J. Appl. & Pure Math. Vol. **4***(2022), No. 3 - 4, pp. 85 - 97*
https://doi.org/10.23091/japm.2022.085

PAIR MEAN CORDIAL LABELING OF GRAPHS OBTAINED FROM PATH AND CYCLE

R. PONRAJ*, S. PRABHU

Abstract. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we investigate the pair mean cordial labeling of graphs which are obtained from path and cycle.

AMS Mathematics Subject Classification : 05C78.
Key words and phrases : Path, cycle, crown, middle graph, split graph and friendship graph.

1. Introduction

In this paper, the graph $G = (V, E)$ is a finite, simple, undirected connected and non trivial graph. We follow the graph theory related basic terminologies and different notations by the book of Harary [12]. For a detailed survey on graph labeling, we refer the survey of Gallian [11]. The notion of cordial labeling was introduced by Cahit [5]. Also cordial related labeling was studied in [1-4,6-10,13]. Recently pair mean cordial labeling has been introduced in [16] and investigate the pair mean cordial labeling behavior of path, cycle, comb, wheel, helm, ladder.

Received April 4, 2022. Revised June 22, 2022. Accepted June 30, 2022. *Corresponding author.

In this paper, we discussed the pair mean cordial labeling of graphs obtained from path and cycle like crown, $C_n \odot K_2$, $P_n \odot K_2$, middle graph of the path and cycle, splitting graph of the path and cycle, friendship graph, subdivision of comb.

2. Pair Mean Cordial Labeling Of Graphs

Definition 2.1. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called a pair mean cordial graph.

3. Preliminaries

Definition 3.1. The *Middle graph* $M(G)$ of a graph G is the graph whose vertex set is $\{u : u \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1e_2 : e_1, e_2 \in E(G)$ and e_1 and e_2 are adjacent edges of $G\}$ $\cup\{ue : u \in V(G), e \in E(G)$ and e is incident with $u\}$.

Definition 3.2. For a graph G, the *split graph* which is denoted by $spl(G)$ is obtained by adding to each vertex u a new vertex v such that v is adjacent to every vertex that is adjacent to u in G.

Definition 3.3. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then the join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is given by $E_1 \cup E_2 \cup \{uv : u \in V_1$ and $v \in V_2\}$.

Definition 3.4. The subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G.

Definition 3.5. The *crown* $C_n \odot K_1$ is obtained by joining a pendent edge to each vertex of C_n

Definition 3.6. The *friendship graph* F_n is a graph which consists of n triangles with a common vertex.

Definition 3.7. The *double fan graph* $D(f_n) = P_n + 2K_1, n \geq 2$ is a graph which consists of two fan graph that have a common path.

4. Main Results

Theorem 4.1. $P_n \odot 2K_1$ is pair mean cordial for all $n \geq 1$.

