq \delta_k\|x - y\| \qquad x, y \in \mathcal{B}_h.$$

(H7):

(1)

$$\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{-\beta}\|\Big[\hat{\mathcal{M}}_1(1 + \|\phi\|_{\mathcal{B}_h}) + (1 + \mathscr{D}_1^* r + c_n)\Big] + K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}$$
$$(1 + \mathscr{D}_1^* r + c_n) + \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)}\aleph_{\mathscr{F}}[(\mathscr{D}_1^* r + c_n) + TN_1(1 + (\mathscr{D}_1^* r + c_n))]$$
$$\sup\nolimits_{s\in\mathscr{I}} \ c_{\mathscr{F}}(s) + m\hat{\mathcal{M}}_1 L_0 H[\mathscr{D}_1^* r + \widetilde{c_n}] < 1.$$

(2)

$$\beta_1 = \Big(\mathcal{W}_{\mathscr{G}}(\mathscr{D}_1^*(r)) + K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1 + \mathscr{D}_1^*(r)) + \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)}\mathcal{N}_{\mathscr{F}}$$
$$\Big(\mathscr{D}_1^*(r)\Big(1 + TN_1\Big) + TN_1\Big)\sup\nolimits_{s\in\mathscr{I}} \ c_{\mathscr{F}}(s) + \hat{\mathcal{M}}_1\delta_k\Big) < 1$$

Theorem 3.1. *Assume that the hypotheses (H1) to (H7) are satisfied, then there exists at least one fixed point and a unique solution of system (1.1)-(1.3) on $\mathscr{I}$.*

Proof. We will modify the structure (1.1)-(1.3) into a fixed point problem. Consider the operator $\Upsilon : \mathcal{B}_h \to \mathcal{B}_h$ defined by,

$$(\Upsilon \ x)(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0] \\ \mathcal{R}_q(t)\Big[\phi(0) - \mathscr{G}(0,\phi)\Big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ + \int_0^t \mathcal{A}\mathcal{S}_q(t)(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ + \int_0^t \mathcal{S}_q(t)(t-s)\mathscr{F}(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau)ds \\ + \sum_{0<t_k<t}\mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(x(t_k)), \quad t \in \mathscr{I}. \end{cases}$$

It is evident that the fixed points of the operator Υ are mild solutions of the model (1.1)-(1.3). We define the function $y(.) : (-\infty, T] \to \mathbb{X}$ by

$$y(t) = \begin{cases} \phi(t), & t \leq 0; \\ \mathcal{R}_q(t)\phi(0), & t \in \mathscr{I}; \end{cases}$$

then, $y_0 = \phi$. For every function $z \in C(\mathscr{I}, \mathbb{R})$ with $z(0) = 0$, we allocate that $\hat{z}$ is characterized by:

$$\hat{z}(t) = \begin{cases} 0, & t \leq 0, \\ z(t), & t \in \mathscr{I}. \end{cases}$$

If $x(.)$ satisfies equation (2.4), we are able to decompose it as $x(t) = z(t) + y(t),\ t \in \mathscr{I}$, which suggests that $x_t = z_t + y_t \ \ for \ \ t \in \mathscr{I}$, and the function $z(.)$ satisfies

$$z(t) = \begin{cases} -\mathcal{R}_q(t)\mathscr{G}(0,\phi) + \mathscr{G}\Big(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}\Big) \\ + \int_0^t \mathcal{A}\mathcal{S}_q(t)(t-s)\mathscr{G}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\Big)ds \\ + \int_0^t \mathcal{S}_q(t)(t-s)\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\ \int_0^s \mathfrak{W}(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds \\ + \sum_{0<t_k<t} \mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k)), \quad t \in \mathscr{I}. \end{cases}$$

Let $\mathcal{B}_h'' = \{z \in \mathcal{B}_h' : z_0 = 0\}$. Let $\|.\|_{\mathcal{B}_h''}$ be the seminorm in $\mathcal{B}_h''$ described by

$$\begin{aligned} \|z\|_{\mathcal{B}_h''} &= \sup_{s\in\mathscr{I}} \|z(t)\|_{\mathbb{X}} + \|z_0\|_{\mathcal{B}_h} \\ &= \sup_{s\in\mathscr{I}} \|z(t)\|_{\mathbb{X}}, \quad z \in \mathcal{B}_h''. \end{aligned}$$

As a result, $(\mathcal{B}_h'', \|.\|_{\mathcal{B}_h''})$ is a Banach space. We define the operator $\hat{\Upsilon} : \mathcal{B}_h'' \to \mathcal{B}_h''$ by

$$(\hat{\Upsilon}z)(t) = \begin{cases} -\mathcal{R}_q(t)\mathscr{G}(0,\phi) + \mathscr{G}\Big(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}\Big) \\ + \int_0^t \mathcal{A}\mathcal{S}_q(t)(t-s)\mathscr{G}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\Big)ds \\ + \int_0^t \mathcal{S}_q(t)(t-s)\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\ \int_0^s \mathfrak{W}(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds \\ + \sum_{0<t_k<t} \mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k)), \quad t \in \mathscr{I}. \end{cases}$$

We see that the operator Υ has a fixed point if and only if $\hat{\Upsilon}$ has a fixed point. Let us demonstrate that $\hat{\Upsilon}$ has a fixed point.

Remark 3.1. *From Lemma 2.1 and above assumptions, we have the following estimates:*

$$\begin{aligned} (i)\ & \|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathcal{B}_h} \\ & \leq \|z_{\varrho(s,z_s+y_s)}\|_{\mathcal{B}_h} + \|y_{\varrho(s,z_s+y_s)}\|_{\mathcal{B}_h} \\ & \leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s} \|z(\tau)\|_{\mathbb{X}} + (\mathscr{D}_2^* + J^\phi)\|z_0\|_{\mathcal{B}_h} + \mathscr{D}_1^*|y(s)| \end{aligned}$$

$$+ (\mathscr{D}_2^* + J^{\phi})\|y_0\|_{\mathcal{B}_h}$$
$$\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s}\|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\|\mathcal{S}_q(t)\|_{L(\mathbb{X})}|\phi(0)| + (\mathscr{D}_2^* + J^{\phi})\|\phi\|_{\mathcal{B}_h}$$
$$\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s}\|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\hat{\mathcal{M}}_1 H\|\phi\|_{\mathcal{B}_h} + (\mathscr{D}_2^* + J^{\phi})\|\phi\|_{\mathcal{B}_h}$$
$$\leq \mathscr{D}_1^* \sup_{0\leq\tau\leq s}\|z(\tau)\|_{\mathbb{X}} + (\mathscr{D}_1^*\hat{\mathcal{M}}_1 H + \mathscr{D}_2^* + J^{\phi})\|\phi\|_{\mathcal{B}_h}.$$

If $\|z\|_{\mathbb{X}} < r$, $r > 0$, *then*

$$\left\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\right\|_{\mathcal{B}_h} \leq \mathscr{D}_1^* r + c_n,$$

where $c_n = (\mathscr{D}_1^*\hat{\mathcal{M}}_1 H + \mathscr{D}_2^* + J^{\phi})\|\phi\|_{\mathcal{B}_h}$.

$$(ii)\quad \Big\|\sum_{k=1}^{m}\mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k))\Big\|_{\mathbb{X}} \leq m\hat{\mathcal{M}}_1\|\mathcal{I}_k(z(t_k)+y(t_k))\|_{\mathbb{X}}. \quad (3.1)$$

Since,

$$|\mathcal{I}_k(z(t_k)+y(t_k))| \leq L_k(t)(|z(t_k)+y(t_k)|)$$
$$\leq L_k(t)(\sup_{s\in\mathscr{I}}|z(t)+y(t)|)$$
$$\leq L_0 H\|z_t+y_t\|_{\mathcal{B}_h},$$

where $L_0 = \max\{\mathrm{L_k(t)}|\mathrm{t}\in\mathscr{I},\quad \mathrm{k}=1,2,3,...,\mathrm{m}\}$.

Now,

$$\|z_t+y_t\|_{\mathcal{B}_h} \leq \|z_t\|_{\mathcal{B}_h} + \|y_t\|_{\mathcal{B}_h}$$
$$\leq \mathscr{D}_1(t)\sup_{0\leq\tau\leq t}\|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_2(t)\|z_0\|_{\mathcal{B}_h} + \mathscr{D}_1(t)\sup_{0\leq\tau\leq t}\|y(\tau)\|_{\mathbb{X}}$$
$$+ \mathscr{D}_2(t)\|y_0\|_{\mathcal{B}_h}$$
$$\leq \mathscr{D}_1(t)\sup_{0\leq\tau\leq t}\|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1(t)\Big[\|\mathcal{S}_q(t)\|_{L(\mathbb{X})}|\phi(0)|\Big]$$
$$\leq \mathscr{D}_1^* r + (\mathscr{D}_1^*\hat{\mathcal{M}}_1 H + \mathscr{D}_2^*)\|\phi\|_{\mathcal{B}_h}$$
$$\leq \mathscr{D}_1^* r + \widetilde{c_n};$$

where $\widetilde{c_n} = (\mathscr{D}_1^*\hat{\mathcal{M}}_1 H + \mathscr{D}_2^*)\|\phi\|_{\mathcal{B}_h}$. Hence, Equation (3.1) becomes

$$\Big\|\sum_{k=1}^{m}\mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k))\Big\|_{\mathbb{X}} \leq m\hat{\mathcal{M}}_1 L_0 H[\mathscr{D}_1^* r + \widetilde{c_n}].$$

Let $\mathcal{B}_r = \{z \in \mathcal{B}_h'' : z(0) = 0; \quad \|z\|_{\mathcal{B}_h''} \leq r\}$ for some $r > 0$,
where r is any fixed finite real number that fulfills the inequality. The proof will be given in several steps.

