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MIAO-TAM EQUATION ON ALMOST COKÄHLER MANIFOLDS

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Abstract. In the present paper, we have studied Miao-Tam equation on three dimensional almost coKähler manifolds. We have also proved that there does not exist non-trivial solution of Miao-Tam equation on the said manifolds if the dimension is greater than three. Also we give an example to verify the deduced results.

1. Introduction

Recently, the study of differentiable manifolds endowed with certain structures, namely almost contact and almost complex structures has become a subject of growing interest due to their applications in relativity, cosmology, string theory etc. An odd dimensional differentiable manifold M equipped with a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1 \tag{1}$$

for any $X \in \chi(M)$, the set of all vector fields on M, is known as an almost contact manifold [4].

If the 2-form Φ given by $\Phi(X,Y) = g(\phi X, Y)$ for any X, $Y \in \chi(M)$ and the 1-form η both are closed, then the almost contact manifold is called an almost coKähler manifold [3]. Due to additional properties that Φ and η are closed almost coKähler manifolds show some special properties which are not found generally in almost contact manifolds. Therefore almost coKähler manifolds need special attention. In 1967, Blair [3] introduced the notion of almost coKähler manifolds. The almost coKähler manifolds are odd dimensional analogues of the almost Kähler manifolds [17]. So many examples of almost coKähler manifolds have been constructed by various authors. For instance, the Riemannian product of a real line and an almost Kähler manifold admits an almost coKähler manifold [19, 23, 24]. Many authors such as Blair [5], De and Sardar [11], De, Majhi and Suh [10], Dacko [8], Dacko and Olszak [9],

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Goldberg and Yano [14], Olszak [23, 24], and Wang [27, 28] have been studied almost coKähler manifolds.

In [18], Miao-Tam equation has been studied on normal almost contact manifold. Due to distinguishing nature of almost coKähler manifolds we feel the necessity of investigation of Miao-Tam equation on almost coKähler manifolds. Miao-Tam equation on a compact Riemannian manifold M^n of dimension greater than two is given by

$$-(\Delta_g \lambda)g + \nabla_g^2 \lambda - \lambda S = g, \tag{2}$$

where $\lambda : M^n \to R$ is smooth on the manifold M^n and $\lambda = 0$ on the boundary ∂M^n. Δ_g, ∇_g^2 are the negative Laplacian and Hessain operator, respectively, with respect to the metric g. For details see ([2, 20]). In the paper [21], the authors studied conformally flat Riemannian manifolds satisfying Miao-Tam equation. Miao-Tam equation on different manifolds has been studied by several other authors ([1, 7, 12, 13, 16, 25, 26]).

The present paper is organized as follows: After introduction, we give some preliminaries in Section 2. In Section 3, we have studied three dimensional (κ, μ)-almost coKähler manifolds satisfying Miao-Tam equation. In the same section we have proved that there does not exist a non-trivial solution of the Miao-Tam equation if the dimension of the (κ, μ)-almost coKähler manifold is greater than three. In the last section, we give an example to verify the deduced results.

2. Preliminaries

An almost contact manifold satisfying (1) is called an almost contact metric manifold if it admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

As a consequence of (1) and the above equation, we have

$$\phi\xi = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0,$$

$$g(\phi X, Y) = -g(X, \phi Y)$$

for any vector fields X, $Y \in \chi(M)$ ([11, 15]).

For an almost coKähler manifold the 2-form Φ and the 1-form η are closed. That is, $d\Phi = 0$ and $d\eta = 0$.

On an almost coKähler manifold, we also have ([11, 15])

$$h\xi = 0, \quad tr(h) = 0, \quad h\phi = -\phi h, \tag{3}$$

$$\nabla_X \xi = h\phi X, \tag{4}$$

$$\phi l \phi - l = 2h^2$$

for any vector field $X \in \chi(M)$. Here the operators h and l are defined by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $l = R(\cdot,\xi)\xi$, respectively, where R is the Riemannian curvature tensor and $\mathcal{L}$ is the Lie differentiation operator. 'tr' denotes trace. The operators h and l both are symmetric.

