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CHEN INVARIANTS AND STATISTICAL SUBMANIFOLDS

Hitoshi Furuhata, Izumi Hasegawa, and Naoto Satoh

Abstract. We define a kind of sectional curvature and δ-invariants for statistical manifolds. For statistical submanifolds the sum of the squared mean curvature and the squared dual mean curvature is bounded below by using the δ-invariant. This inequality can be considered as a generalization of the so-called Chen inequality for Riemannian submanifolds.

1. Introduction

For a Riemannian manifold, B.-Y. Chen introduced functions $\delta_{(m_1,\ldots,m_k)}$, new kinds of curvatures, which are defined in terms of sectional curvature and its generalizations. They are now called Chen's delta-invariants. He established inequalities for Riemannian submanifolds which involve his delta-invariant and the squared mean curvature. His work inspires many geometers and derives inequalities for various settings. A general expression can be found in [2] for example (see also Corollary 3.7).

The submanifold theory in statistical manifolds is a developing research field. A statistical structure on a manifold is a pair of a Riemannian metric and an affine connection satisfying certain conditions. By definition, a pair of a Riemannian metric and its Levi-Civita connection is a basic example. Accordingly, it is a natural problem to build corresponding inequalities for statistical submanifolds. In fact, many geometers recently give various inequalities for statistical submanifolds (for example, see [1, 3, 6, 7, 9] and references therein). In particular, A. Mihai and I. Mihai [7] obtained an inequality for statistical submanifolds corresponding to the one in terms of the $\delta_{(2,2)}$-invariant, though they did not define the delta-invariant for a statistical manifold.

In this paper, we reformulate and generalize their inequality by defining delta-invariants for a statistical manifold. To define the delta-invariants, we use a new notion of sectional curvature for a statistical manifold, which is different from the ones defined from the so-called the statistical curvature tensor field S or the so-called the K-curvature tensor field $[K, K]$ (see Section 2 and [4, 8]

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for sectional curvatures of statistical manifolds). We can define another delta-invariant by using each of those sectional curvatures for a statistical manifold. However, our delta-invariant δ^U is suitable for obtaining the relation between the sum of the squared mean curvature and the squared dual mean curvature. In this paper, we have:

Theorem 1.1. *Let $\widetilde{M}$ be an $(m+p)$-dimensional statistical manifold of constant U sectional curvature $\widetilde{\kappa}$. Let (M, ∇, g) be an m-dimensional statistical submanifold in $\widetilde{M}$ with the mean curvature vector field H and the dual mean curvature vector field H^*. Then the following inequality holds at each point of M:*

$$\|H\|^2 + \|H^*\|^2 \geq 2c(m_1, \ldots, m_k)^{-1}\{\delta^U_{(m_1,\ldots,m_k)} - b(m_1, \ldots, m_k)\widetilde{\kappa}\},$$

where $\delta^U_{(m_1,\ldots,m_k)}$ is the delta-invariant of (∇, g) for U of type $(m_1, \ldots, m_k)$, and $b(m_1, \ldots, m_k)$, $c(m_1, \ldots, m_k)$ are positive constants defined in Definition 3.2.

The precise statement will be given as Proposition 3.4 and Theorem 3.6, which will be presented in the style same to [2]. The statistical submanifolds characterized by the equality will be stated there. The definitions concerning U are presented in Section 2. For example, a Hessian manifold of constant Hessian curvature is of constant U sectional curvature. If $\widetilde{M}$ is such a manifold and if $k = 2$ and $m_1 = m_2 = 2$, then the theorem is reduced to the inequality in [7]. If $\widetilde{M}$ is a Riemannian manifold, that is, if the considering affine connection coincides with the Levi-Civita connection, then the theorem is reduced to the inequality in [2]. A key of the proof is the algebraic identity (3.6), which seems easier to understand than the proof of the known Riemannian version. As an application, we have the non-existence of doubly minimal statistical submanifolds in statistical manifolds of non-positive U sectional curvature (Corollary 4.4).

2. Curvatures for statistical structures

Throughout this paper, M denotes a smooth manifold of dimension $m \geq 2$, and all the objects are assumed to be smooth. $\Gamma(E)$ denotes the set of sections of a vector bundle $E \to M$. For example, $\Gamma(TM^{(p,q)})$ means the set of all the tensor fields on M of type (p, q).

Let ∇ be an affine connection on M, and $g \in \Gamma(TM^{(0,2)})$ a Riemannian metric. We denote the Levi-Civita connection of g by ∇^g.

We will start with the review of statistical structures.

Definition 2.1. A pair (∇, g) is called a *statistical structure* on M if ∇ is of torsion free, and the Codazzi equation

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$$

holds for any $X, Y, Z \in \Gamma(TM)$.

Remark 2.2. For an affine connection ∇ on a Riemannian manifold (M, g), define ∇^* by the formula

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{2.1}$$

for any $X, Y, Z \in \Gamma(TM)$. Then ∇^* is an affine connection on M which is called the *dual* connection of ∇ with respect to g. Moreover, if (∇, g) is a statistical structure, then (∇^*, g) is also a statistical structure and $\nabla^g = \frac{1}{2}(\nabla + \nabla^*)$ as well.

