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ON PARTIAL SUMS OF FOUR PARAMETRIC WRIGHT FUNCTION

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Abstract. Special functions and Geometric function theory are close related to each other due to the surprise use of hypergeometric function in the solution of the Bieberbach conjecture. The purpose of this paper is to provide a set of sufficient conditions under which the normalized four parametric Wright function has lower bounds for the ratios to its partial sums and as well as for their derivatives. The sufficient conditions are also obtained by using Alexander transform. The results of this paper are generalized and also improved the work of M. Din et al. [15]. Some examples are also discussed for the sake of better understanding of this article.

1. Introduction and preliminaries

The four parametric Wright function

$$\mathcal{W}_{(\beta,b)}^{(\alpha,a)}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(a+m\alpha)\,\Gamma(b+m\beta)},\quad a,b\in\mathbb{C},\quad \alpha,\beta\in\mathbb{R}, \tag{1.1}$$

was studied by E. M. Wright for α, $\beta>0$. The series defined in (1.1) converges absolutely and it is an entire function. For details see [18, 20, 38, 39]. The Wright function is the particular case of the four parametric Wright function.

The Wright function

$$W_{\zeta,\eta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{m!\Gamma(\zeta m+\eta)},\ \zeta>-1,\ \eta\in\mathbb{C},$$

was introduced by E. M. Wright [37] and have been used in the asymptotic theory of partitions, in the theory of integral transforms of Hankel type and in Mikusinski operational calculus. Recently, Wright functions have been found in the solution of partial differential equations of fractional order. It was found that the corresponding Green functions can be expressed in terms of Wright

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functions [27, 30]. Mainardi [19] involved Wright functions in the solution of fractional diffusion wave equation. Luchko et al. [13, 18] obtained the scale variant solutions of partial differential equations of fractional order in terms of Wright functions. For positive rational number ζ, the Wright functions can be expressed in terms of generalized hypergeometric functions. For some details see [16, Section 2.1]. In particular, the functions $W_{1,v+1}(-z^2/4)$ can be expressed in terms of the Bessel functions J_v, given as:

$$J_v(z) = \left(\frac{z}{2}\right)^2 W_{1,v+1}(-z^2/4) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+v}}{m!\Gamma(m+v+1)}.$$

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Consider the Alexander transform given as:

$$\mathbb{A}[f](z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{m=2}^{\infty} \frac{a_m}{m} z^m.$$

Special functions and Geometric function theory are close related to each other due to the surprise use of hypergeometric function in the solution of the Bieberbach conjecture. Due to this special functions become more important for the researchers. Recently, many researchers studied some geometric properties such as univalency, starlikeness, convexity and close-to-convexity of special functions. Several researchers have studied the geometric properties of hypergeometric functions [22, 29], Bessel functions [5–11, 31, 35, 36], Struve functions [25, 40], Lommel functions [14], Wright functions [15, 28], q-Bessel functions [1, 4], and Fox-Wright functions [20, 21]. This study motivated Sourav Das and Khaled Mehrez [21] to study some geometric properties of four parametric Wright function. The series (1.1) is not normalized. To make it normalized we consider the following form

$$\begin{aligned} \mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z) &= z\Gamma(a)\Gamma(b)\mathcal{W}_{(\beta,b)}^{(\alpha,a)}(z) \\ &= z + \sum_{m=1}^{\infty} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)} z^{m+1}. \end{aligned} \tag{1.2}$$

In this note, we study the ratio of a function of the form (1.2) to its sequence of partial sums $\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)_n(z) = z + \sum\limits_{m=1}^{\infty} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)} z^{m+1}$ when the coefficients of $\mathbb{W}_{(\beta,b)}^{(\alpha,a)}$ satisfy certain conditions. We determine the lower bounds of $\Re\left\{\frac{\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)}{\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)_n(z)}\right\}$, $\Re\left\{\frac{\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)_n(z)}{\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)}\right\}$, $\Re\left\{\frac{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)(z)}{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)_n(z)}\right\}$, $\Re\left\{\frac{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)_n(z)}{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)(z)}\right\}$,

$\Re\left\{\frac{\mathbb{A}\left[\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)\right](z)}{\left(\mathbb{A}\left[\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right]\right)_n(z)}\right\}$, $\Re\left\{\frac{\left(\mathbb{A}\left[\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right]\right)_n(z)}{\mathbb{A}\left[\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right](z)}\right\}$, where $\mathbb{A}\left[\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right]$ is the Alexander transform of $\mathbb{W}_{(\beta,b)}^{(\alpha,a)}$. Some similar results are obtained for the function $\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)$. For some works on partial sums, we refer [2–4,12,17,23,24,26,32–34].