For an almost coKähler manifold the 1-form η satisfies

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0$$

for any vector fields X, $Y \in \chi(M)$.

In [6], Blair et al. introduced the idea of (κ,μ)-nullity distribution in the context of contact geometry. The vector field ξ is said to belong to (κ,μ)-nullity distribution if

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{5}$$

for any vector fields X, $Y \in \chi(M)$ and κ, μ being real numbers. If κ, μ both are smooth functions on the manifold, then the nullity distribution is called generalized (κ,μ)-nullity distribution.

If the Reeb vector field ξ of an almost coKähler manifold belongs to (κ,μ)-nullity distribution, then it is called a (κ,μ)-almost coKähler manifold. For details see ([11, 15]). On a (κ,μ)-almost coKähler manifold, one obtains

$$h^2X = \kappa\phi^2X, \tag{6}$$

$$S(X,\xi) = 2n\kappa\eta(X), \tag{7}$$

$$Q\xi = 2n\kappa\xi \tag{8}$$

for any vector field $X \in \chi(M)$, where S and Q are the Ricci tensor of type (0,2) and Ricci operator, respectively.

Now let us recall two known lemmas:

Lemma 2.1 ([27]). *Let M^{2n+1} be a (κ,μ)-almost coKähler manifold of dimension greater than 3 with $\kappa < 0$. Then the Ricci operator Q is given by*

$$QX = \mu hX + 2n\kappa\eta(X)\xi \tag{9}$$

for all vector field X on M^{2n+1}, where κ is a non-zero constant and μ is a smooth function satisfying $d\mu \wedge \eta = 0$. Moreover, the scalar curvature of M^{2n+1} is $2n\kappa$.

Lemma 2.2 ([13]). *Let (M^{2n+1}, g) be a Riemannian manifold of dimension $(2n+1)$ satisfying the Miao-Tam equation. Then the curvature tensor R can be expressed as*

$$\begin{aligned} R(X,Y)D\lambda = {} & (X\lambda)QY - (Y\lambda)QX + \lambda((\nabla_X Q)Y - (\nabla_Y Q)X) \\ & + (Xf)Y - (Yf)X \end{aligned} \tag{10}$$

for any vector fields X, Y on M, where $f = -\frac{r\lambda+1}{2n}$ and D is the gradient operator.

Moreover,

$$\nabla_X D\lambda = \lambda QX + fX \tag{11}$$

for any vector field X.

3. Almost coKähler manifolds satisfying Miao-Tam equation

In the present section, we study (κ, μ)-almost coKähler manifolds with $\kappa < 0$ which satisfy Miao-Tam equation.

Theorem 3.1. *If a (κ, μ)-almost coKähler manifold of dimension three satisfy Miao-Tam equation, then either $\mu^2 = -\frac{(6\kappa - r)^2}{4\kappa}$ or the potential function is constant, i.e., the manifold is Einstein.*

Proof. Taking inner product of (10) with the vector field ξ and using (7), we get

$$\begin{aligned} g(R(X,Y)D\lambda, \xi) &= 2\kappa(X\lambda)\eta(Y) - 2\kappa(Y\lambda)\eta(X) \\ &\quad + \lambda[g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X)] \\ &\quad + (Xf)\eta(Y) - (Yf)\eta(X). \end{aligned} \tag{12}$$

Since any Riemannian metric g satisfying the Miao-Tam equation must have constant scalar curvature [20], r is constant. Thus we have $(Xf) = -\frac{r}{2}(X\lambda)$ and $(Yf) = -\frac{r}{2}(Y\lambda)$. Therefore from (12), we get

$$\begin{aligned} g(R(X,Y)D\lambda, \xi) &= (2\kappa - \frac{r}{2})[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] \\ &\quad + \lambda[g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X)]. \end{aligned} \tag{13}$$