Remark 2.3. For a statistical structure (∇, g), we set

$$K_X Y = \nabla_X Y - \nabla^g_X Y \tag{2.2}$$

for any $X, Y \in \Gamma(TM)$. Then $K \in \Gamma(TM^{(1,2)})$ satisfies

$$K_X Y = K_Y X, \quad g(K_X Y, Z) = g(Y, K_X Z). \tag{2.3}$$

Conversely, for a Riemannian metric g if a given $K \in \Gamma(TM^{(1,2)})$ satisfies (2.3), then a pair $(\nabla = \nabla^g + K, g)$ becomes a statistical structure.

Besides, we have $K = \nabla^g - \nabla^* = \frac{1}{2}(\nabla - \nabla^*)$. We often denote $K_X Y$ by $K(X, Y)$ as well.

Definition 2.4. Let (∇, g) be a statistical structure on M. We denote the curvature tensor field of ∇ by R^∇ or R for short, and denote R^{∇^*} by R^*, R^{∇^g} by R^g in the similar fashion.

(1) We define

$$S(X, Y)Z = \frac{1}{2}\left\{R(X, Y)Z + R^*(X, Y)Z\right\}$$

for $X, Y, Z \in \Gamma(TM)$, and call $S \in \Gamma(TM^{(1,3)})$ the *statistical curvature tensor field* of (∇, g).

(2) Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $T_x M$. For a 2-dimensional subspace $e_i \wedge e_j$, $1 \leq i < j \leq m$, spanned by $e_i, e_j \in T_x M$,

$$\mathcal{K}^S(e_i \wedge e_j) = g(S(e_i, e_j)e_j, e_i)$$

is called the *statistical sectional curvature* of (∇, g) for a plane $e_i \wedge e_j$, which is denoted by $\mathcal{K}(e_i \wedge e_j)$ for short. We remark that $\mathcal{K}(\Pi)$ for a 2-dimensional subspace Π of $T_x M$ is well defined (see [4]). We denote by $\mathcal{K}^g$ the sectional curvature of g, which is given by using R^g instead of S.

(3) We define a global scalar field

$$\rho = \sum_{1 \leq i, j \leq m} g(S(e_i, e_j)e_j, e_i) = 2 \sum_{1 \leq i < j \leq m} \mathcal{K}(e_i \wedge e_j),$$

and call ρ the *statistical scalar curvature* of (∇, g). The scalar curvature of g is written by $\rho^g = 2\sum_{1 \leq i < j \leq m} \mathcal{K}^g(e_i \wedge e_j)$.

Remark 2.5. For a statistical structure (∇, g), the following holds:

$$S(X, Y)Z = R^g(X, Y)Z + [K_X, K_Y]Z$$

for $X, Y, Z \in \Gamma(TM)$. If $K = 0$, that is, if ∇ is the Levi-Civita connection of g, then we have $S = R^g$, and so $\mathcal{K} = \mathcal{K}^g$, $\rho = \rho^g$.

Definition 2.6. Let (∇, g) be a statistical structure on M. We set $U \in \Gamma(TM^{(1,3)})$ as

$$U(X,Y)Z = R^g(X,Y)Z - [K_X, K_Y]Z$$
$$= 2R^g(X,Y)Z - S(X,Y)Z$$

for $X, Y, Z \in \Gamma(TM)$. As $\mathcal{K}^S$ is well defined, we can define the U *sectional curvature* $\mathcal{K}^U(e_i \wedge e_j)$ of (∇, g) for a plane $e_i \wedge e_j$ of T_xM:

$$\mathcal{K}^U(e_i \wedge e_j) = g(U(e_i, e_j)e_j, e_i),$$

and the U *scalar curvature*:

$$\rho^U = 2 \sum_{1 \le i < j \le m} \mathcal{K}^U(e_i \wedge e_j)$$
$$= 2\rho^g - \rho.$$

Remark that if $K = 0$, then $U = R^g$, and so $\mathcal{K}^U = \mathcal{K}^g$, $\rho^U = \rho^g$. We also remark that an m-dimensional Hessian manifold (M, ∇, g) of constant Hessian curvature κ is of constant U sectional curvature $-\kappa/2$, particularly, $\rho^U = -\kappa m(m-1)/4$.

For integers $m \ge 3$, $k \ge 1$, let us denote by $\mathcal{S}(m,k)$ the set consisting of unordered k-tuples $(m_1, \dots, m_k)$ of integers which satisfies

$$2 \le m_q < m \text{ for } q = 1, \dots, k, \quad m \ge l_k, \tag{2.4}$$

where $l_k = m_1 + \cdots + m_k$.

Definition 2.7. Let (M, ∇, g) be a statistical manifold of dimension $m \ge 3$.

(1) Let L be a subspace of T_xM of dimension $l \ge 2$ and $\{e_1, \dots, e_l\}$ an orthonormal basis of L. We denote

$$\rho^U(L) = 2 \sum_{1 \le i < j \le l} \mathcal{K}^U(e_i \wedge e_j).$$

Remark that $\rho^U(T_xM) = \rho^U(x)$.