Step 1 : We show that there exists some $r > 0$, such that $\hat{\Upsilon}(\mathcal{B}_r) \subset \mathcal{B}_r$. If it is not true for each positive number r, there exists a function $z^r(.) \in \mathcal{B}_r$ and some $t \in \mathscr{I}$ such that $\|(\hat{\Upsilon}z^r)(t)\| > r$. On the other hand, from hypotheses (H2)(i)-(ii), (H3), Lemma 2.3 (i) and Hölder's inequality, we obtain

$$r < \|(\hat{\Upsilon}z^r)(t)\|$$

$$
\begin{aligned}
&\le \| -\mathcal{R}_q(t)\mathscr{G}(0,\phi)\| + \|\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})\Big\| \\
&+ \Big\| \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\Big\| \\
&+ \Big\| \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \\
&\quad \int_0^s \mathfrak{W}(s,\tau, z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)ds\Big\| \\
&+ \Big\| \sum_{0<t_k<t} \mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k))\Big\| \\
&= \sum_{i=1}^{4} J_i. \qquad (3.2)
\end{aligned}
$$

Let us estimate, $J_i, i = 1,2,3,4$. By assumption $(H1)$, we have

$$
\begin{aligned}
J_1 &\le \Big\|\mathcal{R}_q(t)\mathscr{G}(0,\phi)\Big\| \\
&\le \hat{\mathcal{M}}_1\|\mathcal{A}^{-\beta}\|\Big\|\mathcal{A}^{\beta}\mathscr{G}(0,\phi)\Big\| \\
&\le \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{-\beta}\|(1+\|\phi\|_{\mathcal{B}_h}) \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
J_2 &\le \|\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})\| \\
&\le \|\mathcal{A}^{-\beta}\|\Big\|\mathcal{A}^{\beta}\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})\Big\| \\
&\le \|\mathcal{A}^{-\beta}\|\mathcal{W}_{\mathscr{G}}(1+\Big\|z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}\Big\|) \\
&\le \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{-\beta}\|(1+\mathscr{D}_1^*r + c_n). \qquad (3.4)
\end{aligned}
$$

By using Lemma 2.3 and Hölder's inequality, one can deduce that

$$
\begin{aligned}
J_3 &\le \Big\| \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})ds\Big\| \\
&\le \int_0^t \Big\|(t-s)^{q-1}\mathcal{A}^{1-\beta}\mathcal{S}_q(t-s)\mathcal{A}^{\beta}\mathscr{G}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})\Big\|ds \\
&\le \int_0^t \Big\|(t-s)^{q-1}\mathcal{A}^{1-\beta}\mathcal{S}_q(t-s)\Big\|\times \qquad (3.5) \\
&\quad \Big[\Big\|\mathcal{A}^{\beta}\mathscr{G}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})\Big\|\Big]ds \\
&\le \int_0^t \Big\{q\int_0^\infty \theta\psi_q(\theta)(t-s)^{q-1}\mathcal{A}^{1-\beta}\mathcal{R}_q((t-s)^q\theta)d\theta\Big\} \qquad (3.6) \\
&\quad \Big[\mathcal{W}_{\mathscr{G}}(1+\mathscr{D}_1^*r + c_n)\Big]ds \\
&\le \int_0^t q\mathcal{M}_{1-\beta}(t-s)^{q\beta-1}\Big[\int_0^\infty \theta^{\beta}\psi_q(\theta)d\theta\Big]\Big[\mathcal{W}_{\mathscr{G}}(1+\mathscr{D}_1^*r + c_n)\Big]ds
\end{aligned}
$$

$$\leq \frac{q\mathcal{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}\int_0^t (t-s)^{q\beta-1}\Big[\mathcal{W}_{\mathscr{G}}(1+\mathscr{D}_1^* r + c_n)\Big]ds$$
$$\leq K(q,\beta)\int_0^t (t-s)^{q\beta-1}\mathcal{W}_{\mathscr{G}}(1+\mathscr{D}_1^* r + c_n)ds$$
$$\leq K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^* r + c_n). \tag{3.7}$$

Using assumptions $(H2)$ and $(H3)$, we have

$$J_4 \leq \Big\| \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \tag{3.8}$$
$$\int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)\Big\| ds$$
$$\leq \int_0^t \Big\|(t-s)^{q-1}\mathcal{S}_q(t-s)\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \tag{3.9}$$
$$\int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)\Big\| ds$$
$$\leq \frac{\hat{\mathcal{M}}_2}{\Gamma(q)}\int_0^t (t-s)^{q-1}c_{\mathscr{F}}(s)\aleph_{\mathscr{F}}\Big(\Big\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\Big\|$$
$$+\Big\|\int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big\|\Big)ds$$
$$\leq \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)}c_{\mathscr{F}}(s)\aleph_{\mathscr{F}}\Big(\|z_s+y_s\| + \Big\|\int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big\|\Big)$$
$$\leq \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)}c_{\mathscr{F}}(s)\aleph_{\mathscr{F}}(\mathscr{D}_1^* r + c_n) + TN_1(1+\mathscr{D}_1^* r + c_n)$$
$$\leq \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)}\aleph_{\mathscr{F}}[(\mathscr{D}_1^* r + c_n) + TN_1(1+(\mathscr{D}_1^* r + c_n))]\sup_{s\in\mathscr{I}} c_{\mathscr{F}}(s). \tag{3.10}$$

Combining the estimate $(J_1)-(J_4)$ and (3.1) together with (3.2), we obtain

$$r \leq \|\hat{\Upsilon}(z^r)(t)\|$$
$$\leq \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{-\beta}\|(1+\|\phi\|_{\mathcal{B}_h}) + \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{-\beta}\|(1+\mathscr{D}_1^* r + c_n)$$
$$+ K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^* r + c_n)$$
$$+ \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)}\aleph_{\mathscr{F}}[(\mathscr{D}_1^* r + c_n) + TN_1(1+(\mathscr{D}_1^* r + c_n))]\mathrm{sup}_{\mathrm{s}\in\mathscr{I}}\ c_{\mathscr{F}}(s)$$
$$+ m\hat{\mathcal{M}}_1 L_0 H[\mathscr{D}_1^* r + \widetilde{c_n}].$$

Dividing both sides by r and taking $r\to\infty$, we get that

$$\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{-\beta}\|\Big[\hat{\mathcal{M}}_1(1+\|\phi\|_{\mathcal{B}_h}) + (1+\mathscr{D}_1^* r + c_n)\Big] + K(q,\beta)\mathcal{W}_{\mathscr{G}}$$

$$\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^* r + c_n) + \frac{\hat{\mathcal{M}}_2 T^q}{\Gamma(q+1)} \aleph_{\mathscr{F}}[(\mathscr{D}_1^* r + c_n)$$
$$+ TN_1(1+(\mathscr{D}_1^* r + c_n))]\sup_{s\in\mathscr{I}}\ c_{\mathscr{F}}(s) + m\hat{\mathcal{M}}_1 L_0 H[\mathscr{D}_1^* r + \widetilde{c_n}] \geq 1,$$

which is a contradiction to (H7)(1). For this reason, for some positive number r, we have $\hat{\Upsilon}(\mathcal{B}_r) \subset \mathcal{B}_r$.

Step 2 : $\hat{\Upsilon}$ is continuous on $\mathcal{B}_r$.

Let ${z^n}_{n\in N} \subset \mathcal{B}_r$ with $z^n \to z$ in $\mathcal{B}_r$ as $n \to \infty$.

Denote

$$\mathcal{F}_n(s) = \mathscr{F}\Big(s, z^n_{\varrho(s,z^n_s+y_s)} + y_{\varrho(s,z^n_s+y_s)}, \int_0^s \mathfrak{W}(s,\tau,z^n_{\varrho(\tau,z^n_\tau+y_\tau)} + y_{\varrho(\tau,z^n_\tau+y_\tau)})d\tau\Big)$$
$$\mathcal{F}(s) = \mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big).$$

Then by using the hypotheses (H3), (H6) and Lebesgue's Dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{q-1}\|\mathcal{F}_n(s) - \mathcal{F}(s)\|ds \to 0 \quad as \quad n\to\infty, \quad t\in\mathscr{I}. \tag{3.11}$$

Now,

$$\|\hat{\Upsilon}z^n - \hat{\Upsilon}z\|_{\mathcal{C}} \leq \frac{\hat{\mathcal{M}}_2 q}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|\mathcal{F}_n(s)-\mathcal{F}(s)\|ds \tag{3.12}$$

Observing that (3.4) and (3.6), we have

$$\|\hat{\Upsilon}z^n - \hat{\Upsilon}z\|_{\mathcal{C}} \to 0 \quad as \quad n\to\infty,$$

which implies that $\hat{\nu}$ is continuous on $\mathcal{B}_r$.