Proof. Let P_n be the path $u_1u_2\dots u_n$. Let $V(P_n \odot 2K_1) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(P_n \odot 2K_1) = E(P_n) \cup \{u_iv_i, u_iw_i : 1 \leq i \leq n\}$. Then $P_n \odot K_1$ has $3n$ vertices and $3n-1$ edges. This proof is divided into four cases:
Case 1: $n \equiv 0(mod 4)$
First assign the labels $-1, -3, -5, \dots, \frac{-3n+2}{2}$ to the vertices $u_1, u_2, u_3, \dots, u_{\frac{3n}{4}}$ respectively. We assign the labels $-2, -4, -6, \dots, \frac{-n}{2}$ respectively to the vertices $u_{\frac{3n+4}{4}}, u_{\frac{3n+8}{4}}, u_{\frac{3n+12}{4}}, \dots, u_n$. Then we assign the labels $2, 4, 6, \dots, \frac{3n}{2}$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{3n}{4}}$ respectively. Next we assign the labels $\frac{-n-8}{2}, \frac{-n-16}{2}, \dots, \frac{-3n}{2}$ respectively to the vertices $v_{\frac{3n+4}{4}}, v_{\frac{3n+8}{4}}, \dots, v_n$. Now we give labels $3, 5, 7, \dots, \frac{3n-2}{2}$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{3n-4}{4}}$ respectively. We give labels $\frac{-n-4}{2}, \frac{-n-12}{2}, \dots, \frac{-3n+4}{2}$ respectively to the vertices $w_{\frac{3n}{4}}, w_{\frac{3n+4}{4}}, \dots, w_{n-1}$. Finally assign the label 1 to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.
Case 2: $n \equiv 1(mod 4)$
First assign the labels $-1, -3, -5, \dots, \frac{-3n+1}{2}$ to the vertices $u_1, u_2, u_3, \dots, u_{\frac{3n+1}{4}}$ respectively. We assign the labels $-2, -4, -6, \dots, \frac{-n+1}{2}$ respectively to the vertices $u_{\frac{3n+5}{4}}, u_{\frac{3n+9}{4}}, u_{\frac{3n+13}{4}}, \dots, u_n$. Then we assign the labels $2, 4, 6, \dots, \frac{3n-3}{2}$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{3n-3}{4}}$ respectively. Next we assign the labels $\frac{-n-3}{2}, \frac{-n-11}{2}, \frac{-n-19}{4}, \dots, \frac{-3n+5}{2}$ respectively to the vertices $v_{\frac{3n+1}{4}}, v_{\frac{3n+5}{4}}, v_{\frac{3n+9}{4}}, \dots, v_{n-1}$. Assign the label $\frac{n+3}{2}$ to the vertex v_n. Now we give labels $3, 5, 7, \dots, \frac{3n-1}{2}$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{3n-3}{4}}$ respectively. We give labels $\frac{-n-7}{2}, \frac{-n-15}{2}, \frac{-n-23}{2}, \dots, \frac{-3n+2}{2}$ respectively to the vertices $w_{\frac{3n+1}{4}}, w_{\frac{3n+5}{4}}, w_{\frac{3n+9}{4}}, \dots, w_{n-1}$. Finally assign the label 1 to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.
Case 3: $n \equiv 2(mod 4)$
First assign the labels $-1, -3, -5, \dots, \frac{-3n}{2}$ to the vertices $u_1, u_2, u_3, \dots, u_{\frac{3n+2}{4}}$ respectively. We assign the labels $-2, -4, -6, \dots, \frac{-n+2}{2}$ respectively to the vertices $u_{\frac{3n+6}{4}}, u_{\frac{3n+10}{4}}, u_{\frac{3n+14}{4}}, \dots, u_n$. Then we assign the labels $2, 4, 6, \dots, \frac{3n-2}{2}$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{3n-2}{4}}$ respectively. Next we assign the labels $\frac{-n-2}{2}, \frac{-n-10}{2}, \frac{-n-18}{4}, \dots, \frac{-3n+2}{2}$ respectively to the vertices $v_{\frac{3n+2}{4}}, v_{\frac{3n+6}{4}}, v_{\frac{3n+10}{4}}, \dots, v_n$. Now we give labels $3, 5, 7, \dots, \frac{3n}{2}$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{3n-2}{4}}$ respectively. We give labels $\frac{-n-6}{2}, \frac{-n-14}{2}, \frac{-n-22}{2}, \dots, \frac{-3n+4}{2}$ respectively to the vertices $w_{\frac{3n+2}{4}}, w_{\frac{3n+6}{4}}, w_{\frac{3n+10}{4}}, \dots, w_{n-1}$. Finally assign the label 1 to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.
Case 4: $n \equiv 3(mod 4)$
First assign the labels $-1, -3, -5, \dots, \frac{-3n+3}{2}$ to the vertices $u_1, u_2, u_3, \dots, u_{\frac{3n-1}{4}}$ respectively. We assign the labels $-2, -4, -6, \dots, \frac{-n-1}{2}$ respectively to the vertices $u_{\frac{3n+3}{4}}, u_{\frac{3n+7}{4}}, u_{\frac{3n+11}{4}}, \dots, u_n$. Then we assign the labels $2, 4, 6, \dots, \frac{3n-1}{2}$ to

the vertices $v_1, v_2, v_3, \ldots, v_{\frac{3n-1}{4}}$ respectively. Next we assign the labels $\frac{-n-9}{2}$, $\frac{-n-17}{2}, \frac{-n-25}{2}, \ldots, \frac{-3n-1}{2}$ respectively to the vertices $v_{\frac{3n+3}{4}}, v_{\frac{3n+7}{4}}, v_{\frac{3n+11}{4}}, \ldots,$ v_{n-1}. Assign the label $\frac{n+3}{2}$ to the vertex v_n. Now we give labels $3, 5, 7, \ldots,$ $\frac{3n-3}{2}$ to the vertices $w_1, w_2, w_3, \ldots, w_{\frac{3n-5}{4}}$ respectively. We give labels $\frac{-n-5}{2}$, $\frac{-n-13}{2}, \frac{-n-21}{2}, \ldots, \frac{-3n+1}{2}$ respectively to the vertices $w_{\frac{3n-1}{4}}, w_{\frac{3n+3}{4}}, w_{\frac{3n+7}{4}}, \ldots,$ w_{n-1}. Finally assign the label 1 to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$. □

Theorem 4.2. $P_n \odot K_2$ is pair mean cordial for all $n \geq 1$.

Proof. Let P_n be the path $u_1u_2 \ldots u_n$. Let $V(P_n \odot K_2) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_2) = E(P_n) \cup \{u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\}$. Then $P_n \odot K_2$ has $3n$ vertices and $4n-1$ edges. This proof is divided into two cases:

Case 1: n is odd

First assign the label -1 to the vertex u_1. We assign the labels $3, 6, 9, \ldots, \frac{3n-3}{2}$ to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$ respectively. Then we assign the labels $-3, -6, -9,$ $\ldots, \frac{-3n+3}{2}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_n$. Next we assign the labels $1, 4, 7, \ldots, \frac{3n-1}{2}$to the vertices $v_1, v_3, v_5, \ldots, v_n$ respectively. Now we give labels $-1, -4, -7, \ldots, \frac{-3n+7}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$. Also assign the label $2, 5, 8, \ldots, \frac{3n-5}{2}$ to the vertices $w_1, w_3, w_5, \ldots, w_{n-2}$ respectively. We give labels $-2, -5, -8, \ldots, \frac{-3n+5}{2}$ respectively to the vertices $w_2, w_4, w_6, \ldots,$ w_{n-1}. Finally assign the label $\frac{-3n+1}{2}$ to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = 2n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n$.