(2) For $(m_1, \dots, m_k) \in \mathcal{S}(m,k)$, we define a function $\delta^U_{(m_1,\dots,m_k)} : M \to \mathbb{R}$ by

$$\delta^U_{(m_1,\dots,m_k)}(x) = \frac{1}{2}\left[\rho^U(x) - \inf\left\{\sum_{q=1}^{k} \rho^U(L_q) \mid L_1, \dots, L_k\right\}\right], \tag{2.5}$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_xM with $\dim L_q = m_q$, $q = 1, \dots, k$. We call $\delta^U_{(m_1,\dots,m_k)}$ the *delta-invariant* of (∇, g) for U of type $(m_1, \dots, m_k)$. Furthermore, we write $\delta^U_{(\emptyset)}(x) = \rho^U(x)/2$ for convenience sake.

Remark 2.8. For (M, ∇^g, g), our $\delta^U_{(m_1,\dots,m_k)}$ coincides with $\delta_{(m_1,\dots,m_k)}$ defined by B.-Y. Chen for a Riemannian manifold (M, g). We put 1/2 on the right hand side of (2.5) because his scalar curvature is a half of ours.

3. Chen inequalities

We give an algebraic preliminary, which is a key lemma to prove our theorems.

Lemma 3.1. *For $(m_1,\dots,m_k) \in \mathcal{S}(m,k)$, set $l_0 = 0$ and $l_q = m_1 + \cdots + m_q$ for $q = 1,\dots,k$. Suppose that $m \geq l_k + 1$. We have the following inequalities* (3.1) *and* (3.3) *for arbitrary $a_1,\dots,a_m \in \mathbb{R}$:*

$$(m - l_k - 1)\Big(\sum_{i=l_k+1}^{m} a_i\Big)^2 \geq 2(m - l_k) \sum_{l_k+1 \leq i < j \leq m} a_i a_j. \tag{3.1}$$

The equality holds if and only if

$$a_{l_k+1} = \cdots = a_m. \tag{3.2}$$

It also holds for $m \geq 2$ and $k = 0$.

$$\begin{aligned}(m + k - l_k - 1)\Big(\sum_{i=1}^{m} a_i\Big)^2 \\ \geq 2(m + k - l_k)\Big(\sum_{1 \leq i < j \leq m} a_i a_j - \sum_{q=1}^{k} \sum_{l_{q-1}+1 \leq i < j \leq l_q} a_i a_j\Big).\end{aligned} \tag{3.3}$$

The equality holds if and only if

$$A_1 = \cdots = A_k = a_{l_k+1} = \cdots = a_m, \tag{3.4}$$

where $A_q = a_{l_{q-1}+1} + \cdots + a_{l_q}$.

Proof. These are obtained directly from the following two identities:

$$\begin{aligned}&\sum_{l_k+1 \leq i < j \leq m} (a_i - a_j)^2 \\ &= (m - l_k - 1)\Big(\sum_{i=l_k+1}^{m} a_i\Big)^2 - 2(m - l_k) \sum_{l_k+1 \leq i < j \leq m} a_i a_j,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\sum_{l_k+1 \leq i < j \leq m} (a_i - a_j)^2 + \sum_{q=1}^{k} \sum_{i=l_k+1}^{m} (A_q - a_i)^2 + \sum_{1 \leq q < r \leq k} (A_q - A_r)^2 \\ &= (m + k - l_k - 1)\Big(\sum_{i=1}^{m} a_i\Big)^2\end{aligned} \tag{3.6}$$

$$- 2(m+k-l_k)\Big(\sum_{1\le i<j\le m} a_i a_j - \sum_{q=1}^{k} \sum_{l_{q-1}+1\le i<j\le l_q} a_i a_j\Big).$$

The proof of (3.5) is as follows: We calculate

$$\begin{aligned}\sum_{l_k+1\le i<j\le m} (a_i-a_j)^2 &= \frac{1}{2} \sum_{l_k+1\le i,j\le m} (a_i-a_j)^2 \\ &= (m-l_k) \sum_{i=l_k+1}^{m} a_i^2 - \big(\sum_{i=l_k+1}^{m} a_i\big)^2,\end{aligned}$$

and

$$\begin{aligned}&\sum_{l_k+1\le i<j\le m} (a_i-a_j)^2 \\ &= \frac{1}{2}\Big(\sum_{l_k+1\le i,j\le m} - \sum_{l_k+1\le i=j\le m}\Big)(a_i^2+a_j^2) - 2 \sum_{l_k+1\le i<j\le m} a_i a_j \\ &= (m-l_k-1) \sum_{i=l_k+1}^{m} a_i^2 - 2 \sum_{l_k+1\le i<j\le m} a_i a_j\,.\end{aligned}$$

Deleting the term $\sum {a_i}^2$ from these two identities implies (3.5).