Step 3: $\hat{\Upsilon}(\mathcal{B}_r)$ is equicontinuous on $\mathscr{I}$. Let $Z\in\hat{\Upsilon}(\mathcal{B}_r)$ and $0\leq t_1\leq t_2\leq T$. Then there is $z\in\mathcal{B}_r$ such that

$$\begin{aligned}
\|Z(t_2)-Z(t_1)\| &\leq \Big\|\mathcal{R}_q(t_2)-\mathcal{R}_q(t_1)\Big\| + \Big[\|\phi(0)\| + \|\mathscr{G}(0,\phi(0))\|\Big] \\
&\quad + \Big\|\int_0^{t_2}(t_2-s)^{q-1}\mathcal{S}_q(t)(t_2-s)\mathcal{F}(s)ds \\
&\qquad - \int_0^{t_1}(t_1-s)^{q-1}\mathcal{S}_q(t)(t_1-s)\mathcal{F}(s)ds\Big\| \\
&\leq \Big\|\mathcal{R}_q(t_2)-\mathcal{R}_q(t_1)\Big\| + \Big[\|\phi(0)\| + \|\mathscr{G}(0,\phi(0))\|\Big] \\
&\qquad + \Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathcal{S}_q(t)(t_2-s)\mathcal{F}(s)ds\Big\| \\
&\quad + \Big\|\int_{t_1-\in}^{t_1}(t_2-s)^{q-1}\Big[\mathcal{S}_q(t)(t_2-s) - \mathcal{S}_q(t)(t_1-s)\Big]\mathcal{F}(s)ds\Big\| \\
&\quad + \Big\|\int_{t_1-\in}^{t_2}\Big[(t_2-s)^{q-1} - (t_1-s)^{q-1}\Big]\mathcal{S}_q(t)(t_1-s)\mathcal{F}(s)ds\Big\|
\end{aligned}$$

$$+\Big\|\int_0^{t_1-\in}(t_2-s)^{q-1}\Big[\mathcal{S}_q(t)(t_2-s)-\mathcal{S}_q(t)(t_1-s)\Big]\mathcal{F}(s)ds\Big\|$$
$$+\Big\|\int_0^{t_1-\in}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]\mathcal{S}_q(t)(t_1-s)\mathcal{F}(s)ds\Big\|.$$

Using Lemma 2.3, we can verify that the right hand side of the above inequality tends to zero as $t_2 \to t_1$.

As a consequence of steps 1-3 together with the Ascoli-Arzela Theorem, we conclude that $\hat{\Upsilon} : \mathcal{B}''_h \to \mathcal{B}''_h$ is completely continuous. As a consequence of Schaefer's fixed theorem we deduce that Z has a fixed point, which is a mild solution of the problem (1.1)-(1.3).
Further,

$$\|\Upsilon x(t) - \Upsilon y(t)\|$$
$$\leq \Big\|\Big[\mathcal{R}_q(t)\Big[\phi(0)-\mathscr{G}(0,\phi(0))\Big]+\mathscr{G}(t,x_{\varrho(t,x_t)})$$
$$+\int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t)(t-s)\mathscr{G}(s,x_{\varrho(s,x_s)})ds$$
$$+\int_0^t (t-s)^{q-1}\mathcal{S}_q(t)(t-s)$$
$$\mathscr{F}\Big(s,x_{\varrho(s,x_s)},\int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big)ds$$
$$+\sum_{0<t_k<t}\mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(x(t_k))\Big]-\Big[\mathcal{R}_q(t)\Big[\phi(0)-\mathscr{G}(0,\phi(0))\Big]$$
$$+\mathscr{G}(t,y_{\varrho(t,y_t)})+\int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t)(t-s)\mathscr{G}(s,y_{\varrho(s,y_s)})ds$$
$$+\int_0^t (t-s)^{q-1}\mathcal{S}_q(t)(t-s)$$
$$\mathscr{F}\Big(s,y_{\varrho(s,y_s)},\int_0^s \mathfrak{W}(s,\tau,y_{\varrho(\tau,y_\tau)})d\tau\Big)ds$$
$$+\sum_{0<t_k<t}\mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(y(t_k))\Big]\Big\|$$

$$\leq \Big\|\mathcal{A}^\beta\mathscr{G}(t,x_{\varrho(t,x_t)})-\mathcal{A}^\beta\mathscr{G}(t,y_{\varrho(t,y_t)})\Big\|$$
$$+\int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t)(t-s)ds\Big\|\mathcal{A}^\beta\mathscr{G}(s,x_{\varrho(s,x_s)})-\mathcal{A}^\beta\mathscr{G}(s,y_{\varrho(s,y_s)})\Big\|$$
$$+\int_0^t (t-s)^{q-1}\mathcal{S}_q(t)(t-s)ds$$

$$
\begin{aligned}
&\Big\|\mathscr{F}\Big(s,x_{\varrho(s,x_s)},\int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big)-\\
&\mathscr{F}\Big(s,y_{\varrho(s,y_s)},\int_0^s \mathfrak{W}(s,\tau,y_{\varrho(\tau,y_\tau)})d\tau\Big)\Big\|\\
&+\Big\|\sum_{0<t_k<t}\mathcal{R}_q(t)(t-t_k)\Big[\mathcal{I}_k(x(t_k))-\mathcal{I}_k(y(t_k))\Big]\Big\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ &\Big\|\mathcal{A}^\beta\mathscr{G}(t,x_{\varrho(t,x_t)})-\mathcal{A}^\beta\mathscr{G}(t,y_{\varrho(t,y_t)})\Big\|\\
&+\int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t)(t-s)ds\Big\|\mathcal{A}^\beta\mathscr{G}(s,x_{\varrho(s,x_s)})-\mathcal{A}^\beta\mathscr{G}(s,y_{\varrho(s,y_s)})\Big\|\\
&+\int_0^t (t-s)^{q-1}\mathcal{S}_q(t)(t-s)ds\\
&\Big\|\mathscr{F}\Big(s,x_{\varrho(s,x_s)},\int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big)-\\
&\mathscr{F}\Big(s,y_{\varrho(s,y_s)},\int_0^s \mathfrak{W}(s,\tau,y_{\varrho(\tau,y_\tau)})d\tau\Big)\Big\|\\
&+\hat{\mathcal{M}}_1\sum_{0<t_k<t}\Big\|\mathcal{I}_k(x(t_k))-\mathcal{I}_k(y(t_k))\Big\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ &\mathcal{W}_{\mathscr{G}}\mathscr{D}_1^*(r)\|x-y\|\\
&+K(q,\beta)\int_0^t (t-s)^{q\beta-1}ds\mathcal{W}_{\mathscr{G}}(1+\mathscr{D}_1^*(r))\|x-y\|\\
&+\frac{\hat{\mathcal{M}}_2}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\ c_{\mathscr{F}}(s)\mathcal{N}_{\mathscr{F}}\Big(\mathscr{D}_1^*(r)\|x-y\|+TN_1(1+\mathscr{D}_1^*(r))\|x-y\|\Big)\\
&+\hat{\mathcal{M}}_1\delta_k\|x-y\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ &\mathcal{W}_{\mathscr{G}}\mathscr{D}_1^*(r)\|x-y\|\\
&+K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^*(r))\|x-y\|\\
&+\frac{\hat{\mathcal{M}}_2T^q}{\Gamma(q+1)}\mathcal{N}_{\mathscr{F}}\Big(\mathscr{D}_1^*(r)\|x-y\|+TN_1(1+\mathscr{D}_1^*(r))\|x-y\|\Big)\sup_{s\in\mathscr{I}}\ c_{\mathscr{F}}(s)\\
&+\hat{\mathcal{M}}_1\delta_k\|x-y\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ &\Big(\mathcal{W}_{\mathscr{G}}\mathscr{D}_1^*(r)+K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^*(r))\\
&+\frac{\hat{\mathcal{M}}_2T^q}{\Gamma(q+1)}\mathcal{N}_{\mathscr{F}}\Big(\mathscr{D}_1^*(r)\Big(1+TN_1\Big)+TN_1\Big)\sup_{s\in\mathscr{I}}\ c_{\mathscr{F}}(s)+\hat{\mathcal{M}}_1\delta_k\Big)\|x-y\|
\end{aligned}
$$

$\|\Upsilon x(t)-\Upsilon y(t)\| \leq \beta_1\|x-y\|$,
where

$$\beta_1=\Big(\mathcal{W}_{\mathscr{G}}\mathscr{D}_1^*(r)+K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^*(r))$$
$$+\frac{\hat{\mathcal{M}}_2T^q}{\Gamma(q+1)}\mathcal{N}_{\mathscr{F}}\Big(\mathscr{D}_1^*(r)\Big(1+TN_1\Big)+TN_1\Big)\sup_{s\in\mathscr{I}}\, c_{\mathscr{F}}(s)+\hat{\mathcal{M}}_1\delta_k\Big),\ \beta_1<1$$

Thus Υ is a contraction mapping on $\mathcal{B}_h''$. By applying the well known Banach contraction principle, the operator Υ has a unique fixed point in $\mathcal{B}_h''$. Hence, the problem (1.1)-(1.3) has a unique solution in $\mathcal{B}_h''$. □

Remark 3.2. *The existence and uniqueness results of Theorem 3.1 is proved using the Lipschitz condition given in (H1). The same result can also be proved by replacing the first part of condition of* $(H1)$ *by the following growth conditions.*

(i) $\mathscr{G}:\mathscr{I}\times\mathcal{B}_h\to\mathbb{X}$ *is continuous function and verifies the following condition:*
There exists an integrable function $m_{\mathscr{G}}:\mathscr{I}\to[0,\infty)$ *and a continuous non-decreasing function* $\mathscr{W}_{\mathscr{G}}:[0,\infty)\to(0,\infty)$ *such that* $\|\mathscr{G}(t,x)\|_{\mathcal{B}_h}\leq m_{\mathscr{G}}(t)\mathscr{W}_{\mathscr{G}}(\|x\|),(t,x)\in\mathscr{I}\times\mathcal{B}_h$.