Taking covariant derivative of (8) along the vector field X, we obtain

$$(\nabla_X Q)\xi = 2\kappa h\phi X - Qh\phi X. \tag{14}$$

Using (14) in (13), we get

$$\begin{aligned} g(R(X,Y)D\lambda, \xi) &= (2\kappa - \frac{r}{2})[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] \\ &\quad - \lambda[S(h\phi X, Y) - S(X, h\phi Y)]. \end{aligned}$$

Putting $X = \xi$ in the above equation and using (7), we get

$$g(R(\xi,Y)D\lambda, \xi) = (2\kappa - \frac{r}{2})[(\xi\lambda)\eta(Y) - (Y\lambda)]. \tag{15}$$

Putting $X = \xi$ in (5) and then taking inner product with $D\lambda$, we get

$$g(R(\xi,Y)\xi, D\lambda) = \kappa[(\xi\lambda)\eta(Y) - (Y\lambda)] - \mu(hY\lambda). \tag{16}$$

Since $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$ for all vector fields X, Y, Z and W, from (15) and (16), we get

$$(3\kappa - \frac{r}{2})[(\xi\lambda)\eta(Y) - (Y\lambda)] = \mu(hY\lambda). \tag{17}$$

Replacing Y by hY in (17), we obtain

$$(hY\lambda) = -\frac{2\kappa\mu}{6\kappa - r}[(\xi\lambda)\eta(Y) - (Y\lambda)]. \tag{18}$$

Therefore, from (17) and (18), we obtain

$$\frac{(6\kappa - r)^2 + 4\kappa\mu^2}{2(6\kappa - r)}[(\xi\lambda)\eta(Y) - (Y\lambda)] = 0.$$

Thus, there arise two cases
Case-I: $(6\kappa - r)^2 + 4\kappa\mu^2 = 0$, which gives $\mu^2 = -\frac{(6\kappa - r)^2}{4\kappa}$.
Case-II: $(Y\lambda) = (\xi\lambda)\eta(Y)$, which gives

$$D\lambda = (\xi\lambda)\xi. \tag{19}$$

Taking differentiation of (19) covariantly along the vector field X and using (4), we get

$$\nabla_X D\lambda = X(\xi\lambda)\xi + (\xi\lambda)h\phi X. \tag{20}$$

Comparing (11) and (20), we get

$$X(\xi\lambda)\xi + (\xi\lambda)h\phi X = \lambda QX + fX.$$

Tracing over X in the above equation and using $tr(h\phi) = 0$, we obtain

$$\xi(\xi\lambda) = r\lambda + 3f, \tag{21}$$

where r is the scalar curvature.

Putting $X = \xi$ in (11) and taking inner product with ξ, we get

$$\xi(\xi\lambda) = \lambda S(\xi, \xi) + f. \tag{22}$$

From (7), we get

$$S(\xi, \xi) = 2\kappa. \tag{23}$$

From (22) and (23), we obtain

$$\xi(\xi\lambda) = 2\kappa\lambda + f. \tag{24}$$

Comparing (21) and (24), we get

$$\lambda = -\frac{1}{2\kappa},$$

which is a non-zero constant. □

Theorem 3.2. *There does not exist a non-trivial solution of Miao-Tam equation on a (κ, μ)-almost coKähler manifold M^{2n+1} of dimension greater than three with $\kappa < 0$.*

Proof. Taking inner product of (10) with the vector field ξ and using (7), we get

$$g(R(X,Y)D\lambda,\xi) = 2n\kappa(X\lambda)\eta(Y) - 2n\kappa(Y\lambda)\eta(X) + \lambda[g((\nabla_X Q)\xi,Y) - g((\nabla_Y Q)\xi,X)] + (Xf)\eta(Y) - (Yf)\eta(X). \tag{25}$$

Since r is constant, we have $(Xf) = -\frac{r}{2n}(X\lambda)$ and $(Yf) = -\frac{r}{2n}(Y\lambda)$. Therefore from (25), we get

$$g(R(X,Y)D\lambda,\xi) = (2n\kappa - \frac{r}{2n})[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] + \lambda[g((\nabla_X Q)\xi,Y) - g((\nabla_Y Q)\xi,X)]. \tag{26}$$