The proof of (3.6) is as follows: We have

$$\begin{aligned}&\sum_{1\le q<r\le k} A_q A_r + \sum_{q=1}^{k} A_q \sum_{i=l_k+1}^{m} a_i + \sum_{l_k+1\le i<j\le m} a_i a_j \\ &= \sum_{1\le i<j\le m} a_i a_j - \sum_{q=1}^{k} \sum_{l_{q-1}+1\le i<j\le l_q} a_i a_j,\end{aligned}$$

which implies that

$$\begin{aligned}&\text{[the left-hand side of (3.6)]} \\ &= \Big\{(m-l_k-1) \sum_{i=l_k+1}^{m} {a_i}^2 - 2 \sum_{l_k+1\le i<j\le m} a_i a_j\Big\} \\ &\quad + \Big\{(m-l_k)\sum_{q=1}^{k} {A_q}^2 + k \sum_{i=l_k+1}^{m} {a_i}^2 - 2\sum_{q=1}^{k} A_q \sum_{i=l_k+1}^{m} a_i\Big\} \\ &\quad + \Big\{(k-1)\sum_{q=1}^{k} {A_q}^2 - 2 \sum_{1\le q<r\le k} A_q A_r\Big\} \\ &= (m+k-l_k-1)\Big(\sum_{q=1}^{k} {A_q}^2 + \sum_{i=l_k+1}^{m} {a_i}^2\Big)\end{aligned}$$

$$
\begin{aligned}
&- 2\Big(\sum_{1 \le q < r \le k} A_q A_r + \sum_{q=1}^{k} A_q \sum_{i=l_k+1}^{m} a_i + \sum_{l_k+1 \le i < j \le m} a_i a_j \Big) \\
&= (m + k - l_k - 1)\Big\{ \big(\sum_{q=1}^{k} A_q\big)^2 + \big(\sum_{i=l_k+1}^{m} a_i\big)^2 \Big\} \\
&\quad - 2(m + k - l_k)\Big(\sum_{1 \le q < r \le k} A_q A_r + \sum_{l_k+1 \le i < j \le m} a_i a_j \Big) - 2 \sum_{q=1}^{k} A_q \sum_{i=l_k+1}^{m} a_i \\
&= (m + k - l_k - 1)\big(\sum_{q=1}^{k} A_q + \sum_{i=l_k+1}^{m} a_i \big)^2 \\
&\quad - 2(m + k - l_k)\Big(\sum_{1 \le q < r \le k} A_q A_r + \sum_{q=1}^{k} A_q \sum_{i=l_k+1}^{m} a_i + \sum_{l_k+1 \le i < j \le m} a_i a_j \Big) \\
&= [\text{the right-hand side of (3.6)}].
\end{aligned}
$$

□

Following [2], we adopt the symbols below for later use.

Definition 3.2. For $(m_1, \dots, m_k) \in \mathcal{S}(m, k)$, we set the positive constants as follow:

$$
b(m_1, \dots, m_k) = \frac{1}{2} m(m-1) - \frac{1}{2} \sum_{q=1}^{k} m_q (m_q - 1), \tag{3.7}
$$

$$
\begin{aligned}
c(m_1, \dots, m_k) &= \frac{m^2}{2} \frac{m + k - \sum_{q=1}^{k} m_q - 1}{m + k - \sum_{q=1}^{k} m_q} \\
&= \frac{m^2}{2} \frac{m + k - l_k - 1}{m + k - l_k},
\end{aligned} \tag{3.8}
$$

and moreover,

$$
b(\emptyset) = c(\emptyset) = \frac{1}{2} m(m-1). \tag{3.9}
$$

Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical manifold of dimension $m + p$. Let (M, ∇, g) be a statistical submanifold of $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$. For detail, refer to [4,9] for example. By definition, we have $h, h^* \in \Gamma(T^{\perp}M \otimes TM^{(0,2)})$, $A, A^* \in \Gamma((T^{\perp}M)^* \otimes TM^{(1,1)})$ and connections D, D^* of the normal bundle $T^{\perp}M$ satisfying the Gauss and Weingarten formulas:

$$
\left\{ \begin{array}{l} \widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \\ \widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \end{array} \right. \qquad \left\{ \begin{array}{l} \widetilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \\ \widetilde{\nabla}^*_X \xi = -A^*_\xi X + D^*_X \xi \end{array} \right.
$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$. We denote the mean curvature vector fields of M for $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively, by

$$H = \frac{1}{m}\mathrm{tr}_g h, \quad H^* = \frac{1}{m}\mathrm{tr}_g h^*, \tag{3.10}$$

and write

$$\|H\|^2 = \widetilde{g}(H, H), \quad \|H^*\|^2 = \widetilde{g}(H^*, H^*).$$

The inclusion map $\iota : M \to \widetilde{M}$ can be considered as a *statistical immersion* of (M, ∇, g) into $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.

Definition 3.3. A statistical immersion is said to be *doubly totally-geodesic* if $h = h^* = 0$, and *doubly totally-umbilical* if $h = g \otimes H, h^* = g \otimes H^*$. Furthermore, a statistical immersion is said to be *doubly minimal* if $H = H^* = 0$.