Lemma 1.1. *Let* $a,b,\alpha,\beta\in\mathbb{R}$ *and* $\lambda=a+b+ab$, $\mu=ab$ *with inequality* $\lambda\mu>0$. *Then the function* $\mathbb{W}_{(\beta,b)}^{(\alpha,a)}:\mathcal{U}\to\mathbb{C}$ *defined by* (1.2) *satisfies the following inequalities*:

(i)

$$\left|\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)\right|\le\frac{\lambda\mu+\lambda+1}{\lambda\mu},\quad z\in\mathcal{U},$$

(ii)

$$\left|\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)\right|\le\frac{\lambda\mu+2\lambda-\mu+2}{\lambda\mu},\quad z\in\mathcal{U},$$

Proof. (i) By using the well-known triangle inequality

$$|z_1+z_2|\le|z_1|+|z_2|$$

with the inequality $\Gamma(a+m)\le\Gamma(a+m\alpha)$, $m\in\mathbb{N}$, which is equivalent to $\frac{\Gamma(a)}{\Gamma(a+m\alpha)}\le\frac{1}{a(a+1)\cdots(a+m-1)}=\frac{1}{(a)_m}$, $m\in\mathbb{N}$, and $(a)_m\ge(a)^m$, $m\in\mathbb{N}$, we obtain

$$\begin{aligned}\left|\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)\right|&=\left|z+\sum_{m=1}^{\infty}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)}z^{m+1}\right|\\&\le1+\sum_{m=1}^{\infty}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)}\\&\le1+\sum_{m=1}^{\infty}\frac{1}{(a)_m(b)_m}\\&\le1+\frac{1}{ab}\sum_{m=1}^{\infty}\left(\frac{1}{(a+1)(b+1)}\right)^{m-1}\\&=\frac{(a+1)(b+1)(ab+1)-ab}{ab(a+b+ab)},\quad a>\frac{-b}{b+1},\ z\in\mathcal{U}\\&=\frac{\lambda\mu+\lambda+1}{\lambda\mu}.\end{aligned}$$

(ii) To prove (ii), we use the well-known triangle inequality with the inequality $\frac{\Gamma(a)}{\Gamma(a+m\alpha)}\le\frac{1}{a(a+1)\cdots(a+m-1)}=\frac{1}{(a)_m}$, $m\in\mathbb{N}$, with inequalities

$$\begin{aligned}&(m+1)\le2^m,\ m\in\mathbb{N},\\&(a)_m\ge(a)^m,\ m\in\mathbb{N},\end{aligned}$$

we have

$$
\begin{aligned}
\left|\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)\right| &= \left|1+\sum_{m=1}^{\infty}\frac{\Gamma(a)\Gamma(b)(m+1)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)}z^{m}\right| \\
&\leq 1+\sum_{m=1}^{\infty}\frac{\Gamma(a)\Gamma(b)(m+1)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)} \\
&\leq 1+\sum_{m=1}^{\infty}\frac{m+1}{(a)_m(b)_m} \\
&= 1+\frac{1}{ab}\sum_{m=1}^{\infty}\frac{m+1}{(a+1)_{m-1}(b+1)_{m-1}} \\
&\leq 1+\frac{2}{ab}\sum_{m=1}^{\infty}\left(\frac{2}{(a+1)(b+1)}\right)^{m-1} \\
&= \frac{(a+1)(b+1)(ab+2)-2ab}{ab(a+b+ab-1)}, \quad a>\frac{1-b}{b+1},\ z\in\mathcal{U} \\
&= \frac{\lambda\mu+\lambda+1}{\lambda\mu}.
\end{aligned}
$$

□

Lemma 1.2. *Let $a,b,\alpha,\beta\in\mathbb{R}$ and $M=(a+1)(b+1)$, $N=ab$ with inequality $MN>0$. Then the function $\mathbb{W}^{(\alpha,a)}_{(\beta,b)}:\mathcal{U}\to\mathbb{C}$ defined by (1.2) satisfies the following*

$$
\left|\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right](z)\right| \leq \frac{2MN+M-2N}{2MN-2N}, \quad z\in\mathcal{U}.
$$