Taking covariant derivative of equation (8) along the vector field X, we obtain

$$(\nabla_X Q)\xi = 2n\kappa h\phi X - Qh\phi X. \tag{27}$$

Using (27) in (26), we get

$$g(R(X,Y)D\lambda,\xi) = (2n\kappa - \frac{r}{2n})[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] - \lambda[S(h\phi X,Y) - S(X,h\phi Y)]. \tag{28}$$

Replacing X by ϕX and Y by ϕY in (28), we obtain

$$g(R(\phi X,\phi Y)D\lambda,\xi) = -\lambda[S(h\phi^2 X,\phi Y) - S(\phi X,h\phi^2 Y)]. \tag{29}$$

From (3), (6) and (9), we obtain

$$S(h\phi^2 X,\phi Y) = \mu\kappa g(X,\phi Y). \tag{30}$$

Therefore, from (29) and (30), we get

$$g(R(\phi X,\phi Y)D\lambda,\xi) = -2\mu\kappa\lambda g(X,\phi Y). \tag{31}$$

Also, replacing X by ϕX and Y by ϕY in (5) and then taking inner product with $D\lambda$, we obtain

$$g(R(\phi X,\phi Y)\xi,D\lambda) = 0. \tag{32}$$

Since $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$ for all vector fields X, Y, Z and W, from (31) and (32), we get

$$\mu\kappa\lambda g(X,\phi Y) = 0. \tag{33}$$

Since $\kappa < 0$, from (33), we get

$$\mu\lambda = 0.$$

Let us assume that $\lambda \neq 0$ in some open subset Ω of the manifold. Then $\mu = 0$ on Ω.

Now, putting $X = \xi$ in (5) and then taking inner product with $D\lambda$, we get

$$g(R(\xi,Y)\xi,D\lambda) = \kappa[(\xi\lambda)\eta(Y) - (Y\lambda)]. \tag{34}$$

Again, putting $X = \xi$ in (28) and using (7), we get

$$g(R(\xi,Y)D\lambda,\xi) = (2n\kappa - \frac{r}{2n})[(\xi\lambda)\eta(Y) - (Y\lambda)]. \tag{35}$$

Since $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$ for all vector fields X, Y, Z and W, from (34) and (35), we get

$$2n\kappa[(\xi\lambda)\xi - (Y\lambda)] = 0, \tag{36}$$

since $r = 2n\kappa$.

Therefore, from (36), we get

$$D\lambda = (\xi\lambda)\xi, \tag{37}$$

since $\kappa < 0$.

Taking covariant derivative of (37) and using (4), we get

$$\nabla_X D\lambda = X(\xi\lambda)\xi + (\xi\lambda)h\phi X. \tag{38}$$

Also, from (9) and (11), we get

$$\nabla_X D\lambda = 2n\kappa\lambda\eta(X)\xi - (\kappa\lambda + \frac{1}{2n})X. \tag{39}$$

Comparing (38) and (39), we get

$$X(\xi\lambda)\xi + (\xi\lambda)h\phi X = 2n\kappa\lambda\eta(X)\xi - (\kappa\lambda + \frac{1}{2n})X. \tag{40}$$

Tracing over X in (40) and using $tr(h\phi) = 0$, we obtain

$$\xi(\xi\lambda) = -(\kappa\lambda + \frac{2n+1}{2n}). \tag{41}$$

Again, putting $X = \xi$ in (39) and then taking inner product with ξ, we get

$$\xi(\xi\lambda) = 2n\kappa\lambda - (\kappa\lambda + \frac{1}{2n}). \tag{42}$$

From (41) and (42), we get

$$2n\kappa\lambda + 1 = 0. \tag{43}$$

Again, operating (40) by ϕ and using (3) and (43), we get

$$(\xi\lambda) = 0.$$

Therefore, from (37), λ is a constant.

Thus, from (2), we get

$$S(X,Y) = -\frac{1}{\lambda}g(X,Y). \tag{44}$$

Putting $X = Y = e_i$ in (44), where $\{e_i\}$ is the orthonormal basis of the tangent space of the manifold and summing over i, $1 \leq i \leq 2n+1$, we obtain

$$\lambda = -\frac{2n+1}{2n\kappa},$$

since $r = 2n\kappa$.