A doubly totally-geodesic statistical submanifold is also called a *doubly autoparallel* statistical submanifold. Remark that a doubly minimal statistical immersion of (M, ∇, g) into $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is an isometric minimal immersion of (M, g) into $(\widetilde{M}, \widetilde{g})$.

Our Gauss equations are the following:

$$\begin{aligned}
\widetilde{g}(\widetilde{R}(X,Y)Z, W) &= g(R(X,Y)Z, W) \\
&\quad - \widetilde{g}(h(Y,Z), h^*(X,W)) + \widetilde{g}(h(X,Z), h^*(Y,W)), \\
2\widetilde{g}(\widetilde{S}(X,Y)Z, W) &= 2g(S(X,Y)Z, W) \\
&\quad - \widetilde{g}(h(Y,Z), h^*(X,W)) + \widetilde{g}(h(X,Z), h^*(Y,W)) \\
&\quad - \widetilde{g}(h^*(Y,Z), h(X,W)) + \widetilde{g}(h^*(X,Z), h(Y,W)), \\
4\widetilde{g}(R^{\widetilde{g}}(X,Y)Z, W) &= 4g(R^g(X,Y)Z, W) \\
&\quad - \widetilde{g}(h(Y,Z) + h^*(Y,Z),\ h(X,W) + h^*(X,W)) \\
&\quad + \widetilde{g}(h(X,Z) + h^*(X,Z),\ h(Y,W) + h^*(Y,W))
\end{aligned}$$

for $X, Y, Z, W \in \Gamma(TM)$.

Proposition 3.4. *Let (M, ∇, g) be an $m(\geq 2)$-dimensional statistical submanifold in an $(m+p)$-dimensional statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ with the U sectional curvature $\mathcal{K}^{\widetilde{U}}$. Then*

$$\delta^U_{(\emptyset)} \leq b(\emptyset) \max \mathcal{K}^{\widetilde{U}} + c(\emptyset)\,(\|H\|^2 + \|H^*\|^2)/2, \tag{3.11}$$

where $\max \mathcal{K}^{\widetilde{U}} = \max \left\{\mathcal{K}^{\widetilde{U}}(\Pi) \mid \Pi : \textit{plane section of } TM\right\}$.

Suppose that $(\widetilde{\nabla}, \widetilde{g})$ is of constant U sectional curvature. The equality holds at $x \in M$ if and only if $h_x = g_x \otimes H_x$, $h^*_x = g_x \otimes H^*_x$.

Proof. Using an orthonormal frame $\{e_1, \ldots, e_m, \xi_1, \ldots, \xi_p\}$ adapted for M, we express

$$h(e_i, e_j) = \sum h^\alpha_{ij}\xi_\alpha, \quad h^*(e_i, e_j) = \sum h^{*\alpha}_{ij}\xi_\alpha.$$

As in the proof of Lemma 3.1 in [9], by the Gauss equations we have

$$\begin{aligned}
&2\sum_{1\le i<j\le m}\mathcal{K}^U(e_i\wedge e_j)\\
&=2\sum_{1\le i<j\le m}(2\mathcal{K}^g-\mathcal{K})(e_i\wedge e_j)\\
&=2\sum_{1\le i<j\le m}(2\mathcal{K}^{\widetilde{g}}-\widetilde{\mathcal{K}})(e_i\wedge e_j)\\
&\quad+\sum_{\alpha=1}^{p}\sum_{1\le i<j\le m}\big(h_{ii}^\alpha h_{jj}^\alpha+h_{ii}^{*\alpha}h_{jj}^{*\alpha}-(h_{ij}^\alpha)^2-(h_{ij}^{*\alpha})^2\big)\\
&\le m(m-1)\max(2\mathcal{K}^{\widetilde{g}}-\widetilde{\mathcal{K}})+\sum_{\alpha=1}^{p}\sum_{1\le i<j\le m}\big(h_{ii}^\alpha h_{jj}^\alpha+h_{ii}^{*\alpha}h_{jj}^{*\alpha}\big).
\end{aligned}$$

Considering h_{ii}^α and $h_{ii}^{*\alpha}$ as a_i in (3.1) with $k=0$, respectively, we have

$$2\delta_{(\emptyset)}^U\le m(m-1)\Big\{\max\mathcal{K}^{\widetilde{U}}+(\|H\|^2+\|H^*\|^2)/2\Big\}.$$

The latter part of the proposition is easy to obtain from (3.2). □

Remark 3.5. In [9], we had the following inequality (Theorem 3.7):

$$\delta_{(\emptyset)}^U\le b(\emptyset)\,\max\mathcal{K}^{\widetilde{U}}+(m^3/8)\,(\|H\|^2+\|H^*\|^2)/2, \tag{3.12}$$

which characterizes doubly totally-umbilical surfaces and doubly totally-geodesic submanifolds as the equality holding cases at every point. It is easy to see that (3.11) coincides (3.12) in the case where $m=2$. The inequality (3.12) was obtained from the relation between the Ricci curvature and the squared mean curvatures.