Proof. Making the use of triangle inequality with $\frac{\Gamma(a)}{\Gamma(a+m\alpha)} \leq \frac{1}{a(a+1)\cdots(a+m-1)} = \frac{1}{(a)_m}$, $m\in\mathbb{N}$, and the inequality

$$
(m+1)\geq 2,\ m\in\mathbb{N},
$$

we have

$$
\begin{aligned}
\left|\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right](z)\right| &= \left|z+\sum_{m=1}^{\infty}\frac{\Gamma(a)\Gamma(b)}{(m+1)\Gamma(a+m\alpha)\Gamma(b+m\beta)}z^{m+1}\right| \\
&\leq 1+\sum_{m=1}^{\infty}\frac{1}{(m+1)(a)_m(b)_m} \\
&\leq 1+\frac{1}{2ab}\sum_{m=1}^{\infty}\left(\frac{1}{(a+1)(b+1)}\right)^{m-1} \\
&= \frac{2ab\{(a+1)(b+1)-1\}+(a+1)(b+1)}{2ab\{(a+1)(b+1)-1\}}, \ a>\frac{-b}{b+1},\ z\in\mathcal{U} \\
&= \frac{2MN+M-2N}{2MN-2N}.
\end{aligned}
$$

□

2. Partial sums of $\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)$ and $\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)$

Theorem 2.1. *Let $a, b, \alpha, \beta \in \mathbb{R}$ and $\lambda = a + b + ab$, $\mu = ab$ with inequalities $a > \frac{-b}{b+1}$, $\lambda\mu > 0$ and $\lambda\mu > \lambda + 1$. Then*

$$\Re\left\{\frac{\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)}{\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)_n(z)}\right\} \geq \frac{\lambda\mu - \lambda - 1}{\lambda\mu}, \quad z \in \mathcal{U}, \tag{2.1}$$

and

$$\Re\left\{\frac{\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)_n(z)}{\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)}\right\} \geq \frac{\lambda\mu}{\lambda\mu + \lambda + 1}, \quad z \in \mathcal{U}. \tag{2.2}$$

Proof. By using (i) of Lemma 1.1, it is clear that

$$1 + \sum_{m=1}^{\infty} |a_m| \leq \frac{\lambda\mu + \lambda + 1}{\lambda\mu},$$

which is equivalent to

$$\frac{\lambda\mu}{\lambda + 1} \sum_{m=1}^{\infty} |a_m| \leq 1,$$

where $a_m = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)}$. Now, we may write

$$\begin{aligned}
&\frac{\lambda\mu}{\lambda + 1}\left\{\frac{\mathcal{W}_{\lambda,\mu}(z)}{(\mathcal{W}_{\lambda,\mu})_n(z)} - \frac{\lambda\mu - \lambda - 1}{\lambda\mu}\right\} \\
&= \frac{1 + \sum\limits_{m=1}^{n} a_m z^m + \left(\frac{\lambda\mu}{\lambda+1}\right) \sum\limits_{m=n+1}^{\infty} a_m z^m}{1 + \sum\limits_{m=1}^{n} a_m z^m} =: \frac{1 + w(z)}{1 - w(z)}.
\end{aligned}$$

Then it is clear that

$$w(z) = \frac{\left(\frac{\lambda\mu}{\lambda+1}\right) \sum\limits_{m=n+1}^{\infty} a_m z^m}{2 + 2\sum\limits_{m=1}^{n} a_m z^m + \left(\frac{\lambda\mu}{\lambda+1}\right) \sum\limits_{m=n+1}^{\infty} a_m z^m}$$

and

$$|w(z)| \leq \frac{\left(\frac{\lambda\mu}{\lambda+1}\right) \sum\limits_{m=n+1}^{\infty} |a_m|}{2 - 2\sum\limits_{m=1}^{n} |a_m| - \left(\frac{\lambda\mu}{\lambda+1}\right) \sum\limits_{m=n+1}^{\infty} |a_m|}.$$

This implies that $|w(z)| \leq 1$ if and only if

$$2\left(\frac{\lambda\mu}{\lambda + 1}\right) \sum_{m=n+1}^{\infty} |a_m| \leq 2 - 2\sum_{m=1}^{n} |a_m|.$$

Which further implies that

$$\sum_{m=1}^{n}|a_m|+\left(\frac{\lambda\mu}{\lambda+1}\right)\sum_{m=n+1}^{\infty}|a_m|\leq 1. \tag{2.3}$$