Remark 3.3. *In Theorem 3.1, we apply two fixed point theorems (Schaefer's and Banach fixed point theorems) to prove existence and uniqueness of solutions. Note that Banach fixed point theorem alone gives existence and uniqueness of solutions with only the assumption that the appeared functions are contractions. On the other hand, Schaefer's fixed point theorem alone gives existence result with different assumptions usually not so strong as contraction.*

4. Controllability Result

We consider the controllability of fractional impulsive neutral integro-differential systems with state-dependent delay of the form

$$^{C}D_t^q\Big[x(t)-\mathscr{G}(t,x_{\varrho(t,x_t)})\Big]=\mathcal{A}x(t)+\mathcal{B}u(t)+\mathscr{F}\Big(t,x_{\varrho(t,x_t)},\int_0^t\mathfrak{W}(t,s,x_{\varrho(s,x_s)})ds\Big),\quad t\in\mathscr{I} \tag{4.1}$$

$$\Delta x|_{t=t_k}=\mathcal{I}_k(x(t_k));\quad k=1,2,..,m,\quad t\neq t_k \tag{4.2}$$

$$x_0=\phi\in\mathcal{B}_h,\quad t\in(-\infty,0], \tag{4.3}$$

where $^{C}D_t^q$ denote the Caputo derivative of order $0<q<1$. The control function $u(.)\in\mathcal{L}^2(\mathscr{I},u)$, a Banach space of admissible control function with u as a Banach space. Furthermore, $\mathcal{B}_h$ is a phase space endowed with a seminorm $\|.\|_{\mathcal{B}_h}$ and $\mathcal{B}$ is a bounded linear operator u to X. For almost any continuous function x characterized on $(-\infty,T]$ and for almost any $t>0$, we designate by x_t the part of $\mathcal{B}_h$ characterized by $x_t(\theta)=x(t+\theta)$ for $\theta\geq0$. Now $x_t(.)$ belongs to some abstract space $\mathcal{B}_h$ defined with $\mathscr{G}:\mathscr{I}\times\mathcal{B}_h\to\mathbb{X}$, $\mathscr{F}:\mathscr{I}\times\mathcal{B}_h\times\mathbb{X}\to\mathbb{X}$,

$\mathfrak{W} : \mathscr{D} \times \mathcal{B}_h \to \mathbb{X}$, $i = 1, 2$; $\mathscr{D} = \{(t, s) \in \mathscr{I} \times \mathscr{I} : 0 \leq s \leq t \leq T\}$, $\varrho : \mathscr{I} \times \mathcal{B}_h \to (-\infty, T]$, $\mathcal{I}_k : \mathbb{X} \to \mathbb{X}(k = 1, 2, \ldots, m)$ are suitable functions.

Definition 4.1. *A function $x : (-\infty, T] \to \mathbb{X}$ is a mild solution of the model* (4.1) − (4.3) *on* $[0, T]$ *if* $x_0 = \phi$, $x_{\varrho(s,x_s)} \in \mathcal{B}_h$ *for every* $s \in [0, T]$*, the restriction of* $x(.)$ *to* $[0, T]$ *is continuous for each* $0 \leq t \leq T$*, the function* $\mathcal{A}S_q(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})$, $s \in [0, T]$ *is integrable and the following integral equation is satisfied.*

$$\begin{aligned} x(t) =& \mathcal{R}_q(t)\Big[\phi(0) - \mathscr{G}(0, \phi)\Big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ &+ \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ &+ \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\Big[Bu(s) + \mathscr{F}\Big(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s, \tau, x_{\varrho(\tau,x_\tau)})d\tau\Big)\Big]ds \\ &+ \sum_{k=1}^m \mathcal{R}_q(t-t_k)\mathcal{I}_k(x(t_k)), \quad t \in \mathscr{I}. \end{aligned} \tag{4.4}$$

Definition 4.2. *The system (4.1)-(4.3) is said to be controllable on the interval* $\mathscr{I}$ *if and only if for* x_0, $x_1 \in \mathbb{X}$*, there exists a control* $u \in \mathcal{L}^2(\mathscr{I}, u)$ *such that the mild solution* $x(t)$ *of (4.1)-(4.3) satisfies* $x(0) = x_0$ *and* $x(T) = x_1$*.*

To establish our controllability results, we introduce the following assumptions:
(H8): The linear operator $\mathcal{W} : \mathcal{L}^2(\mathscr{I}, u) \to X$ defined by

$$\mathcal{W}u : \int_0^T (T-s)^{q-1}\mathscr{G}_q(T-s)\mathcal{B}u(s)ds$$

has an inverse operator $\mathcal{W}^{-1}$ that takes values in $\mathcal{L}^2(\mathscr{I}, u)/ker\mathcal{W}$ and there exists two constants $\mathcal{M}_2$, $\mathcal{M}_3 > 0$ such that $\|\mathcal{B}\| \leq \mathcal{M}_2$ and $\|\mathcal{B}\mathcal{W}^{-1}\| \leq \mathcal{M}_3$.
For convenience, let us take

$$\mathcal{M}_4 := K_1\|m\|_{\mathcal{L}^{\frac{1}{q_1}}}(\mathscr{I}, R^+)$$

Theorem 4.3. *Assume that the hypotheses (H1) to (H8) are satisfied. Then the system (4.1)-(4.3) is controllable on* $\mathscr{I}$*.*

Proof. Using the hypotheses $(H2)$, for arbitrary function $x(.) \in \mathcal{C}$, we define the control $u_x(t)$ by

$$\begin{aligned} x(t) = \mathcal{R}_q(t)&\Big[\phi(0) - \mathscr{G}(0, \phi(0))\Big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ &+ \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ &+ \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\Bigg[\mathcal{B}u(s)+ \end{aligned}$$

$$\mathscr{F}\Big(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big)\Big]ds$$
$$+ \sum_{0<t_k<t} \mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(x(t_k)), \quad t \in \mathscr{I}.$$

$$\begin{aligned} u_x(t) = \mathcal{W}^{-1}\Bigg[&x_1 - \mathcal{R}_q(T)\Big[\phi(0) - \mathscr{G}(0,\phi(0))\Big] - \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ &- \int_0^T (T-s)^{q-1}\mathcal{A}\mathcal{S}_q(T-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ &- \int_0^T (T-s)^{q-1}\mathcal{S}_q(T-s)\Big[\mathscr{F}\Big(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big)\Big]ds \\ &- \sum_{0<t_k<t} \mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(x(t_k))\Bigg](t), \quad t \in \mathscr{I}. \end{aligned}$$

We show that, using the control and the operator $\Upsilon : \mathcal{B}_h^{'} \to \mathcal{B}_h^{''}$

$$\Upsilon\, x(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0] \\ \mathcal{R}_q(t)\Big[\mathscr{G}(0,\phi)\Big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) \\ \quad - \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\ \quad - \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\Big[\mathscr{F}\Big(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big) \\ \quad +\mathcal{B}u_x(s)\Big]ds - \sum_{0<t_k<t} \mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(x(t_k)), \quad t \in \mathscr{I} \end{cases} \tag{4.5}$$

has a fixed point. This fixed point is a mild solution of a given system. Clearly, $\Upsilon x(T) = x_1$, which implies the fractional system (4.1)-(4.3) is controllable on $\mathscr{I}$.
For $\Upsilon \in \mathcal{B}_h$, we define $y : (\infty, T] \to \mathbb{X}$ by

$$y(t) = \begin{cases} \phi(t), \quad t \le 0 \\ \mathcal{R}_q(t)\phi(0), \quad t \in \mathscr{I}. \end{cases}$$

If $x(.)$ satisfies above equation, then we can split it as $x(t) = z(t) + y(t)$, $t \in \mathscr{I}$, which implies that $x_t = z_t + y_t$ for $t \in \mathscr{I}$ and the function $z(.)$ satisfies

$$\begin{aligned} z(t) = &-\mathcal{R}_q(t)\mathscr{G}(0,\phi) + \mathscr{G}\Big(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}\Big) \\ &+ \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\Big)ds \\ &+ \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\Bigg[\mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \end{aligned}$$

$$\int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)+\mathcal{B}u_{z+y}(s)\Big]ds$$
$$+\sum_{0<t_k<t}\mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k)),\quad t\in\mathscr{I},$$

where

$$\begin{aligned}u_{z+y}(s)=\mathcal{W}^{-1}\Bigg[&x_1-\mathcal{R}_q(T)\mathscr{G}(0,\phi)-\mathscr{G}(t,z_{\varrho(t,z_t+y_t)}+y_{\varrho(t,z_t+y_t)})\\&-\int_0^T(T-s)^{q-1}\mathcal{A}\mathcal{S}_q(T-s)\mathscr{G}(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)})ds\\&-\int_0^T(T-s)^{q-1}\mathcal{S}_q(T-s)\Big[\mathscr{F}(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},\\&\int_0^s\mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)})d\tau)\Big]ds-\sum_{0<t_k<t}\mathcal{R}_q(t-t_k)\mathcal{I}_k(x(t_k))\Bigg](t),\quad t\in\mathscr{I}.\end{aligned}$$