Theorem 3.6. *Let (M,∇,g) be an $m(\ge 3)$-dimensional statistical submanifold in an $(m+p)$-dimensional statistical manifold $(\widetilde{M},\widetilde{\nabla},\widetilde{g})$ with the U sectional curvature $\mathcal{K}^{\widetilde{U}}$. For $(m_1,\dots,m_k)\in\mathcal{S}(m,k)$, we have*

$$\begin{aligned}\delta^U_{(m_1,\dots,m_k)}\le{}& b(m_1,\dots,m_k)\max\mathcal{K}^{\widetilde{U}}\\&+c(m_1,\dots,m_k)(\|H\|^2+\|H^*\|^2)/2,\end{aligned}\tag{3.13}$$

where $\max\mathcal{K}^{\widetilde{U}}=\max\big\{\mathcal{K}^{\widetilde{U}}(\Pi)\mid\Pi:\textit{plane section of }TM\big\}$.

Suppose that $(\widetilde{\nabla},\widetilde{g})$ is of constant U sectional curvature. The equality holds at $x\in M$ if and only if there exist mutually orthogonal subspaces $L_1,\dots,L_k$ of T_xM with $\dim L_q=m_q$, $q=1,\dots,k$, and adapted orthonormal basis satisfying

$$L_q=\operatorname{span}\{e_{l_{q-1}+1},\dots,e_{l_q}\},\tag{3.14}$$

$$\sum_{i=1}^{l_1}h_{ii}^\alpha=\cdots=\sum_{i=l_{k-1}+1}^{l_k}h_{ii}^\alpha=h^\alpha_{l_k+1\,l_k+1}=\cdots=h^\alpha_{mm},\tag{3.15}$$

$$\sum_{i=1}^{l_1} h_{ii}^{*\alpha} = \cdots = \sum_{i=l_{k-1}+1}^{l_k} h_{ii}^{*\alpha} = h_{l_k+1\, l_k+1}^{*\alpha} = \cdots = h_{mm}^{*\alpha}, \tag{3.16}$$

$$h_{ij}^{\alpha} = h_{ij}^{*\alpha} = 0 \quad \textit{for} \quad i \le l_q < l_q + 1 \le j, \quad q = 1, \dots, k, \\ \textit{or} \quad l_k + 1 \le i < j \le m. \tag{3.17}$$

Proof. Let $L_1, \dots, L_k$ be mutually orthogonal subspaces of T_xM with $\dim L_q = m_q$, $q = 1, \dots, k$ and $\{e_{l_{q-1}+1}, \dots, e_{l_q}\}$ an orthonormal basis of L_q. As in the proof of Proposition 3.4, by the Gauss equations we have

$$\begin{aligned} &2 \sum_{1 \le i < j \le m} \mathcal{K}^U(e_i \wedge e_j) - 2 \sum_{q=1}^{k} \sum_{l_{q-1}+1 \le i < j \le l_q} \mathcal{K}^U(e_i \wedge e_j) \\ &= 2 \sum_{1 \le i < j \le m} \mathcal{K}^{\widetilde{U}}(e_i \wedge e_j) \\ &\quad + \sum_{\alpha=1}^{p} \sum_{1 \le i < j \le m} \big(h_{ii}^{\alpha} h_{jj}^{\alpha} + h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{\alpha})^2 - (h_{ij}^{*\alpha})^2\big) \\ &\quad - 2 \sum_{q=1}^{k} \sum_{l_{q-1}+1 \le i < j \le l_q} \mathcal{K}^{\widetilde{U}}(e_i \wedge e_j) \\ &\quad - \sum_{\alpha=1}^{p} \sum_{q=1}^{k} \sum_{l_{q-1}+1 \le i < j \le l_q} \big(h_{ii}^{\alpha} h_{jj}^{\alpha} + h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{\alpha})^2 - (h_{ij}^{*\alpha})^2\big) \\ &\le 2b(m_1, \dots, m_k) \max \mathcal{K}^{\widetilde{U}} \\ &\quad + \sum_{\alpha=1}^{p} \Big\{ \sum_{1 \le i < j \le m} (h_{ii}^{\alpha} h_{jj}^{\alpha} + h_{ii}^{*\alpha} h_{jj}^{*\alpha}) - \sum_{q=1}^{k} \sum_{l_{q-1}+1 \le i < j \le l_q} (h_{ii}^{\alpha} h_{jj}^{\alpha} + h_{ii}^{*\alpha} h_{jj}^{*\alpha}) \Big\}. \end{aligned}$$

In the case where $\mathcal{K}^{\widetilde{U}}$ is constant, we remark that the equality holds if and only if (3.17) holds.