It suffices to show that the left hand side of (2.3) is bounded above by

$$\left(\frac{\lambda\mu}{\lambda+1}\right)\sum_{m=1}^{\infty}|a_m|,$$

which is equivalent to

$$\frac{\lambda\mu-\lambda-1}{\lambda+1}\sum_{m=1}^{n}|a_m|\geq 0.$$

To prove (2.2) , we write

$$\frac{\lambda\mu+\lambda+1}{\lambda+1}\left\{\frac{\left(\mathbb{W}_{(\beta,b)}^{(\alpha,a)}\right)_n(z)}{\mathbb{W}_{(\beta,b)}^{(\alpha,a)}(z)}-\frac{\lambda\mu}{\lambda\mu+\lambda+1}\right\}$$

$$=\frac{1+\sum\limits_{m=1}^{n}a_m z^m-\left(\frac{\lambda\mu}{\lambda+1}\right)\sum\limits_{m=n+1}^{\infty}a_m z^m}{1+\sum\limits_{m=1}^{\infty}a_m z^m}=\frac{1+w(z)}{1-w(z)}.$$

Therefore

$$|w(z)|\leq\frac{\left(\frac{\lambda\mu}{\lambda+1}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}{2-2\sum\limits_{m=1}^{n}|a_m|-\left(\frac{\lambda\mu}{\lambda+1}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}\leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}|a_m|+\left(\frac{\lambda\mu}{\lambda+1}\right)\sum_{m=n+1}^{\infty}|a_m|\leq 1. \tag{2.4}$$

Since the left hand side of (2.4) is bounded above by $\left(\frac{\lambda\mu}{\lambda+1}\right)\sum\limits_{m=1}^{\infty}|a_m|$, this completes the proof. □

Remark 2.2. When we put $a=1$ and $\alpha=1$ in Theorem 2.1, we get Theorem 2.1 of [15] as corollary.

Corollary 2.3. *Let* β, $b\in\mathbb{R}$ *such that* $\beta\geq 1$, $b>1.280776406\ldots$. *Then*

$$\Re\left\{\frac{\mathcal{W}_{\beta,b}(z)}{\left(\mathcal{W}_{\beta,b}\right)_n(z)}\right\}\geq\frac{2b^2-b-2}{2b^2+b},\quad z\in\mathcal{U}, \tag{2.5}$$

and

$$Re\left\{\frac{\left(\mathcal{W}_{\beta,b}\right)_n(z)}{\mathcal{W}_{\beta,b}(z)}\right\}\geq\frac{2b^2+b}{2b^2+3b+2},\quad z\in\mathcal{U}. \tag{2.6}$$

Theorem 2.4. *Let $a, b, \alpha, \beta \in \mathbb{R}$ and $\lambda = a + b + ab$, $\mu = ab$ with inequalities $a > \frac{1-b}{b+1}$, $\lambda\mu > 0$ and $\lambda\mu > 2\lambda - \mu + 2$. Then*

$$\Re\left\{\frac{\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)}{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)_n(z)}\right\} \geq \frac{\lambda\mu - 2\lambda + \mu - 2}{\lambda\mu}, \quad z \in \mathcal{U}, \tag{2.7}$$

$$\Re\left\{\frac{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)_n(z)}{\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)}\right\} \geq \frac{\lambda\mu}{\lambda\mu + 2\lambda - \mu + 2}, \quad z \in \mathcal{U}. \tag{2.8}$$

Proof. From part (ii) of Lemma 1.1, we observe that

$$1 + \sum_{m=1}^{\infty}(m+1)\left|a_m\right| \leq \frac{\lambda\mu + 2\lambda - \mu + 2}{\lambda\mu},$$

where $a_m = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+m\alpha)\Gamma(b+m\beta)}$. This implies that

$$\left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum_{m=1}^{\infty}(m+1)\left|a_m\right| \leq 1.$$

Consider

$$\left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\left\{\frac{\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)}{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)_n(z)} - \frac{\lambda\mu - 2\lambda + \mu - 2}{\lambda\mu}\right\}$$

$$= \frac{1 + \sum\limits_{m=1}^{n}(m+1)a_m z^m + \left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum\limits_{m=n+1}^{\infty}(m+1)a_m z^m}{1 + \sum\limits_{m=1}^{n}(m+1)a_m z^m} = \frac{1 + w(z)}{1 - w(z)}.$$