Let $\mathcal{B}''_h=\{z\in\mathcal{B}'_h : z_0=0\}$. Let $\|.\|_{\mathcal{B}''_h}$ be the seminorm in $\mathcal{B}''_h$ described by :

$$\begin{aligned}\|z\|_{\mathcal{B}''_h}&=\sup_{s\in\mathscr{I}}\|z(t)_{\mathbb{X}}\|+\|z_0\|_{\mathcal{B}_h}\\&=\sup_{s\in\mathscr{I}}\|z(t)\|_{\mathbb{X}},\quad z\in\mathcal{B}''_h.\end{aligned}$$

Define the operator $\hat{\Upsilon}:\mathcal{B}''_h\to\mathcal{B}''_h$ by

$$(\hat{\nu}z)(t)=\begin{cases}0,\quad t\in(-\infty,0]\\-\mathcal{R}_q(t)\mathscr{G}(0,\phi)+\mathscr{G}\Big(t,z_{\varrho(t,z_t+y_t)}+y_{\varrho(t,z_t+y_t)}\Big)\\+\int_0^t\mathcal{A}\mathcal{S}_q(t)(t-s)\mathscr{G}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)}\Big)ds\\+\int_0^t\mathcal{S}_q(t)(t-s)\Big[\mathscr{F}\Big(s,z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)},\\\int_0^s\mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)}+y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big)+\mathcal{B}u_y(s)\Big]ds\\+\sum_{0<t_k<t}\mathcal{R}_q(t-t_k)\mathcal{I}_k(z(t_k)+y(t_k)),\quad t\in\mathscr{I}.\end{cases}$$

So our main goal is to show that $\hat{\Upsilon}$ has a fixed point and the proof is given in the following steps:

Step 1 : We show that there exists some $r>0$, such that $\hat{\Upsilon}(\mathcal{B}_r)\subset\mathcal{B}_r$. If it is not true for each positive number r, there exists a function $z^r(.)\in\mathcal{B}_r$ and some $t\in\mathscr{I}$ such that $\|\hat{\Upsilon}(z^r)(t)\|>r$.
Then by hypotheses $(H2)(i)$,(ii), $(H3)$, Lemma 2.3(i), and Hölder's inequality, we obtain

$$\begin{aligned}r&<\|(\hat{\Upsilon}\ z)^r(t)\|\\&\le\Big\|-\mathcal{R}_q(t)\Big[\mathscr{G}(0,\phi)\Big]+\mathscr{G}(t,x_{\varrho(t,x_t)})\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{S}_q(t-s)\mathscr{G}(s, x_{\varrho(s,x_s)})ds \\
&- \int_0^t (t-s)^{q-1}\mathcal{S}_q(t-s)\Big[\mathscr{F}\Big(s, x_{\varrho(s,x_s)}, \int_0^s \mathfrak{W}(s,\tau,x_{\varrho(\tau,x_\tau)})d\tau\Big) + \mathcal{B}u_y(s)\Big]ds \\
&- \sum_{0<t_k<t} \mathcal{R}_q(t)(t-t_k)\mathcal{I}_k(x(t_k))\Big\|, \quad t \in \mathscr{I}
\end{aligned}
$$

$$
\begin{aligned}
\leq \hat{\mathcal{M}}_1 \mathcal{W}_{\mathscr{G}} \|\mathcal{A}^\beta\|(1 + \|\phi\|_{\mathcal{B}_h}) &+ \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \mathscr{D}_1^* r + c_n) \\
&+ K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1 + \mathscr{D}_1^* r + c_n) \\
&+ \frac{\hat{\mathcal{M}}_2\mathcal{M}_4 q}{\Gamma(1+q)}\aleph_{\mathscr{G}}\Big[(\mathscr{D}_1^* r + c_n) + TN_1(1 + (\mathscr{D}_1^* r + c_n))\Big] \\
&+ \frac{\hat{\mathcal{M}}_2\mathcal{M}_2 q}{\Gamma(1+q)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_y^r\|_{\mathcal{L}^2} + m\hat{\mathcal{M}}_1 L_0 H\Big[\mathscr{D}_1^* r + \tilde{c}_n\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\|u_y^r\|_{\mathcal{L}^2} \leq \mathcal{M}_3\Bigg[&\|x_1\| + \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \|\phi\|_{\mathcal{B}_h}) + \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \mathscr{D}_1^* r + c_n) \\
&+ K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1 + \mathscr{D}_1^* r + c_n) \\
&+ \frac{\hat{\mathcal{M}}_2\mathcal{M}_4 q}{\Gamma(1+q)}\aleph_{\mathscr{G}}\Big[(\mathscr{D}_1^* r + c_n) + TN_1(1 + (\mathscr{D}_1^* r + c_n))\Big] \\
&+ m\hat{\mathcal{M}}_1 L_0 H[\mathscr{D}_1^* r + \widetilde{c_n}]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \|\phi\|_{\mathcal{B}_h}) + \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \mathscr{D}_1^* r + c_n) \\
&\quad + K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1 + \mathscr{D}_1^* r + c_n) + \frac{\hat{\mathcal{M}}_2\mathcal{M}_4 q}{\Gamma(1+q)}\aleph_{\mathscr{G}}\Big[(\mathscr{D}_1^* r + c_n) \\
&+ TN_1(1 + (\mathscr{D}_1^* r + c_n))\Big] + \frac{\hat{\mathcal{M}}_2\mathcal{M}_2 q}{\Gamma(1+q)}\sqrt{\frac{b^{2q-1}}{2q-1}}\Bigg[\|x_1\| + \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \|\phi\|_{\mathcal{B}_h}) \\
&+ \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \mathscr{D}_1^* r + c_n) + K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1 + \mathscr{D}_1^* r + c_n) \\
&+ \frac{\hat{\mathcal{M}}_2\mathcal{M}_4 q}{\Gamma(1+q)}\aleph_{\mathscr{G}}\Big[(\mathscr{D}_1^* r + c_n) + TN_1(1 + (\mathscr{D}_1^* r + c_n))\Big] + m\hat{\mathcal{M}}_1 L_0 H[\mathscr{D}_1^* r + \widetilde{c_n}]\Bigg] \\
&+ m\hat{\mathcal{M}}_1 L_0 H\Big[\mathscr{D}_1^* r + \tilde{c}_n\Big].
\end{aligned}
$$

$$r \leq \left(1 + \frac{\hat{\mathcal{M}}_2\mathcal{M}_2\mathcal{M}_3 q}{\Gamma(1+q)}\sqrt{\frac{b^{2q-1}}{2q-1}}\right)\Bigg[\|x_1\| + \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{\beta}\|(1+\|\phi\|_{\mathcal{B}_h}) +$$
$$\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{\beta}\|(1+\mathscr{D}_1^* r + c_n) + K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^* r + c_n)$$
$$+ \frac{\hat{\mathcal{M}}_2\mathcal{M}_4 q}{\Gamma(1+q)}\aleph_{\mathscr{G}}\Big[(\mathscr{D}_1^* r + c_n) + TN_1(1+(\mathscr{D}_1^* r + c_n))\Big]$$
$$+ m\hat{\mathcal{M}}_1 L_0 H\Big[\mathscr{D}_1^* r + \tilde{c}_n\Big]\Bigg]. \tag{4.6}$$

Let $\iota = \Big[(\mathscr{D}_1^* r + c_n) + TN_1(1+(\mathscr{D}_1^* r + c_n))\Big]$ and $N_{\mathfrak{F}} = \dfrac{\hat{\mathcal{M}}_2\mathcal{M}_4 q}{\Gamma(1+q)}$. Note that $\iota \to \infty$ as $r \to \infty$. Now dividing both sides by r, taking limit as $r \to \infty$, we get,

$$1 \leq \frac{1}{r}\left(1 + \frac{\hat{\mathcal{M}}_2\mathcal{M}_2\mathcal{M}_3 q}{\Gamma(1+q)}\sqrt{\frac{b^{2q-1}}{2q-1}}\right)\Bigg[\|x_1\| + \hat{\mathcal{M}}_1\mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{\beta}\|(1+\|\phi\|_{\mathcal{B}_h})$$
$$+ \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^{\beta}\|(1+\mathscr{D}_1^* r + c_n) + K(q,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{q\beta}}{q\beta}(1+\mathscr{D}_1^* r + c_n)$$
$$+ N_{\mathfrak{F}} \lim_{\iota\to\infty} \inf \frac{\aleph_{\mathscr{G}}(\iota)}{\iota} + m\hat{\mathcal{M}}_1 L_0 H\Big[\mathscr{D}_1^* r + \tilde{c}_n\Big]\Bigg]. \tag{4.7}$$

We get $1 \leq 0$. This is a contradiction. Hence, for some positive integer r, $\hat{\Upsilon}(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

Step 2 : $\hat{\Upsilon}$ is continuous on $\mathcal{B}_r$.