Case 2: n is even

First assign the label -1 to the vertex u_1. We assign the labels $3, 6, 9, \ldots, \frac{3n}{2}$ to the vertices $u_2, u_4, u_6, \ldots, u_n$ respectively. Then we assign the labels $-3, -6, -9,$ $\ldots, \frac{-3n+6}{2}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_{n-1}$. Next we assign the labels $1, 4, 7, \ldots, \frac{3n-4}{2}$to the vertices $v_1, v_3, v_5, \ldots, v_{n-1}$ respectively. Assign the label $\frac{-3n}{2}$ to the vertex v_2. Now we give labels $-1, -4, -7, \ldots, \frac{-3n+4}{2}$ respectively to the vertices $v_4, v_6, v_8, \ldots, v_n$. Also assign the label $2, 5, 8, \ldots, \frac{3n-2}{2}$ to the vertices $w_1, w_3, w_5, \ldots, w_{n-1}$ respectively. Finally we give labels $-2, -5, -8,$ $\ldots, \frac{-3n+2}{2}$ respectively to the vertices $w_2, w_4, w_6, \ldots, w_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = 2n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n$. □

Theorem 4.3. The crown $C_n \odot K_1$ is pair mean cordial for all $n \geq 3$.

Proof. Let C_n be the cycle $u_1u_2 \ldots u_nu_1$. Let $V(C_n \odot K_1) = V(C_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = E(C_n) \cup \{u_iv_i, : 1 \leq i \leq n\}$. Then $C_n \odot K_1$ has $2n$ vertices and $2n$ edges. This proof is divided into four cases:

Case 1: $n = 3$

Define $\lambda(u_1) = -1, \lambda(u_2) = 3, \lambda(u_3) = -3, \lambda(v_1) = 2, \lambda(v_2) = -2, \lambda(v_3) = 1$. Hence $\bar{\mathbb{S}}_{\lambda_1} = 3$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 3$.

Case 2: $n = 4$

Define $\lambda(u_1) = -1, \lambda(u_2) = 3, \lambda(u_3) = -3, \lambda(u_4) = -4, \lambda(v_1) = 2, \lambda(v_2) =$

$-2, \lambda(v_3) = 4, \lambda(v_4) = 1$. Hence $\bar{\mathbb{S}}_{\lambda_1} = 4$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 4$.

Case 3: $n = 5$

Define $\lambda(u_1) = -1, \lambda(u_2) = 3, \lambda(u_3) = -3, \lambda(u_4) = 5, \lambda(u_5) = -5, \lambda(v_1) = 2, \lambda(v_2) = -2, \lambda(v_3) = 4, f(v_4) = 1 f(v_5) = -4$. Hence $\bar{\mathbb{S}}_{\lambda_1} = 5$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 5$.

Case 4: $n > 5$

There are four subcases arises:

Subcase 1: $n \equiv 0(mod4)$

First assign the labels $-1, -3, -5, \ldots, -n+1$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-1}$ respectively. We assign the labels $3, 5, 7, \ldots, n-1$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-2}$. Next we assign the label 1 to the vertex u_n. Then we assign the labels $2, 4, 6, \ldots, \frac{n+4}{2}$ to the vertices $v_1, v_3, v_5, \ldots, v_{\frac{n+2}{2}}$ respectively. Also we assign the labels $-2, -4, -6, \ldots, \frac{-n}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{\frac{n}{2}}$. Now we give labels $\frac{-n-8}{2}, \frac{-n-4}{2}$ to the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}$ respectively. We give labels $\frac{-n-12}{2}, \frac{-n-16}{2}, \ldots, -n$ respectively to the vertices $v_{\frac{n+8}{2}}, v_{\frac{n+10}{2}}, \ldots, v_{\frac{3n+4}{4}}$. Finally we give labels $\frac{n+8}{2}, \frac{n+12}{2}, \ldots, n$ to the vertices $v_{\frac{3n+8}{4}}, v_{\frac{3n+12}{4}}, \ldots, v_n$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = n$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.

Subcase 2: $n \equiv 1(mod4)$

First assign the labels $-1, -3, -5, \ldots, -n$ to the vertices $u_1, u_3, u_5, \ldots, u_n$ respectively. We assign the labels $3, 5, 7, \ldots, n$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$. Then we assign the labels $2, 4, 6, \ldots, \frac{n+3}{2}$ to the vertices $v_1, v_3, v_5, \ldots, v_{\frac{n+1}{2}}$ respectively. Next we assign the labels $-2, -4, -6, \ldots, \frac{-n+1}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{\frac{n-1}{2}}$. Now we give labels $\frac{-n-7}{2}, \frac{-n-3}{2}$ to the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}$ respectively. We give labels $\frac{-n-11}{2}, \frac{-n-15}{2}, \ldots, -n+1$ respectively to the vertices $v_{\frac{n+7}{2}}, v_{\frac{n+9}{2}}, \ldots, v_{\frac{3n+1}{4}}$. Finally we give labels $\frac{n+7}{2}, \frac{n+11}{2}, \ldots, n-1$ to the vertices $v_{\frac{3n+5}{4}}, v_{\frac{3n+9}{4}}, \ldots, v_{n-1}$ respectively. Then assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.