Considering h_{ii}^{α} and $h_{ii}^{*\alpha}$ as a_i in (3.3), respectively, we have

$$\begin{aligned} &\rho^U - 2 \sum_{q=1}^{k} \sum_{l_{q-1}+1 \le i < j \le l_q} \mathcal{K}^U(e_i \wedge e_j) \\ &\le 2b(m_1, \dots, m_k) \max \mathcal{K}^{\widetilde{U}} + c(m_1, \dots, m_k)\big(\|H\|^2 + \|H^*\|^2\big). \end{aligned}$$

The latter part of the proposition is easy to obtain from (3.4). □

Corollary 3.7 ([2]). *Let $(\widetilde{M}, \widetilde{g})$ be an $(m+p)$-dimensional Riemannian manifold of constant curvature $\widetilde{c}$, and (M, g) an m-dimensional Riemannian submanifold with the mean curvature vector field $\widehat{H}$. For $(m_1, \dots, m_k) \in \mathcal{S}(m, k)$, we have*

$$\delta_{(m_1, \dots, m_k)} \le b(m_1, \dots, m_k)\widetilde{c} + c(m_1, \dots, m_k)\|\widehat{H}\|^2. \tag{3.18}$$

Proof. In Theorem 3.6, consider the case where $\widetilde{\nabla} = \nabla^{\widetilde{g}}$. Remark that $\nabla = \nabla^g$ and $H = H^* = \widehat{H}$. Since $\mathcal{K}^{\widetilde{U}} = \widetilde{c}$, we have (3.18). □

Corollary 3.8. *Let (M, ∇, g) be an m-dimensional statistical submanifold in an $(m+p)$-dimensional Hessian manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ of constant Hessian curvature κ. For $(m_1, \dots, m_k) \in \mathcal{S}(m, k)$, we have*

$$\begin{aligned} \delta^U_{(m_1,\dots,m_k)} \le\ & b(m_1, \dots, m_k)(-\kappa/2) \\ & + c(m_1, \dots, m_k)\big(\|H\|^2 + \|H^*\|^2\big)/2. \end{aligned} \tag{3.19}$$

Proof. By definition, $R^{\widetilde{\nabla}} = 0$ and $\widetilde{g}$ is of constant curvature $-\kappa/4$ (see [10]). Therefore, we have $\mathcal{K}^{\widetilde{U}} = -\kappa/2$. Theorem 3.6 implies (3.19). □

In the case where $k = 2$ and $m_1 = m_2 = 2$, the inequality was essentially obtained by [7].

4. Examples

Example 4.1. The triple $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ defined below is an n-dimensional statistical manifold such that the U sectional curvature vanishes.

$$\begin{aligned} &\widetilde{M} = (\mathbb{R}^+)^n = \big\{y = (y^1, \dots, y^n) \in \mathbb{R}^n \mid y^1 > 0, \dots, y^n > 0\big\}, \\ &\widetilde{g} = \sum_{i=1}^n (dy^i)^2, \\ &\widetilde{\nabla}_{\widetilde{\partial}_j} \widetilde{\partial}_i = \widetilde{K}(\widetilde{\partial}_j, \widetilde{\partial}_i) = -\delta_{ji}(y^i)^{-1}\widetilde{\partial}_i, \end{aligned}$$

where $\widetilde{\partial}_i = \partial/\partial y^i$. In fact, it is a Hessian manifold of constant Hessian curvature 0. For $(n_1, \dots, n_k) \in \mathcal{S}(n, k)$, we have $\delta^{\widetilde{U}}_{(n_1,\dots,n_k)} = 0$.

Example 4.2. For $\alpha \in \mathbb{R}$, the triple $(\widetilde{M}, \widetilde{\nabla}^{(\alpha)}, \widetilde{g})$ defined below is an n-dimensional statistical manifold such that the U sectional curvature is negative constant $-(1 + \alpha^2)$.

$$\begin{aligned} &\widetilde{M} = \mathbb{H}^n = \big\{y = (y^1, \dots, y^{n-1}, y^n) \in \mathbb{R}^n \mid y^n > 0\big\}, \\ &\widetilde{g} = (y^n)^{-2} \sum_{A=1}^n (dy^A)^2, \\ &\widetilde{K}(\widetilde{\partial}_i, \widetilde{\partial}_j) = \delta_{ij}(y^n)^{-1}\widetilde{\partial}_n, \\ &\widetilde{K}(\widetilde{\partial}_i, \widetilde{\partial}_n) = \widetilde{K}(\widetilde{\partial}_n, \widetilde{\partial}_i) = (y^n)^{-1}\widetilde{\partial}_i, \\ &\widetilde{K}(\widetilde{\partial}_n, \widetilde{\partial}_n) = 2(y^n)^{-1}\widetilde{\partial}_n, \end{aligned}$$

and $\widetilde{\nabla}^{(\alpha)} = \nabla^{\widetilde{g}} + \alpha\widetilde{K}$ as in Remark 2.3, where $\widetilde{\partial}_A = \partial/\partial y^A$, $A = 1, \dots, n$ and $i, j = 1, \dots, n-1$. Then we have

$$[\widetilde{K}, \widetilde{K}](X, Y)Z = \widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y,$$

$$\widetilde{U}(X,Y)Z = -(1+\alpha^2)\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}$$

for $X, Y, Z \in \Gamma(T\widetilde{M})$.