Let ${z^n}_{n\in N} \subset \mathcal{B}_r$ with $z^n \to z$ in $\mathcal{B}_r$ as $n \to \infty$.

Denote

$$\mathcal{F}_n(s) = \mathscr{F}\Big(s, z^n_{\varrho(s,z^n_s+y_s)} + y_{\varrho(s,z^n_s+y_s)}, \int_0^s \mathfrak{W}(s,\tau,z^n_{\varrho(\tau,z^n_\tau+y_\tau)} + y_{\varrho(\tau,z^n_\tau+y_\tau)})d\tau\Big)$$
$$\mathcal{F}(s) = \mathscr{F}\Big(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}, \int_0^s \mathfrak{W}(s,\tau,z_{\varrho(\tau,z_\tau+y_\tau)} + y_{\varrho(\tau,z_\tau+y_\tau)})d\tau\Big).$$

Then by using the hypotheses $(H3), (H6)$ and Lebesgue's Dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{q-1}\|\mathcal{F}_n(s) - \mathcal{F}(s)\|ds \to 0 \quad as \quad n\to\infty, \quad t\in\mathscr{I}. \tag{4.8}$$

Now,

$$\|\hat{\Upsilon}z^n - \hat{\Upsilon}z\|_{\mathcal{C}} \leq \frac{\hat{\mathcal{M}}_2 q}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|\mathcal{F}_n(s) - \mathcal{F}(s)\|ds$$
$$+ \sqrt{\frac{b^{2q-1}}{2q-1}}\ \frac{q\hat{\mathcal{M}}_1\mathcal{M}_2}{\Gamma(1+q)}\|u^n_y - u_y\|_{\mathcal{L}^2}. \tag{4.9}$$

Observing that (4.8), (4.9) and (4.10), we have $\|\hat{\Upsilon}z^n - \hat{\Upsilon}z\|_{\mathcal{C}} \to 0 \quad as \quad n \to \infty$, which implies that $\hat{\Upsilon}$ is continuous on $\mathcal{B}_r$.

Step 3: $\hat{\Upsilon}(\mathcal{B}_r)$ is equicontinuous on $\mathscr{I}$.

Let $Z \in \hat{\Upsilon}(\mathcal{B}_r)$ and $0 \leq t_1 \leq t_2 \leq T$. Then there is $z \in \mathcal{B}_r$ such that

$$\begin{aligned}
&\|Z(t_2) - Z(t_1)\| \\
&\quad \leq \| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{S}_q(t)(t_2 - s)\Big[F(s) + Bu_y(s)\Big]ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{S}_q(t)(t_1 - s)\Big[F(s) + Bu_y(s)\Big]ds\| \\
&\quad \leq \| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{G}_q(t)(t_2 - s)\Big[F(s) + Bu_y(s)\Big]ds\| \\
&\quad + \| \int_{t_1-\in}^{t_1} (t_2 - s)^{q-1} \Big[\mathcal{S}_q(t)(t_2 - s) - \mathcal{G}_q(t)(t_1 - s)\Big]\Big[F(s) + Bu_y(s)\Big]ds\| \\
&\quad + \| \int_{t_1-\in}^{t_1} \Big[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\Big]\mathcal{S}_q(t)(t_1 - s)\Big[F(s) + Bu_y(s)\Big]ds\| \\
&\quad + \| \int_0^{t_1-\in} (t_2 - s)^{q-1} \Big[\mathcal{S}_q(t)(t_2 - s) - \mathcal{S}_q(t)(t_1 - s)\Big]\Big[F(s) + Bu_y(s)\Big]ds\| \\
&\quad + \| \int_0^{t_1-\in} \Big[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\Big]\mathcal{S}_q(t)(t_1 - s)\Big[F(s) + Bu_y(s)\Big]ds\|.
\end{aligned}$$

Using Lemma 2.3, we can verify that the right hand side of the above inequality tends to zero as $t_2 \to t_1$.

As a consequence of steps 1-3 together with the Ascoli-Arzela Theorem, we conclude that $\hat{\Upsilon} : \mathcal{B}_h'' \to \mathcal{B}_h''$ is completely continuous. As a consequence of Schaefer's fixed theorem we deduce that Z has a fixed point, which is a mild solution of the problem (1.1)-(1.3). Hence, the system (4.1)-(4.3) is controllable on $\mathscr{I}$. □

5. Applications

Example 1. Consider the fractional neutral impulsive integro-differential equations with state dependent delay of the form:

$$\begin{aligned}
&{}^C D_t^q \Big[u(t,x) - \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \varrho_1(s)\varrho_2(\|u(s)\|), x)}{49} ds\Big] \\
&\quad = \frac{\partial^2}{\partial x^2} u(t,x) + \int_{-\infty}^{t} e^{2(s-t)} \frac{u(\tau - \varrho_1(\tau)\varrho_2(\|u(s)\|), x)}{9} ds \\
&\quad + \mu(t,x) + \int_0^t sin(t-s) \int_{-\infty}^{s} e^{2(\tau - s)} \frac{u(\tau - \varrho_1(\tau)\varrho_2(\|u(\tau)\|), x)}{25} d\tau ds
\end{aligned} \tag{5.1}$$

$$u(t,0) = 0 = u(t,\pi), \quad t \geq 0, \tag{5.2}$$

$$u(t,x) = \phi(t,x) \quad t \in [-\infty, 0], \; x \in [0,\pi] \tag{5.3}$$

$$\Delta u(t_k)(x) = \int_{-\infty}^{t_k} \mathscr{Q}_k(t_k - s)u(s,x)dx, \; x \in [0,\pi]. \tag{5.4}$$

where ${}^C D_t^q$ is Caputo's fractional of order $0 < q < 1, \; 0 < t_1 < t_2 < ... < t_m < T$ are positive numbers, and $\phi \in \mathcal{B}_h$. We consider $\mathbb{X} = L^2[0,\pi]$ with the norm $|.|_{L^2}$ and defined the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ by $\mathcal{A}\mathscr{W} = \mathscr{W}''$ with the domain:

$\mathcal{D}(\mathcal{A}) = \{\mathscr{W} \in \mathbb{X} : \mathscr{W}, \mathscr{W}''\text{are absolutely continuous}, \mathscr{W}'' \in \mathbb{X}, \; \mathscr{W}(0) = \mathscr{W}(\pi) = 0\}.$

Then

$$\mathcal{A}\mathscr{W} = \sum_{n=1}^{\infty} n^2 \langle \mathscr{W}, \mathscr{W}_n \rangle \mathscr{W}_n, \quad \mathscr{W} \in \mathcal{D}(\mathcal{A}),$$

where $\mathscr{W}_n(s) = \sqrt{\frac{2}{\pi}} sin(ns)$, n=1,2,...,is the orthogonal set of eigenfunctions of $(-\mathcal{A})$. It is well known that $(-\mathcal{A})$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{R}(t)\}_{t\geq 0}$ in $\mathbb{X}$ and is given by

$$\mathcal{R}(t)\mathscr{W} = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \mathscr{W}, \; \mathscr{W}_n \rangle \mathscr{W}_n, \quad for \;\; all \; \mathscr{W} \in \mathbb{X}, \quad and \;\; t > 0.$$

Then the operator $(-\mathcal{A})^{\frac{1}{2}}$ is given by

$$(-\mathcal{A})^{\frac{1}{2}} \mathscr{W} = \sum_{n=1}^{-\infty} n \langle \mathscr{W}, \mathscr{W}_n \rangle \mathscr{W}_n, \; \mathscr{W} \in \Big(\mathcal{D}(-\mathcal{A})^{\frac{1}{2}}\Big),$$

in which

$$\Big(\mathcal{D}(-\mathcal{A})^{\frac{1}{2}}\Big) = \Big\{\mathscr{W}(.) \in \mathbb{X} : \sum_{n=1}^{\infty} n \langle \mathscr{W}, \mathscr{W}_n \rangle \mathscr{W}_n \in \mathbb{X}\Big\} \text{ and } \Big\|(-\mathcal{A})^{-\frac{1}{2}}\Big\| = 1.$$

Therefore, we conclude that $((-\mathcal{A}))$ is of sectorial type and the corresponding properties hold. For phase space, we choose $\mathcal{H} = e^{2s}, \; s < 0$, then $l = \int_{-\infty}^{0} \mathcal{H}(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$, and determine

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} \mathcal{H}(s) \; \sup_{\theta \in [s,0]} \; \|\phi(\theta)\|_{\mathcal{L}^2} ds.$$

Hence, $(t,\phi) \in [0,T] \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta,x), (\theta,x) \in (-\infty,0] \times [0,\pi]$.
Set

$$u(t)(x) = u(t,x), \quad \rho(t,\phi) = \rho_1(t)\rho_2(\|\phi(0)\|),$$

we have

$$\mathscr{G}(t,\phi)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{\phi}{49} ds,$$

$$\mathscr{F}(t,\phi,\mathscr{H}\phi)(x)=\int_{-\infty}^{0}e^{2(s)}\frac{\phi}{9}ds+\mathscr{H}\phi(x),$$

and

$$\mathcal{I}_k(\phi)(x)=\int_{-\infty}^{0}\mathscr{Q}_k(\theta)\phi(\theta)(x)d\theta,\quad k=1,2,...,m,$$

where

$$(\mathscr{H}\phi)(x)=\int_0^t sin(t-s)\int_{-\infty}^{0}e^{2\tau}\frac{\phi}{25}d\tau ds.$$