Thus

$$S(X,Y)=\frac{2n\kappa}{2n+1}g(X,Y). \tag{45}$$

Putting $X=Y=\xi$ in (45) and using (7), we get $\kappa=0$, which is a contradiction. □

4. Example

In the paper [22], the authors have constructed an example of almost coKähler manifold of dimension three. Now we construct an example of almost coKähler manifold of dimension five following this example.

Let us consider the manifold $M=\{x,y,u,v,z\in\mathbb{R}^5 : z\neq 0\}$ of dimension 5, where $\{x,y,u,v,z\}$ are standard co-ordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1=e^{\frac{z}{2}}\frac{\partial}{\partial x}-e^{-\frac{z}{2}}\frac{\partial}{\partial y},\quad e_2=e^{\frac{z}{2}}\frac{\partial}{\partial x}+e^{-\frac{z}{2}}\frac{\partial}{\partial y},$$
$$e_3=e^{\frac{z}{2}}\frac{\partial}{\partial u}-e^{-\frac{z}{2}}\frac{\partial}{\partial v},\quad e_4=e^{\frac{z}{2}}\frac{\partial}{\partial u}+e^{-\frac{z}{2}}\frac{\partial}{\partial v},\quad e_5=\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. We get the following by direct computations

$$[e_1,e_2]=0,\quad [e_1,e_3]=0,\quad [e_1,e_4]=0,\quad [e_1,e_5]=-\frac{1}{2}e_2,$$
$$[e_2,e_3]=0,\quad [e_2,e_4]=0,\quad [e_2,e_5]=-\frac{1}{2}e_1,$$
$$[e_3,e_4]=0,\quad [e_3,e_5]=-\frac{1}{2}e_4,\quad [e_4,e_5]=-\frac{1}{2}e_3.$$

Let the metric tensor g be defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=g(e_4,e_4)=g(e_5,e_5)=1$$

and $g(e_i,e_j)=0$ for all $i\neq j$; $i,j=1,2,3,4,5$.

The 1-form η is defined by $\eta(X)=g(X,e_5)$ for all X on M. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1)=-e_2,\quad \phi(e_2)=e_1,\quad \phi(e_3)=-e_4,\quad \phi(e_4)=e_3,\quad \phi(e_5)=0.$$

Then we find that

$$\eta(e_5)=1,\quad \phi^2X=-X+\eta(X)e_5,$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for any vector fields X, Y on M. Thus (ϕ,e_5,η,g) defines an almost contact metric structure.

For the Levi-Civita connection ∇ with respect to the metric g on M, we can write

$$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)$$
$$-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),$$

which is known as Koszul's formula.

By Koszul's formula, we get the following expressions:

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = \frac{1}{2}e_5, \quad \nabla_{e_1}e_3 = 0, \quad \nabla_{e_1}e_4 = 0,$$

$$\nabla_{e_1}e_5 = -\frac{1}{2}e_2, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_1 = \frac{1}{2}e_5, \quad \nabla_{e_2}e_3 = 0,$$

$$\nabla_{e_2}e_4 = 0, \quad \nabla_{e_2}e_5 = -\frac{1}{2}e_1, \quad \nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0,$$

$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_4 = \frac{1}{2}e_5, \quad \nabla_{e_3}e_5 = -\frac{1}{2}e_4, \quad \nabla_{e_4}e_1 = 0,$$

$$\nabla_{e_4}e_2 = 0, \quad \nabla_{e_4}e_3 = \frac{1}{2}e_5, \quad \nabla_{e_4}e_4 = 0, \quad \nabla_{e_4}e_5 = -\frac{1}{2}e_3,$$

$$\nabla_{e_5}e_1 = 0, \qquad \nabla_{e_5}e_2 = 0, \qquad \nabla_{e_5}e_3 = 0,$$

$$\nabla_{e_5}e_4 = 0, \quad \nabla_{e_5}e_5 = 0.$$

From the above expressions of ∇, we conclude that the given manifold is an almost coKähler manifold with $he_1 = -\frac{1}{2}e_1$, $he_2 = \frac{1}{2}e_2$, $he_3 = -\frac{1}{2}e_3$, $he_4 = \frac{1}{2}e_4$ and $he_5 = 0$.

