Commun. Korean Math. Soc. **37** (2022), No. 3, pp. 839–849
https://doi.org/10.4134/CKMS.c210097
pISSN: 1225-1763 / eISSN: 2234-3024

NEW VOLUME COMPARISON WITH ALMOST RICCI SOLITON

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Abstract. In this paper we consider a condition on the Ricci curvature involving vector fields which enabled us to achieve new results for volume comparison and Laplacian comparison. These results in special case obtained with considering volume non-collapsing condition. Also, by applying this condition we get new results of volume comparison for almost Ricci solitons.

1. Introduction

Let (M^n, g) be a complete smooth n-dimensional Riemannian manifold with a smooth vector field $X \in \chi(M)$ and a smooth function $\lambda : M \to \mathbb{R}$. Then (M^n, g) is said to define an almost Ricci soliton if

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

and it is called a gradient almost Ricci soliton if the vector field X is the gradient of a smooth function f, i.e., $X = \nabla f$.

Volume comparison has a wide role in differential geometry and analysis on manifolds. In several years, many different results with different conditions on Ricci curvature and scalar curvature on manifolds have been shown for volume comparison. The first result is related to Bishop and Gromov's in [5] that depends on Ricci curvature which is bounded from below. Actually, they compared volume of R-balls in a manifold M^n with R-balls in it's model space M^n_H. Then Qian [9] improved the results with Ricci curvature of the weighted Laplacian Δ_h, i.e., $\Delta_h = \Delta - \nabla h \cdot \nabla$.

John Lott [7] studied on metric measure space $(M, \phi dvol_M)$ with ∞-Bakry-Émery Ricci curvature

$$\mathrm{Ric}_\infty = \mathrm{Ric} - \mathrm{Hess}(\ln \phi),$$

Received March 19, 2021; Revised September 27, 2021; Accepted December 8, 2021.
2020 *Mathematics Subject Classification.* 53C21, 58C25.
Key words and phrases. Volume comparison, Bakry-Émery Ricci curvature.

and obtained important results on comparison geometry. Then Wei and Wylie in [11] expanded the Qian's results on volume comparison with considering the lower bound for this Ricci curvature.

Hu et al. [6] by considering a lower bound for scalar curvature showed that volume of balls with conformally compact metric $\tilde{g}$ is closed to volume of balls with complete noncompact Riemannian metric g. In [14] Zhu and Zhang, under a lower bound for Bakry-Émery Ricci curvature

$$\mathrm{Ric}+\frac{1}{2}\mathcal{L}_V g,$$

obtained volume comparison, Laplacian comparison, isoperimetric inequality and gradient bounds. In fact, they showed that

$$\frac{Vol(B(x,r_2))}{r_2^n}\leq e^{C(n,\lambda,K,\alpha,\rho)[\lambda(r_2^2-r_1^2)+K(r_2-r_1)^{1-\alpha}]}\frac{Vol(B(x,r_1))}{r_1^n}$$

for $0<r_1<r_2\leq 1$, where λ,α,K and C are constants.

Wei Yuan in [13] compared the volume of balls in Riemannian manifold M with a V-static metric $\tilde{g}$ and a Rimannian metric g. For study more about volume comparison with another condition in details you could see [1–4, 8, 10, 12, 15].

It is interesting to extend the Ricci condition in [14] for an almost Ricci soliton.

Let (M,g) be a Riemannian manifold of dimension n with fixed base point $O\in M$. Consider the following basic conditions for Ricci curvature tensor,

$$\mathrm{Ric}+\frac{1}{2}\mathcal{L}_V g\geq -\lambda g, \tag{1}$$

and

$$|V|(y)\leq \frac{K}{d(y,O)^{\alpha}}. \tag{2}$$

Here λ is a non-negative smooth function and V is a smooth vector field on M, also we denote the distance from O to y by $d(y,O)$, and $K\geq 0$ and $0\leq\alpha<1$ are constants. Later we will use a notation $C(a_1,a_2,\dots,a_n)$ for constants that depend on parameters $a_1,a_2,\dots,a_n$ where may it changes line to line.