Subcase 3: $n \equiv 2(mod4)$

First assign the labels $-1, -3, -5, \ldots, -n+1$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-1}$ respectively. We assign the labels $3, 5, 7, \ldots, n-1$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-2}$. Assign the label 1 to the vertex u_n. Then we assign the labels $2, 4, 6, \ldots, \frac{n+2}{2}$ respectively to the vertices $v_1, v_3, v_5, \ldots, v_{\frac{n}{2}}$. Next we assign the labels $-2, -4, -6, \ldots, \frac{-n-2}{2}$ to the vertices $v_2, v_4, v_6, \ldots, v_{\frac{n+2}{2}}$ respectively. Now we give labels $\frac{-n-6}{2}, \frac{-n-10}{2}, \ldots, -n$ respectively to the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, \ldots, v_{\frac{3n+2}{4}}$. Finally we give labels $\frac{n+6}{2}, \frac{n+10}{2}, \ldots, n$ to the vertices $v_{\frac{3n+6}{4}}, v_{\frac{3n+10}{4}}, \ldots, v_n$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = n$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.

Subcase 4: $n \equiv 3(mod4)$

First assign the labels $-1, -3, -5, \ldots, -n$ to the vertices $u_1, u_3, u_5, \ldots, u_n$ respectively. We assign the labels $3, 5, 7, \ldots, n$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$. Then we assign the labels $2, 4, 6, \ldots, \frac{n+1}{2}$ to the vertices $v_1, v_3, v_5,$

$\ldots, v_{\frac{n-1}{2}}$ respectively. Next we assign the labels $-2, -4, -6, \ldots, \frac{-n-1}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{\frac{n+1}{2}}$. Now we give labels $\frac{-n-5}{2}, \frac{-n-9}{2}, \ldots,$ $-n+1$ to the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \ldots, v_{\frac{3n-1}{4}}$ respectively. Next we give labels $\frac{n+5}{2}, \frac{n+9}{2}, \ldots, n-1$ respectively to the vertices $v_{\frac{3n+3}{4}}, v_{\frac{3n+7}{4}}, \ldots, v_{n-1}$. Finally assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$. □

Theorem 4.4. *$C_n \odot 2K_1$ is pair mean cordial for all $n \geq 3$.*

Proof. Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let $V(C_n \odot 2K_1) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(C_n \odot 2K_1) = E(C_n) \cup \{u_iv_i, u_iw_i : 1 \leq i \leq n\}$. Then $C_n \odot 2K_1$ has $3n$ vertices and $3n$ edges. This proof is divided into two cases:

Case 1: n is odd

First assign the label -1 to the vertex u_1. We assign the labels $3, 6, 9, \ldots, \frac{3n-3}{2}$ to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$ respectively. Then we assign the labels $-3, -6, -9,$ $\ldots, \frac{-3n+3}{2}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_n$. Next we assign the labels $1, 4, 7, \ldots, \frac{3n-1}{2}$to the vertices $v_1, v_3, v_5, \ldots, v_n$ respectively. Now we give labels $-2, -5, -8, \ldots, \frac{-3n+5}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$. Assign the label $2, 5, 8, \ldots, \frac{3n-5}{2}$ to the vertices $w_1, w_3, w_5, \ldots, w_{n-2}$ respectively. We give labels $-4, -7, -10, \ldots, \frac{-3n+1}{2}$ respectively to the vertices $w_2, w_4, w_6, \ldots,$ w_{n-1}. Finally assign the label $\frac{-3n+1}{2}$ to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-1}{2}$.

Case 2: n is even

First assign the label -1 to the vertex u_1. We assign the labels $3, 6, 9, \ldots, \frac{3n}{2}$ to the vertices $u_2, u_4, u_6, \ldots, u_n$ respectively. Then we assign the labels $-3, -6, -9,$ $\ldots, \frac{-3n+6}{2}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_{n-1}$. Next we assign the labels $1, 4, 7, \ldots, \frac{3n-4}{2}$to the vertices $v_1, v_3, v_5, \ldots, v_{n-1}$ respectively. Now we give labels $-2, -5, -8, \ldots, \frac{-3n+2}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_n$. Assign the label $2, 5, 8, \ldots, \frac{3n-2}{2}$ to the vertices $w_1, w_3, w_5, \ldots, w_{n-1}$ respectively. We give labels $-4, -7, -10, \ldots, \frac{-3n+4}{2}$ respectively to the vertices $w_2, w_4, w_6,$ $\ldots, w_{n-2}$. Finally assign the label $\frac{-3n}{2}$ to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n}{2}$. □

Theorem 4.5. *$C_n \odot K_2$ is pair mean cordial for all $n \geq 3$.*

Proof. Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let $V(C_n \odot K_2) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_2) = E(C_n) \cup \{u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\}$. Then $C_n \odot K_2$ has $3n$ vertices and $4n$ edges. This proof is divided into two cases:

Case 1: n is odd

First assign the labels $3, 6, 9, \ldots, \frac{3n-3}{2}$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-2}$ respectively. Then we assign the label -2 to the vertex u_2. Next we assign the labels $-6, -9, -12, \ldots, \frac{-3n+3}{2}$to the vertices $u_4, u_6, \ldots, u_{n-1}$ respectively. Assign the label 1 to the vertex u_n. Now we give labels $-1, -4, -7, \ldots, \frac{-3n+1}{2}$ respectively in the vertices $v_1, v_3, v_5, \ldots, v_n$. Then we assign the label -3 to the vertex v_2. We give labels $-5, -8, -11, \ldots, \frac{-3n+5}{2}$ respectively to the vertices $v_4, v_6, v_8, \ldots,$

v_{n-1}. Assign the label $2, 5, 8, \ldots, \frac{3n-5}{2}$ to the vertices $w_1, w_3, w_5, \ldots, w_{n-2}$ respectively. We give labels $4, 7, 10, \ldots, \frac{3n-1}{2}$ respectively to the vertices $w_2, w_4, w_6, \ldots, w_{n-1}$. Finally assign the label 1 to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = 2n$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n$.

Case 2: n is even

First assign the labels $3, 6, 9, \ldots, \frac{3n}{2}$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-1}$ respectively. Then we assign the labels $-2, -5$ respectively to the vertices u_2, u_4. Next we assign the labels $-9, -12, \ldots, \frac{-3n}{2}$ to the vertices $u_6, u_8, \ldots, u_n$ respectively. Now we give labels $-1, -4, -7, \ldots, \frac{-3n+4}{2}$ respectively to the vertices $v_1, v_3, v_5, \ldots, v_{n-1}$. Then we assign the labels $-3, -6$ respectively to the vertices v_2, v_4. We give labels $-8, -11, -14, \ldots, \frac{-3n+2}{2}$ respectively to the vertices $v_6, v_8, v_{10}, \ldots, v_n$. Assign the label $2, 5, 8, \ldots, \frac{3n-2}{2}$ to the vertices $w_1, w_3, w_5, \ldots, w_{n-1}$ respectively. We give labels $4, 7, 10, \ldots, \frac{3n-4}{2}$ respectively to the vertices $w_2, w_4, w_6, \ldots, w_{n-2}$. Finally assign the label 1 to the vertex w_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = 2n$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n$. □

Theorem 4.6. The middle graph of the path P_n, $M(P_n)$ is pair mean cordial for all $n \geq 1$.

Proof. Let $V(M(P_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n-1\}$ and $V(M(P_n)) = \{u_i v_i, v_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\}$. Then $M(P_n)$ has $2n-1$ vertices and $3n-4$ edges. This proof is divided into two cases:

Case 1: n is odd

Now assign the labels $2, 3, 4, \ldots, \frac{n+3}{2}$ to the vertices $u_1, u_2, u_3, \ldots, u_{\frac{n+1}{2}}$ respectively. We assign the labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, -n+1$ respectively to the vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \ldots, u_{n-1}$. Assign the label $-n+1$ to the vertex u_n. Now we give labels $-1, -2, \ldots \frac{-n-1}{2}$ to the vertices $v_1, v_2, \ldots, v_{\frac{n+1}{2}}$ respectively. Next we give labels $\frac{-n-5}{2}, \frac{-n-7}{2}, \ldots, -n+1$ respectively to the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \ldots, v_{n-2}$. Finally assign the label 1 to the vertex v_{n-1}. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-3}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-5}{2}$.

Case 2: n is even

First assign the labels $2, 3, 4, \ldots, \frac{n+2}{2}$ to the vertices $u_1, u_2, u_3, \ldots, u_{\frac{n}{2}}$ respectively. We assign the labels $\frac{-n-2}{2}, \frac{-n-4}{2}, \ldots, -n+1$ respectively to the vertices $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \ldots, u_{n-1}$. Assign the label $-n+1$ to the vertex u_n. Now we give labels $-1, -2, \ldots \frac{-n}{2}$ to the vertices $v_1, v_2, \ldots, v_{\frac{n}{2}}$ respectively. Next we give labels $\frac{n+4}{2}, \frac{n+6}{2}, \ldots, n-1$ respectively to the vertices $v_{\frac{n+2}{2}}, v_{\frac{n+4}{2}}, \ldots, v_{n-2}$. Finally assign the label 1 to the vertex v_{n-1}. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-4}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-4}{2}$. □

Theorem 4.7. *The middle graph of the cycle C_n, $M(C_n)$ is pair mean cordial if only if $n \geq 5$.*

Proof. Let $V(M(C_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $V(M(C_n)) = \{u_i v_i : 1 \leq i \leq n\} \cup \{v_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n u_1, v_n v_1\}$. Then $M(C_n)$ has $2n$ vertices and $3n$ edges. This proof is divided into four cases:

Case 1: $n = 3$

suppose $M(C_3)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 3. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 6$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 6 - 3 = 3 > 1$, a contradiction.

Case 2: $n = 4$

suppose $M(C_4)$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 5. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 5$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 7$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 7 - 5 = 2 > 1$, a contradiction.

Case 3: $n = 5$

Assign the labels $-1, -2, -3, -4, -5$ to the vertices u_1, u_2, u_3, u_4, u_5 respectively. Next we assign the labels $2, 3, 4, 5, 1$ respectively to the vertices v_1, v_2, v_3, v_4, v_5. Hence $\bar{\mathbb{S}}_{\lambda_1} = 7$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 8$.