Further, we define the operator $\mathcal{B}:U\to\mathbb{X}$ by $\mathcal{B}u(t,x)=\mu(t,x),\ \ 0<x<\pi$, $u\in U$, where $\mu:[0,1]\times[0,\pi]\to[0,\pi]$, then using these configurations, the system (4.1)-(4.3) is usually written in the theoretical form of system (1.1)-(1.3).
To treat this system we assume that $\varrho_i:[0,\infty]\to[0,\infty),\ i=1,2$ are continuous. Now, we can see that $t\in[0,1]$, $\phi\in\mathcal{B}_h$, we have

$$\begin{aligned}&\left\|(-\mathcal{A})^{\frac{1}{2}}\mathscr{G}(t,\phi_1)-(-\mathcal{A})^{\frac{1}{2}}\mathscr{G}(t,\phi_2)\right\|\\
&\leq\left(\int_0^{\pi}\left(\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\phi_1}{49}\right\|ds+\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\phi_2}{49}\right\|ds\right)^2dx\right)^{\frac{1}{2}}\\
&\leq\left(\int_0^{\pi}\left(\frac{1}{49}\int_{-\infty}^{0}e^{2(s)}\sup\|\phi_1\|ds+\frac{1}{49}\int_{-\infty}^{0}e^{2(s)}\sup\|\phi_2\|ds\right)^2dx\right)^{\frac{1}{2}}\\
&\leq\frac{\sqrt{\phi}}{49}\left\|\phi_1\right\|_{\mathcal{B}_h}+\frac{\sqrt{\phi}}{49}\left\|\phi_2\right\|_{\mathcal{B}_h}\\
&\leq\frac{\sqrt{\phi}}{49}\left\|(\phi_1-\phi_2)\right\|_{\mathcal{B}_h}.\end{aligned}$$

Similarly, we conclude

$$\begin{aligned}&\left\|\mathscr{F}(t,\pi,\mathscr{H}_\pi)\right\|_{\mathcal{L}^2}\\
&\leq\left(\int_0^{\pi}\left(\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\phi}{9}\right\|ds+\int_0^t\left\|sin(t-s)\right\|\int_{-\infty}^{0}e^{2(\tau)}\left\|\frac{\phi}{25}\right\|d\tau ds\right)^2dx\right)^{\frac{1}{2}}\\
&\leq\left(\int_0^{\pi}\left(\frac{1}{9}\int_{-\infty}^{0}e^{2(s)}\sup\|\phi\|ds+\frac{1}{25}\int_{-\infty}^{0}e^{2(s)}\sup\|\phi\|ds\right)^2dx\right)^{\frac{1}{2}}\\
&\leq\frac{\sqrt{\phi}}{9}\left\|\phi\right\|_{\mathcal{B}_h}+\frac{\sqrt{\phi}}{25}\left\|\phi\right\|_{\mathcal{B}_h}\end{aligned}$$

$$\le [m_1(t) + \sqrt{\pi} m_2(t)]\|\phi\|_{\mathcal{B}_h}$$
$$\le m_{\mathscr{F}}(t)(\|\phi\|_{\mathcal{B}_h}),$$

where $m_{\mathscr{F}}(t) = \dfrac{34\sqrt{\pi}}{225}$.
Therefore the condition $(H1) - (H7)$ are all satisfied and

$$\frac{1}{r}\left(1 + \frac{\hat{\mathcal{M}}_2 \mathcal{M}_2 \mathcal{M}_3 q}{\Gamma(1+q)}\sqrt{\frac{b^{2q-1}}{2q-1}}\right)\Bigg[\|x_1\| + \hat{\mathcal{M}}_1 \mathcal{W}_{\mathscr{G}} \|\mathcal{A}^\beta\|(1 + \|\phi\|_{\mathcal{B}_h})$$
$$+ \mathcal{W}_{\mathscr{G}}\|\mathcal{A}^\beta\|(1 + \mathscr{D}_1^* r + c_n) + K(\alpha,\beta)\mathcal{W}_{\mathscr{G}}\frac{T^{\alpha\beta}}{\alpha\beta}(1 + \mathscr{D}_1^* r + c_n)$$
$$+ N_{\mathfrak{F}} \lim_{\iota\to\infty} \inf \frac{\aleph_{\mathscr{G}}(\iota)}{\iota} + m\hat{\mathcal{M}}_1 L_0 H\Big[\mathscr{D}_1^* r + \tilde{c}_n\Big]\Bigg] < 1.$$

Hence by Theorem (4.1), we realize that the system (4.1)-(4.3) has a unique mild solution on $[0,1]$.

Example 2. In this section, we provide an illustration of the existence results for an impulsive fractional neutral integro differential with state dependent delay of the form

$$^{C}D_t^q\Big[u(t,x) - \int_{-\infty}^{t} \Upsilon_1(t,x,s-t)u(s-\rho_1(t)\rho_2(\|u(t)\|),x)ds\Big]$$
$$= \frac{\partial^2}{\partial x^2}u(t,x) + \int_{-\infty}^{t}\Upsilon_2(t,x,s-t)\mathscr{P}_1(u(s-\rho_1(t)\rho_2(\|u(t)\|),x))ds$$
$$+ \int_0^t\int_{-\infty}^{s}\mathscr{K}_1(s-\tau)\mathscr{P}_2(u(\tau-\rho_1(t)\rho_2(\|u(\tau)\|),x))d\tau\ ds,$$
$$x\in[0,\pi],\quad 0\le t\le T,\quad t\ne t_k, \tag{5.5}$$
$$u(t,0) = 0 = u(t,\pi),\quad t\ge 0, \tag{5.6}$$
$$u(t,x) = \phi(t,x),\quad t\in(-\infty,0],\quad x\in[0,\pi], \tag{5.7}$$
$$\Delta u(t_k)(x) = \int_{-\infty}^{t_k}\mathscr{Q}_k(t_k-s)u(s,x)ds,\quad x\in[0,\pi],\quad k=1,2,...,m, \tag{5.8}$$

where $^{C}D_t^q$ is Caupto's fractional derivative of order $0<q<1$, $0<t_1<t_2<...<t_n<T$ are pre-fixed numbers and $\phi\in\mathcal{B}_h$. We consider $\mathbb{X} = L^2[0,\pi]$ with the norm $|.|_{L^2}$ and defined the operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathbb{X}\to\mathbb{X}$ by $\mathcal{A}\mathscr{W} = \mathscr{W}''$ with the domain:

$\mathcal{D}(\mathcal{A}) = \{\mathscr{W}\in\mathbb{X} : \mathscr{W},\mathscr{W}''$ are absolutely continuous, $\mathscr{W}''\in\mathbb{X}, \mathscr{W}(0) = \mathscr{W}(\pi) = 0\}$.

Then

$$\mathcal{A}\mathscr{W} = \sum_{n=1}^{\infty} n^2\langle\mathscr{W},\mathscr{W}_n\rangle\mathscr{W}_n,\quad \mathscr{W}\in\mathcal{D}(\mathcal{A}),$$

where $\mathscr{W}_n(s) = \sqrt{\frac{2}{\pi}} sin(ns),\ n = 1, 2, ...,$ is the orthogonal set of eigenfunctions of $\mathcal{A}$. It is well known that $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{R}(t)\}_{t\geq 0}$ in $\mathbb{X}$ and is given by:

$$\mathcal{R}(t)\mathscr{W} = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \mathscr{W}, \mathscr{W}_n\rangle \mathscr{W}_n, \quad for \quad all\ \mathscr{W} \in \mathbb{X}, \quad and \quad t > 0.$$

From these outflows, it follows that $(\mathcal{R}(t))_{t\geq 0}$ is a uniformly bounded semigroup, so that $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ is a sectorial operator for all $\lambda \in \rho(\mathcal{A})$; that is, $\mathcal{A} \in \mathbb{A}^{\alpha}(\theta_0, \mathscr{W}_0)$. In addition, the subordination principle of solution operator $(\mathcal{S}_\alpha(t))_{t\geq 0}$ such that $\|\mathcal{S}_\alpha(t)\|_{L(\mathbb{X})} \leq \hat{\mathcal{M}}_2$ for $t \in [0, b]$. For phase space, we choose $\mathcal{H} = e^{2s},\ s < 0$, then $l = \int_{-\infty}^{0} \mathcal{H}(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$, and determine

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} \mathcal{H}(s)\ \sup_{\theta\in[s,0]}\ \|\phi(\theta)\|_{\mathcal{L}^2} ds.$$