Therefore

$$\left|w(z)\right| \leq \frac{\left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum\limits_{m=n+1}^{\infty}(m+1)\left|a_m\right|}{2 - 2\sum\limits_{m=1}^{n}(m+1)\left|a_m\right| - \left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum\limits_{m=n+1}^{\infty}(m+1)\left|a_m\right|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}(m+1)\left|a_m\right| + \left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum_{m=n+1}^{\infty}(m+1)\left|a_m\right| \leq 1. \tag{2.9}$$

It suffices to show that the left hand side of (2.9) is bounded above by

$$\left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum_{m=1}^{\infty}\left|a_m\right|(m+1).$$

Which is equivalent to $\left(\frac{\lambda\mu}{\lambda\mu - \mu + 2}\right)\sum\limits_{m=1}^{n}(m+1)\left|a_m\right| \geq 0$.

To prove the result (2.8), we write

$$\left(\frac{\lambda\mu+2\lambda-\mu+2}{2\lambda-\mu+2}\right)\left\{\frac{\left(\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}\right)_n(z)}{\mathbb{W}'^{(\alpha,a)}_{(\beta,b)}(z)}-\frac{\lambda\mu}{\lambda\mu+2\lambda-\mu+2}\right\}=\frac{1+w(z)}{1-w(z)}.$$

Therefore

$$|w(z)|\leq\frac{\left(\frac{\lambda\mu+2\lambda-\mu+2}{2\lambda-\mu+2}\right)\sum\limits_{m=n+1}^{\infty}(m+1)|a_m|}{2-2\sum\limits_{m=1}^{n}(m+1)|a_m|-\left(\frac{\lambda\mu-2\lambda+\mu-2}{2\lambda-\mu+2}\right)\sum\limits_{m=n+1}^{\infty}(m+1)|a_m|}\leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}|a_m|(m+1)+\frac{\lambda\mu}{2\lambda-\mu+2}\sum_{m=n+1}^{\infty}(m+1)|a_m|\leq 1. \tag{2.10}$$

It suffices to show that the left hand side of (2.10) is bounded above by $\left(\frac{\lambda\mu}{2\lambda-\mu+2}\right)\sum\limits_{m=1}^{\infty}(m+1)|a_m|$, the proof is complete. □

Remark 2.5. When we put $a = 1$ and $\alpha = 1$ in Theorem 2.4, we get the improved version of Theorem 2.2 of [15] as:

Theorem 2.6. *Let β, $b\in\mathbb{R}$, with $\beta\geq 1$ and $b>2$. Then*

$$\Re\left\{\frac{\mathcal{W}'_{\beta,b}(z)}{\left(\mathcal{W}_{\beta,b}\right)'_n(z)}\right\}\geq\frac{2b^2-2b-4}{2b^2+b},\quad z\in\mathcal{U}, \tag{2.11}$$

$$Re\left\{\frac{\left(\mathcal{W}_{\beta,b}\right)'_n(z)}{\mathcal{W}'_{\beta,b}(z)}\right\}\geq\frac{2b^2+b}{2b^2+4b+4},\quad z\in\mathcal{U}. \tag{2.12}$$

3. Partial sums of $\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right](z)$

Theorem 3.1. *Let $a,b,\alpha,\beta\in\mathbb{R}$ and $M=(a+1)(b+1)$, $N=ab$ with inequalities $a>\frac{-b}{b+1}$ and $2MN>M+2N$. Then*

$$Re\left\{\frac{\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right](z)}{\left(\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right]\right)_n(z)}\right\}\geq\frac{2MN-2N-M}{2MN-2N},\quad z\in\mathcal{U}, \tag{3.1}$$

and

$$Re\left\{\frac{\left(\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right]\right)_n(z)}{\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right](z)}\right\}\geq\frac{2MN-2N}{2MN+M-2N},\quad z\in\mathcal{U}, \tag{3.2}$$

where $\mathbb{A}\left[\mathbb{W}^{(\alpha,a)}_{(\beta,b)}\right]$ is the Alexander transform of $\mathbb{W}^{(\alpha,a)}_{(\beta,b)}$.