Case 4: $n > 5$

There are two subcases arises:

Subcase 1: n is odd

First we give labels $-1, -2, -3, \cdots - n$ to the vertices $u_1, u_2, u_3, \ldots, u_n$ respectively. Next we give labels $2, 3, 4, \ldots, \frac{n+7}{2}$ respectively to the vertices $v_1, v_2, v_3, \ldots, v_{\frac{n+5}{2}}$. Assign the label 1 to the vertex $v_{\frac{n+7}{2}}$. Finally we assign the labels $\frac{n+9}{2}, \frac{n+11}{2}, \ldots, n$ to the vertices $v_{\frac{n+9}{2}}, v_{\frac{n+11}{2}}, \ldots, v_n$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n+1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-1}{2}$.

Subcase 2: n is even

First we give labels $-1, -2, -3, \cdots - n$ to the vertices $u_1, u_2, u_3, \ldots, u_n$ respectively. Next we give labels $2, 3, 4, \ldots, \frac{n+6}{2}$ respectively to the vertices $v_1, v_2, v_3, \ldots, v_{\frac{n+4}{2}}$. Assign the label 1 to the vertex $v_{\frac{n+6}{2}}$. Finally we assign the labels $\frac{n+8}{2}, \frac{n+10}{2}, \ldots, n$ respectively to the vertices $v_{\frac{n+8}{2}}, v_{\frac{n+10}{2}}, \ldots, v_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n}{2}$. □

Theorem 4.8. The splitting graph of the path P_n, $spl(P_n)$ is pair mean cordial for all $n \geq 1$.

Proof. Let $V(spl(P_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(spl(P_n)) = \{u_iu_{i+1}, u_iv_{i+1}, u_{i+1}v_i : 1 \leq i \leq n-1\}$. Then $spl(P_n)$ has $2n$ vertices and $3n-3$ edges. This proof is divided into two cases:

Case 1: n is odd

First assign the labels $1, 3, 5, \ldots, n$ to the vertices $u_1, u_3, u_5, \ldots, u_n$ respectively. Next we assign the labels $-1, -3, -5, \ldots, -n+2$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$. Now we give labels $2, 4, 6, \ldots n-1$ to the vertices $v_1, v_3, v_5, \ldots, v_{n-2}$ respectively. Then we give labels $-2, -4, -6, \ldots, -n+1$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$. Finally assign the label $-n$ to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-3}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-3}{2}$.

Case 2: n is even

First assign the labels $1, 3, 5, \ldots, n-1$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-1}$ respectively. Next we assign the labels $-1, -3, -5, \ldots, -n+1$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_n$. Now we give labels $2, 4, 6, \ldots n$ to the vertices $v_1, v_3, v_5, \ldots, v_{n-1}$ respectively. Finally we give labels $-2, -4, -6, \ldots, -n$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-4}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-2}{2}$. □

Theorem 4.9. The splitting graph of the cycle C_n, $spl(C_n)$ is pair mean cordial if and only if $n \geq 5$.

Proof. Let $V(spl(C_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(spl(C_n)) = \{u_iu_{i+1}, u_iv_{i+1}, u_{i+1}v_i : 1 \leq i \leq n-1\} \cup \{u_nu_1, v_nu_1, u_nv_1\}$. Then $spl(C_n)$ has $2n$ vertices and $3n$ edges. This proof is divided into four cases:

Case 1: $n = 3$

suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v) = 1$ or $\lambda(u)+\lambda(v) = 2$. Hence the maximum number of edges label 1 is 3. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 6$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 6-3 = 3 > 1$, a contradiction.

Case 2: $n = 4$

suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v) = 1$ or $\lambda(u)+\lambda(v) = 2$. Hence the maximum number of edges label 1 is 5. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 5$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 7$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 7-5 = 2 > 1$, a contradiction.

Case 3: $n = 5$

First assign the labels $-2, 3, -3, 5, 2$ to the vertices u_1, u_2, u_3, u_4, u_5 respectively. Then we assign the labels $-1, 4, -4, -5, 1$ respectively to the vertices v_1, v_2, v_3, v_4, v_5. Hence $\bar{\mathbb{S}}_{\lambda_1} = 7$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 8$.

Case 4: $n > 5$

There are two subcases arises:

Subcase 1: n is odd

First assign the labels $-2, 2$ to the vertices u_1, u_n respectively. Next we assign the labels $3, 5, 7, \ldots, n$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$. Now we give labels $-3, -5, -7, \cdots - n+2$ to the vertices $u_3, u_5, u_7 \ldots, u_{n-2}$ respectively. Then we give labels $-1, 1$ respectively to the vertices v_1, v_n. We assign the labels $4, 6, 8, \ldots, n-1$ to the vertices $v_2, v_4, v_6, \ldots, v_{n-3}$. Assign the labels $-4, -6, -8, \ldots, -n+1$ to the vertices $v_3, v_5, v_7, \ldots, v_{n-2}$. Finally assign the label $-n$ to the vertex v_{n-1}. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n+1}{2}$.