For $(n_1, \dots, n_k) \in \mathcal{S}(n,k)$, we have $\delta^{\widetilde{U}}_{(n_1,\dots,n_k)} = -b(n_1, \dots, n_k)(1+\alpha^2)$.

Remark that $(\widetilde{M}, \widetilde{\nabla}^{(1)}, \widetilde{g})$ is a Hessian manifold of constant Hessian curvature 4.

Example 4.3 (Example 2.15 in [5])**.** Let $(\mathbb{S}^{2n+1}, g, \phi, \xi)$ be a unit hypersphere in the complex Euclidean space with the standard Sasakian structure. Set $K(X,Y) = g(X,\xi)g(Y,\xi)\xi$ for any $X, Y \in \Gamma(T\mathbb{S}^{2n+1})$, and $\nabla = \nabla^g + K$. Then the statistical manifold $(\mathbb{S}^{2n+1}, \nabla, g)$ is of constant U sectional curvature one. In fact, we have $U = R^g$. For $(m_1, \dots, m_k) \in \mathcal{S}(2n+1, k)$, we have $\delta^U_{(m_1,\dots,m_k)} = b(m_1, \dots, m_k)$.

As an application of Proposition 3.4 and Theorem 3.6, we have the following non-existence of doubly minimal statistical immersions:

Corollary 4.4. *Let $\widetilde{M}$ be a statistical manifold of non-positive U sectional curvature. Let M be an m-dimensional statistical manifold. Suppose that there exist non-negative integer k, $(m_1, \dots, m_k) \in \mathcal{S}(m,k)$ and a point $x \in M$ such that $\delta^U_{(m_1,\dots,m_k)}(x)$ is positive. Then M does not admit doubly minimal statistical immersion into $\widetilde{M}$ for any codimension, in particular, $\widetilde{M}$ as in Examples 4.1 and 4.2.*

We will give basic properties and examples of doubly minimal statistical immersions in another paper.

Examples of doubly totally-umbilical statistical submanifolds, which are submanifolds satisfying the equality in Proposition 3.4, are given in [9]:

Example 4.5. Let $(\widetilde{M}, \widetilde{\nabla}^{(\alpha)}, \widetilde{g})$ be a statistical manifold of dimension $n = m + p$ in Example 4.2.

(1) For $(a^1, \dots, a^p) \in \mathbb{R}^p$, the inclusion map $\iota : \mathbb{H}^m \ni (x^1, \dots, x^{m-1}, x^m) \mapsto (a^1, \dots, a^p, x^1, \dots, x^{m-1}, x^m) \in \mathbb{H}^n$ is doubly totally-geodesic. In fact, we have $h = h^* = 0$, and the induced statistical structure (∇, g) on $\mathbb{H}^m$ is same as in Example 4.2. Accordingly, we have

$$\begin{aligned} &\delta^U_{(m_1,\dots,m_k)} = -b(m_1, \dots, m_k)(1+\alpha^2), \\ &b(m_1, \dots, m_k) \max \mathcal{K}^{\widetilde{U}} + c(m_1, \dots, m_k)(\|H\|^2 + \|H^*\|^2)/2 \\ = &-b(m_1, \dots, m_k)(1+\alpha^2). \end{aligned}$$

(2) For $(a^1, \dots, a^{p-1}, a^p) \in \mathbb{R}^{p-1} \times \mathbb{R}^+$, the inclusion map

$$\iota : \mathbb{R}^m \ni (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, a^1, \dots, a^{p-1}, a^p) \in \mathbb{H}^n$$

is doubly totally-umbilical. In fact, the induced statistical structure (∇, g) on $\mathbb{R}^m$ is given as

$$g = (a^p)^{-2} \sum_{j=1}^{m} (dx^j)^2, \qquad \nabla_{\partial_j} \partial_i = \nabla^g_{\partial_j} \partial_i = 0,$$

and we have

$$\begin{aligned} h &= (1+\alpha) a^p g \otimes (\partial/\partial y^n) = g \otimes H, \\ h^* &= (1-\alpha) a^p g \otimes (\partial/\partial y^n) = g \otimes H^*. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} &\delta^U_{(m_1,\dots,m_k)} = 0, \\ &b(m_1,\dots,m_k) \max \mathcal{K}^{\widetilde{U}} + c(m_1,\dots,m_k)(\|H\|^2 + \|H^*\|^2)/2 \\ = {}&(1+\alpha^2)\{c(m_1,\dots,m_k) - b(m_1,\dots,m_k)\}. \end{aligned}$$

Remark that $c(m_1,\dots,m_k) - b(m_1,\dots,m_k) \geq 0$ and the equality holds if and only if $k = 0$. Therefore, the above inclusion map ι satisfies the equality in (3.11), but does not in (3.13).

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Hitoshi Furuhata
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
Email address: furuhata@math.sci.hokudai.ac.jp

Izumi Hasegawa
Hokkaido University of Education
Sapporo 002-8501, Japan
Email address: hase_izu@kuc.biglobe.ne.jp

Naoto Satoh
National Institute of Technology, Asahikawa College
Asahikawa 071-8142, Japan
Email address: n_satoh@asahikawa-nct.ac.jp