Hence, $(t, \phi) \in [0, T] \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set

$$u(t)(x) = u(t, x), \quad \rho(t, \phi) = \rho_1(t)\rho_2(\|\phi(0)\|),$$

we have

$$\mathscr{G}(t, \phi)(x) = \int_{-\infty}^{0} \Upsilon_1(\theta)\phi(\theta)(x)d\theta,$$

$$\mathscr{F}(t, \phi, \mathscr{H}\phi)(x) = \int_{-\infty}^{0} \Upsilon_2(t, x, \theta)\mathscr{P}_1(\phi(\theta)(x))d\theta + \mathscr{H}\phi(x),$$

and

$$\mathcal{I}_k(\phi)(x) = \int_{-\infty}^{0} \mathscr{Q}_k(\theta)\phi(\theta)(x)d\theta, \quad k = 1, 2, ..., m,$$

where

$$\mathscr{H}\phi(x) = \int_0^t \int_{-\infty}^{0} \mathscr{K}_1(s - \theta)\mathscr{P}_2(\phi(\theta)(x))d\theta\ ds,$$

By all these configurations, system (4.1)-(4.4) can be written in the theoretical form of problem (1.1)-(1.3).
Suppose further that:

(i) the function $\rho_i : [0, \infty)\ to\ [0, \infty),\ i = 1, 2$ are continuous;
(ii) the function $\Upsilon_1(t, x, \theta)$ is continuous in $[0, T] \times [0, \pi] \times (-\infty, 0]$; and $\Upsilon_1(t, x, \theta) \geq 0$;

$$\int_{-\infty}^{0} \Upsilon_1(t, x, \theta)d\theta = p_1(t, x) < \infty.$$

(iii) the functions $\Upsilon_i(t,x,\theta) \geq 0,\ i=1,2,$ are continuous in $[0,T]\times[0,\pi]\times(-\infty,0]$ and satisfy

$$\int_{-\infty}^{0} \Upsilon_2(t,x,\theta)d\theta = p_1(t,x) < \infty. \quad \Big(\int_0^{\pi} p_1^2(t,x)dx\Big)^{\frac{1}{2}} = m_1(t) < \infty,$$

(iv) the function $k(t-s)$ is continuous in $[0,T]$ and $k(t-s)\geq 0$, and

$$\int_0^t \int_{-\infty}^{s} k(s-\theta)d\theta\ ds = m_2(t) < \infty,$$

(v) the functions $q_k : \mathbb{R}\to\mathbb{R},\ k=1,2,...,m$ are continuous and

$$d_i = \int_{-\infty}^{0} h(s)q_i^2(s)ds < \infty, \quad for \ \ i=1,2,...,n.$$

(vi) the function $P(.)$ is continuous and for each $(0,x)\ \in(-\infty,0]\ \times[0,\pi]$;

$$0 \leq P_i(u(\theta)(x)) \leq \Theta_{\mathscr{F}}\Big(\int_{-\infty}^{0} e^{2s}\|u(s,.)\|_{L^2}ds\Big) \text{ with} \lim\inf_{r\to\infty}\frac{\Theta_{\mathscr{F}}(r)}{r} = \Lambda < \infty,$$

where

$\Theta_{\mathscr{F}},\ \Theta_{\mathscr{F}} : [0,\infty)\to(0,\infty)$ is a continuous and nondecreasing functions.

Thus, under all these conditions, we have

$$\begin{aligned}
&\Big\|\mathcal{A}^{\frac{1}{2}}\mathscr{G}(t,\phi_1) - \mathcal{A}^{\frac{1}{2}}\mathscr{G}(t,\phi_2)\Big\|_{L^2}\\
&\leq \left[\int_0^{\pi}\left(\int_{-\infty}^{0}\Upsilon_1(\theta)(\phi_1(\theta)-\phi_2(\theta))(x)d\theta\right)^2 dx\right]^{\frac{1}{2}}\\
&\leq \left(\int_{-\infty}^{0}\Upsilon_1^2(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_0^{\pi}\int_{-\infty}^{0}(\phi_1(\theta)-\phi_2(\theta))^2(x)d\theta\ dx\right)^{\frac{1}{2}}\\
&\leq \left(\int_{-\infty}^{0}\Upsilon_1^2(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}\int_0^{\pi}(\phi_1(\theta)-\phi_2(\theta))^2(x)dx\ d\theta\right)^{\frac{1}{2}}\\
&\leq \left(\int_{-\infty}^{0}\Upsilon_1^2(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}e^{-2s\theta}.\ e^{2s\theta}\int_0^{\pi}(\phi_1(\theta)-\phi_2(\theta))^2(x)dx\ d\theta\right)^{\frac{1}{2}}\\
&\leq \left(\int_{-\infty}^{0}\Upsilon_1^2(\theta)d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}e^{-2s\theta}d\theta\right)^{\frac{1}{2}}.\sup_{-\infty<\theta\leq 0}\\
&\quad\left\{e^{s\theta}\left(\int_0^{\pi}(\phi_1(\theta)-\phi_2(\theta))^2(x)dx\right)^{\frac{1}{2}}\right\}\\
&\leq \left(\frac{-1}{2s}\int_{-\infty}^{0}\Upsilon_1^2(\theta)d\theta\right)^2.\sup_{-\infty<\theta\leq 0}\left\{e^{2s}\Big\|\phi_1(\theta)-\phi_2(\theta)\Big\|_{L^2}\right\}
\end{aligned}$$

$$\leq \mathcal{W}_{\mathscr{G}} \|\phi_1 - \phi_2\|_{\mathcal{B}_h}.$$

Therefore, hypotheses (H1) holds. Similarly, we have

$$\Big\| \mathscr{F}(t, \phi, \mathscr{H}_\phi) \Big\|_{L^2}$$

$$\leq \Bigg[\int_0^\pi \Bigg\{ \int_{-\infty}^0 \Upsilon_2(t, x, \theta) \mathscr{P}_1(\phi(\theta)(x)) d\theta$$

$$+ \int_0^t \int_{-\infty}^0 \mathscr{K}_1(s - \theta) \mathscr{P}_2(\phi(\theta)(x)) d\theta \ ds \Bigg\}^2 dx \Bigg]^{\frac{1}{2}}$$

$$\leq \Bigg[\int_0^\pi \Bigg\{ \int_{-\infty}^0 \Upsilon_2(t, x, \theta) \Theta_{\mathscr{F}}(\big\|\phi(\theta)(.)\big\|_{L^2}) d\theta$$

$$+ \int_0^t \int_{-\infty}^0 k_1(s - \theta) \Theta_{\mathscr{F}}(\big\|\phi(\theta)(.)\big\|_{L^2}) d\theta \ ds \Bigg\}^2 dx \Bigg]^{\frac{1}{2}}$$

$$\leq \Bigg[\int_0^\pi \Bigg\{ \int_{-\infty}^0 \Upsilon_2(t, x, \theta) \Theta_{\mathscr{F}} \Bigg(\sup_{-\infty<\theta\leq 0} \Big\{ e^{2s} \big\|\phi(\theta)\big\|_{L^2} \Big\} \Bigg) d\theta$$

$$+ \int_0^t \int_{-\infty}^0 \mathscr{K}_1(s - \theta) \Theta_{\mathscr{F}} \Bigg(\sup_{-\infty<\theta\leq 0}$$

$$\Big\{ e^{2s} \big\|\phi(\theta)\big\|_{L^2} \Big\} \Bigg) d\theta \ ds \Bigg\}^2 dx \Bigg]^{\frac{1}{2}}$$

$$\leq \Bigg[\int_0^\pi \Bigg\{ \int_{-\infty}^0 \Upsilon_2(t, x, \theta) \mathscr{P}_1(\phi(\theta)(x)) d\theta \Bigg\}^2 dx \Bigg]^{\frac{1}{2}} \Theta_{\mathscr{F}}(\|\phi\|_{\mathcal{B}_h}$$

$$+ \Bigg[\int_0^\pi \Bigg\{ \int_0^t \int_{-\infty}^0 \mathscr{K}_1(s - \theta) d\theta \ ds \Bigg\}^2 dx \Bigg]^{\frac{1}{2}} \Theta_{\mathscr{F}}(\|\phi\|_{\mathcal{B}_h})$$

$$\leq \Big(\int_0^\pi p_1^2(t, x) dx \Big)^{\frac{1}{2}} \Theta_{\mathscr{F}}(\|\phi\|_{\mathcal{B}_h} + \Big(\int_0^\pi q_1^2(t) dx \Big)^{\frac{1}{2}} \Theta_{\mathscr{F}}(\|\phi\|_{\mathcal{B}_h})$$

$$\leq \big[m_1(t) + \sqrt{\pi} m_2(t) \big] \Theta_{\mathscr{F}}(\|\phi\|_{\mathcal{B}_h})$$

$$\leq m_{\mathscr{F}}(t) \Theta_{\mathscr{F}}(\|\phi\|_{\mathcal{B}_h}).$$

Since $\Theta_{\mathscr{F}} : [0, \infty) \to (0, \infty)$ is a continuous and non-decreasing function, we can take $m_1(t) = \bar{m}_1(t)$ with $a = \sqrt{\pi}$ and $\xi_{\mathscr{F}}(r) = \Theta_{\mathscr{F}}(r)$ in (H2). If the bounds in Step (1) are fulfilled, then model (4.1)-(4.4) has a mild solution on $\mathscr{I}$.

6. Conclusion

In this article, we have been detailed the existence and controllability for impulsive fractional neutral integro-differential systems with SDD conditions via sectorial operator in a Banach space. Further accurate, by utilizing the fractional calculus, fractional powers of operators, and the Schaefer's fixed point theorem, we examine the IFNIDE with state dependent delay in a Banach space. The fractional differential equations are every efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability.

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