Subcase 2: n is even

First we assign the labels $-2, 2$ to the vertices u_1, u_n respectively. Assign the labels $3, -3$ respectively to the vertices u_2, u_3. Assign the labels $6, -6$ to the vertices u_4, u_5 respectively. Next we assign the labels $7, 9, \ldots, n-1$ to the vertices $u_6, u_8, \ldots, u_{n-2}$ respectively. Now we give labels $-7, -9, \cdots - n+1$ respectively to the vertices $u_7, u_9, \ldots, u_{n-1}$. Then we give labels $-1, 1$ to the vertices v_1, v_n respectively. Assign the labels $4, -4$ respectively to the vertices v_2, v_3. Assign the labels $5, -5$ to the vertices v_4, v_5 respectively. We assign the labels $8, 10, \cdots - n$ respectively to the vertices $v_6, v_8, \ldots v_{n-2}$. Finally assign the labels $-8, -10, \ldots, -n$ to the vertices $v_7, v_9, \ldots, v_{n-1}$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n}{2}$. □

Theorem 4.10. The subdivision of $P_n \odot K_1$, $S(P_n \odot K_1)$ is pair mean cordial.

Proof. Let P_n be the path $u_1u_2 \ldots u_n$. Let $V(S(P_n \odot K_1)) = \{u_i, v_i, y_i : 1 \leq i \leq n\} \cup \{x_i : 1 \leq i \leq n-1\}$ and $E(S(P_n \odot K_1)) = \{u_iy_i, y_iv_i : 1 \leq i \leq n\} \cup \{u_ix_i, x_iu_{i+1} : 1 \leq i \leq n-1\}$. Then $S(P_n \odot K_1)$ has $4n-1$ vertices and $4n-2$ edges.
Then we define the function $\lambda : V(S(P_n \odot K_1)) \to \{\pm 1, \pm 2, \ldots, \pm 2n-1\}$ by

$$\begin{aligned}
\lambda(u_i) &= -2i, \text{for } 1 \leq i \leq n-1,\\
\lambda(u_n) &= 1,\\
\lambda(v_i) &= -2i+1, \text{for } 1 \leq i \leq n,\\
\lambda(x_i) &= 2i+1, \text{for } 1 \leq i \leq n-1,\\
\lambda(y_i) &= 2i, \text{for } 1 \leq i \leq n-1,\\
\lambda(y_n) &= 1.
\end{aligned}$$

Hence $\bar{\mathbb{S}}_{\lambda_1} = 2n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n-1$. □

Theorem 4.11. P_n^2 is pair mean cordial if only if $n \leq 3$.

Proof. Also $P_1^2 \simeq P_1$ and $P_2^2 \simeq P_2$ are pair mean cordial[16]. This proof is divided into two cases:
friendship graph**Case 1**: $n = 3$
Define $\lambda(u_1) = 1$, $\lambda(u_2) = 1$ and $\lambda(u_1) = -1$. Then $\bar{\mathbb{S}}_{\lambda_1} = 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2$.
Case 2: $n > 3$
suppose P_n^2 is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $n-3$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n - (n-3) = 3 > 1$, a contradiction.
□

Theorem 4.12. The *friendship graph* F_n is pair mean cordial if and only if $n \leq 1$.

Proof. Let $V(F_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(F_n) = \{uu_i, vv_i, u_iv_i : 1 \leq i \leq n\}$. Then F_n has $2n+1$ vertices and $3n$ edges. This proof is divided into three cases:
Case 1: $n = 1$
Now $F_1 \simeq C_3$ is pair mean cordial.
Case 2: $2 \leq n \leq 3$
Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is n. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - n = n > 1$, a contradiction.

Case 3: $n > 3$
Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $n + 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n + 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n - 1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq (2n - 1) - (n + 1) = n - 2 > 1$, a contradiction. □

Theorem 4.13. The *double fan graph* $D(f_n) = P_n + 2K_1, n \geq 2$ is not pair mean cordial.

Proof. Let $V(D(f_n)) = \{u, v, u_i : 1 \leq i \leq n\}$ and $E(D(f_n)) = \{uu_i, vu_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1\}$. Then $D(f_n)$ has $n + 2$ vertices and $3n - 1$ edges. This proof is divided into three cases:
Case 1: $n = 2$
$D(f_2) \simeq C_4$ is not pair mean cordial.
Case 2: $n = 3$
Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 6$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 6 - 2 = 4 > 1$, a contradiction.
Case 3: $n > 3$
There are two subcases arises:
Subcase 1: n is odd
Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is n. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n - 1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq (2n - 1) - n = n - 1 > 1$, a contradiction.
Subcase 2: n is even
Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $n - 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - (n - 1) = n + 1 > 1$, a contradiction. □

Theorem 4.14. The *double cone* $C_n + 2K_1$ is not pair mean cordial.

Proof. Let $V(C_n + 2K_1) = \{u, v, u_i : 1 \leq i \leq n\}$ and $E(C_n + 2K_1) = \{uu_i, vu_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup u_nu_1$. Then $C_n + 2K_1$ has $n + 2$ vertices and $3n$ edges. This proof is divided into two cases:
Case 1: $n = 3$
Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 7$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 7 - 2 = 5 > 1$, a contradiction.
Case 2: $n > 3$
There are two subcases arises:
Subcase 1: n is odd

Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is n. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - n = n > 1$, a contradiction.