Using the formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we get

$$R(e_1,e_2)e_1 = -\frac{1}{4}e_2, \quad R(e_1,e_2)e_2 = \frac{1}{4}e_1, \quad R(e_2,e_5)e_5 = -\frac{1}{4}e_2,$$

$$R(e_5,e_2)e_2 = -\frac{1}{4}e_5, \quad R(e_1,e_5)e_5 = -\frac{1}{4}e_1, \quad R(e_5,e_1)e_1 = -\frac{1}{4}e_5,$$

$$R(e_3,e_4)e_3 = -\frac{1}{4}e_4, \qquad R(e_3,e_4)e_4 = \frac{1}{4}e_3, \quad R(e_3,e_5)e_5 = -\frac{1}{4}e_3,$$

$$R(e_5,e_3)e_3 = -\frac{1}{4}e_5, \quad R(e_4,e_5)e_5 = -\frac{1}{4}e_4, \quad R(e_5,e_4)e_4 = -\frac{1}{4}e_5$$

and the remaining $R(e_i,e_j)e_k = 0$, $i,j,k = 1,2,3,4,5$.

From above, we conclude that the manifold is a (κ,μ)-almost coKähler manifold with $\kappa = -\frac{1}{4}$ and $\mu = 0$.

From the expressions of curvature tensor, we get

$$S(e_1,e_1) = 0, \quad S(e_2,e_2) = 0, \quad S(e_3,e_3) = 0,$$

$$S(e_4,e_4) = 0, \quad S(e_5,e_5) = -1$$

and $S(e_i,e_j) = 0$, for all $i \neq j$; $i,j = 1,2,3,4,5$.

Let r be the scalar curvature. Then from above, we get

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) + S(e_4,e_4) + S(e_5,e_5) = -1.$$

Let $\lambda = e^z + b$, where b is a constant. Therefore, $D\lambda = e^z e_5 = (\lambda - b)e_5$. Thus we get the following:

$$\nabla_{e_1}D\lambda = -\frac{1}{2}(\lambda - b)e_2, \quad \nabla_{e_2}D\lambda = -\frac{1}{2}(\lambda - b)e_1,$$

$$\nabla_{e_3}D\lambda = -\frac{1}{2}(\lambda - b)e_4, \quad \nabla_{e_4}D\lambda = -\frac{1}{2}(\lambda - b)e_3, \quad \nabla_{e_5}D\lambda = (\lambda - b)e_5.$$

From the above expressions, we get

$$\Delta_g \lambda = (\lambda - b).$$

Thus,

$$\begin{aligned}
-(\Delta_g \lambda) g(e_1, e_1) + g(\nabla_{e_1} D\lambda, e_1) - \lambda S(e_1, e_1) &= -(\lambda - b), \\
-(\Delta_g \lambda) g(e_2, e_2) + g(\nabla_{e_2} D\lambda, e_2) - \lambda S(e_2, e_2) &= -(\lambda - b), \\
-(\Delta_g \lambda) g(e_3, e_3) + g(\nabla_{e_3} D\lambda, e_3) - \lambda S(e_3, e_3) &= -(\lambda - b), \\
-(\Delta_g \lambda) g(e_4, e_4) + g(\nabla_{e_4} D\lambda, e_4) - \lambda S(e_4, e_4) &= -(\lambda - b), \\
-(\Delta_g \lambda) g(e_5, e_5) + g(\nabla_{e_5} D\lambda, e_5) - \lambda S(e_5, e_5) &= \lambda.
\end{aligned}$$

From the last five expressions, we conclude that there does not exist any solution of the Miao-Tam equation on the given manifold and this verifies Theorem 3.2.

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