Proof. To prove (3.1), we consider from part (iii) of Lemma 1.1 so that

$$1+\sum_{m=1}^{\infty}\frac{|a_m|}{(m+1)}\leq\frac{2MN-2N+M}{2MN-2N},$$

which is equivalent to

$$\left(\frac{2MN-2N}{M}\right)\sum_{m=1}^{\infty}\frac{|a_m|}{(m+1)}\leq 1,$$

where $a_m=\frac{\Gamma(a)\Gamma(b)}{(m+1)\Gamma(a+m\alpha)\Gamma(b+m\beta)}$. Now, we write

$$\left(\frac{2MN-2N}{M}\right)\left\{\frac{\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right](z)}{\left(\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right]\right)_n(z)}-\frac{2MN-2N-M}{2MN-2N}\right\}$$

$$=\frac{1+\sum\limits_{m=1}^{n}\frac{a_m}{(m+1)}z^m+\left(\frac{2MN-2N}{M}\right)\sum\limits_{m=n+1}^{\infty}\frac{a_m}{(m+1)}z^m}{1+\sum\limits_{m=1}^{n}\frac{a_m}{(m+1)}z^m}=\frac{1+w(z)}{1-w(z)},$$

where

$$|w(z)|\leq\frac{\left(\frac{2MN-2N}{M}\right)\sum\limits_{m=n+1}^{\infty}\frac{|a_m|}{(m+1)}}{2-2\sum\limits_{m=1}^{n}\frac{|a_m|}{(m+1)}-\left(\frac{2MN-2N}{M}\right)\sum\limits_{m=n+1}^{\infty}\frac{|a_m|}{(m+1)}}\leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}\frac{|a_m|}{(m+1)}+\left(\frac{2MN-2N}{M}\right)\sum_{m=n+1}^{\infty}\frac{|a_m|}{(m+1)}\leq 1. \tag{3.3}$$

It suffices to show that the left hand side of (3.3) is bounded above by

$$\left(\frac{2MN-2N}{M}\right)\sum_{m=1}^{\infty}\frac{|a_m|}{(m+1)}.$$

This completes the proof.

The proof of (3.2) is similar to the proof of Theorem 2.1. □

Remark 3.2. When we put $a=1$ and $\alpha=1$ in Theorem 3.1, we get Theorem 2.3 of [15] as corollary.

Corollary 3.3. *Let β, $b\in\mathbb{R}$, with $\beta\geq 1$ and $b>0.70710678\ldots$. Then*

$$Re\left\{\frac{\mathbb{A}\left[\mathcal{W}_{\beta,b}\right](z)}{\left(\mathbb{A}\left[\mathcal{W}_{\beta,b}\right]\right)_n(z)}\right\}\geq\frac{2b^2-1}{2b^2+b},\quad z\in\mathcal{U}, \tag{3.4}$$

and

$$Re\left\{\frac{\left(\mathbb{A}\left[\mathcal{W}_{\beta,b}\right]\right)_n(z)}{\mathbb{A}\left[\mathcal{W}_{\beta,b}\right](z)}\right\}\geq\frac{2b^2+b}{2b^2+2b+1},\quad z\in\mathcal{U}, \tag{3.5}$$

where $\mathbb{A}\left[\mathcal{W}_{\beta,b}\right]$ is the Alexander transform of $\mathcal{W}_{\beta,b}$.

4. Applications

The four parametric Wright function

$$\mathcal{W}_{(\beta,b)}^{(\alpha,a)}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma\left(a+m\alpha\right)\Gamma\left(b+m\beta\right)},\quad a,b\in\mathbb{C},\quad \alpha,\beta\in\mathbb{R},$$

is the generalization of Wright function. When we put $a=1$ and $\alpha=1$ in above series we get the Wright function

$$W_{\beta,b}(z)=\sum_{m=0}^{\infty}\frac{z^m}{m!\Gamma\left(\beta m+b\right)},\ \beta>-1,\ b\in\mathbb{C}.$$

For $\beta=1$, $b=5/2$ we get $W_{1,5/2}(-z)=\frac{3}{4}\left(\frac{\sin(2\sqrt{z})}{2\sqrt{z}}-\cos(2\sqrt{z})\right)$, and for $n=0$, we have $\left(W_{1,5/2}\right)_0(z)=z$, so,

$$\Re\left(\frac{\sin(2\sqrt{z})-2\sqrt{z}\cos(2\sqrt{z})}{2z\sqrt{z}}\right)\geq\frac{8}{15}\cong 0.53333\ldots\quad (z\in\mathcal{U}),\tag{4.1}$$

and

$$\Re\left(\frac{2z\sqrt{z}}{\sin(2\sqrt{z})-2\sqrt{z}\cos(2\sqrt{z})}\right)\geq\frac{15}{22}\cong 0.681818\ldots\quad (z\in\mathcal{U}).\tag{4.2}$$

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