Subcase 2: n is even

Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label 1 is $n - 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n - 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n + 1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq (2n + 1) - (n - 1) = n + 2 > 1$, a contradiction. □

5. Discussion

The concept of pair difference cordial labeling was introduced in [14] and properties of pair difference cordial labeling have studied in [15]. Also Mean labeling of graphs was introduced in [17]. Motivated by these two concepts we have introduced the pair mean cordial labeling in [16]. In this paper we investigate the pair mean cordial labeling of some graphs like crown, $C_n \odot K_2$, $P_n \odot K_2$, middle graph of the path and cycle, splitting graph of the path and cycle, friendship graph and subdivision of comb.

6. Limitation of Research

It is more difficult to investigate the pair mean cordial labeling behavior of broken wheel graph, n-cube graph and grid graph.

7. Future Research

Pair mean cordial labeling behavior of butterfly graph, dumbbell graph, theta graph, windmill graph, shadow graph, shell graph and mobious ladder are the possible future directions of research work.

8. Conclusion

Cahit[5] introduced the concept of cordial labeling of graphs. Since then, cordial labeling of graphs has become an active area of research. Cordial labeling behavior of path, cycle, completed graph, some union of graphs, some cartesian product of graphs, join of some graphs, etc., was studied by several authors[11]. The concept of pair mean cordial labeling of graphs has been introduced in [16]. In the present paper, we have brought out the results concerning the pair mean cordial labeling of graphs obtained from path and cycle like crown, $C_n \odot K_2$, $P_n \odot K_2$, middle graph of the path and cycle, splitting graph of the path and cycle, friendship graph and subdivision of comb. Based on our work, it emerges that a study on the pair mean cordial labeling behavior of some other special graphs like spider graph, multiple shell graph, umbrella graph and torch graph would constitute the open problems for the future research work.

Acknowledgement : The authors thank the Referee for the valuable suggestions towards the improvement of the paper.

References

1. M. Andar, S. Boxwala and N. Limaye, *Cordial labeling of some wheel related graphs*, J. Combin. Math. Combin. Compt. **41** (2002), 203-208.
2. M. Andar, S. Boxwala and N. Limaye, *A note on cordial labeling of multiple shells*, Trends Math. (2002), 77-80.
3. M. Andar, S. Boxwala and N. Limaye, *On the Cordiality of the t-uniform homeomorphs-I*, Ars Combin. **66** (2003), 313-318.
4. M. Andar, S. Boxwala and N. Limaye, *New families of cordial graphs*, J. Combin. Math. Combin. Compt. **53** (2005), 117-154.
5. I. Cahit, *Cordial graphs: a weaker versionof graceful and harmonious graphs*, Ars comb. **23** (1987), 201-207.
6. I. Cahit, *H-Cordial graphs*, Bull. Inst. Combin. Appl. **18** (1996), 87-101.
7. G. Chartrand, S.M. Lee and P. Zhang, *Uniformly cordial graphs*, Discrete Math. **306** (2006), 726-737.
8. A.T. Diab, *A study of some problems of cordial graphs*, Ars Combin. **92** (2009), 255-261.
9. A.T. Diab, On Cordial labeling of some results on cordial graphs, *Ars Combin.* **99** (2011), 161-173.
10. A.T. Diab and E. Elsakhawi, *Some results on cordial graphs*, Proc. Maths. Phys. Soc. Egypt. **77** (2002), 67-87.
11. J.A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics **24** (2021).
12. F. Harary, *Graph theory*, Addison Wesely, Reading Mass., 1972.
13. M. Hovey, *A-cordial graphs*, Discrete Math. **93** (1991), 183-194.
14. R. Ponraj, A. Gayathri and S. Somasundaram, *Pair difference cordial labeling of graphs*, J. Math. Compt. Sci. **11** (2021), 2551-2567.
15. R. Ponraj, A. Gayathri and S. Somasundaram, *Some pair difference cordial graphs*, Ikonion Journal of Mathematics **3** (2021), 17-26.
16. R. Ponraj and S. Prabhu, *Pair mean cordial labeling of graphs*, Journal of Algorithms and Computation **54** (2022), 1-10.
17. S. Somasundaram and R. Ponraj, *Mean labeling of graphs*, National Academy Science Letter **26** (2003), 210-213.

R. Ponraj did his Ph.D. in Manonmaniam Sundaranar University, Tirunelveli, India. He has guided 8 Ph.D. scholars and published around 140 research papers in reputed journals. He is an author of five books for undergraduate students. His research interest in Graph Theory. He is currently an Assistant Professor at Sri Paramakalyani College, Alwarkurichi, India.

Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, Tamilnadu, India.
e-mail: ponrajmaths@gmail.com

S. Prabhu did his M.Sc degree in Sri Kaliswari College, Sivakasi and M.Phil degree at Madurai Kamaraj University, Madurai, India. His research interest is in Graph Theory.

Research Scholar, Register number: 21121232091003, Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, Tamilnadu, India (Affiliated to Manonmaniam Sundaranar University, Abhishekapatti, Tirunelveli– 627 012, Tamilnadu, India).
e-mail: selvaprabhu12@gmail.com