Motivated by above works, we prove:

Theorem 1.1 (Volume comparison 1)**.** *We assume* (1) *and* (2)*, and denote by* $\omega(s,\cdot)$ *the volume element of the metric* g *on* M *in geodesic polar coordinates. Then for any* $0<s_1<s_2$*, we have*

$$\frac{\omega(s_2,\cdot)}{s_2^{n-1}}\leq e^{C(\alpha)Ks_2^{1-\alpha}+\int_{s_1}^{s_2}\frac{1}{s^2}\int_0^s\lambda t^2dtds}\frac{\omega(s_1,\cdot)}{s_1^{n-1}}, \tag{3}$$

and specially by letting $s_1\to 0$*, we get*

$$\omega(s,\cdot)\leq e^{C(\alpha)Ks^{1-\alpha}+\frac{1}{s^2}\int_0^s\lambda t^2dt}s^{n-1},\quad \forall s>0, \tag{4}$$

and hence

$$Vol(B(x,r)) \leq e^{C(\alpha)Kr^{1-\alpha}+\frac{1}{r^2}\int_0^r \lambda t^2 dt}\frac{Vol(S^{n-1})}{n}r^n, \quad \forall r>0. \tag{5}$$

Note that S^{n-1} *is the unit sphere in* $\mathbb{R}^n$.

For the next theorems we considered an upper bound K_1 for λ.

Theorem 1.2 (Volume comparison 2)**.** *Let the assumptions* (1) *and* (2) *hold.*

(a) *Let the conditions on Theorem 1.1 and volume noncollapsing condition* $Vol(B(x,r)) \geq \rho$ *hold for any* $x \in M$. *Then for any* $0 < r_1 < r_2 \leq 1$, *we get the ratio bound as follows*:

$$\frac{Vol(B(x,r_2))}{r_2^n} \leq e^{C(n,K,K_1,\alpha,\rho)[K_1(r_2^2-r_1^2)+K(r_2-r_1)^{1-\alpha}]}\frac{Vol(B(x,r_1))}{r_1^n}. \tag{6}$$

(b) *Specially, assume that the Bakry-Émery Ricci curvature tensor* Ric + Hess$L$ *implies that*

$$\mathrm{Ric} + \mathrm{Hess}L \geq -\lambda g, \tag{7}$$

and

$$|\nabla L(y)| \leq \frac{K}{d(y,O)^\alpha} \tag{8}$$

for all $y \in M$, *a fixed point* $O \in M$, *and constant* $K \geq 0$ *and* $\alpha \in [0,1]$. *Then the results of Theorem 1.1 and part* (a) *of this theorem hold.*

For gradient case, i.e., $V = \nabla L$, we prove:

Theorem 1.3. *Consider the following condition on Bakry-Émery Ricci curvature*

$$\mathrm{Ric} + \mathrm{Hess}L \geq -\lambda g. \tag{9}$$

We assume the following conditions on L:

$$|L(y) - L(z)| \leq K_2 d(y,z)^\alpha \ \textit{and} \ \sup_{x\in M, 0\leq r\leq 1}(r^\beta \|\nabla L\|^*_{q,B(x,r)}) \leq K_3 \tag{10}$$

for any $y, z \in M$ *with* $d(y,z) \leq 1$. *Here* $K_2, K_3 \geq 0$, $0 < \alpha < 1$, $0 < \beta < 1$, *and* $q \geq 1$ *are constants, and*

$$\|\nabla L\|^*_{q,B(x,r)} = \left(\fint_{B(x,r)} |\nabla L|^q(y)dV(y)\right)^{\frac{1}{q}}.$$

(a) *We denote the distance from any point* y *to some fixed point* x *by* $s = d(x,y)$ *and suppose that* $\gamma : [0,s] \to M$ *is a normal minimal geodesic with* $\gamma(0) = x$ *and* $\gamma(s) = y$. *Then in distribution sense we have*

$$\Delta s - \frac{n-1}{s} \leq K_1 s + \frac{4K_2}{s^{1-\alpha}} + \langle \nabla L, \nabla s\rangle, \quad \forall s < 1, \tag{11}$$

moreover

$$\frac{\partial}{\partial s}\frac{\omega(s,\theta)}{s^{n-1}} \leq [K_1 s + \frac{4K_2}{s^{1-\alpha}} + \langle \nabla L, \nabla s\rangle]\frac{\omega(s,\theta)}{s^{n-1}}. \tag{12}$$

Here, we denote the volume element in the geodesic polar coordinates by $\omega = \omega(s, \theta)$ *which is regarded as* 0 *on the cut locus.*

(b) *For any* $0 < r_1 < r_2 \leq 1$*, we have*

$$\frac{Vol(B(x, r_2))}{r_2^n} \leq e^{[K_1(r_2^2 - r_1^2) + K_3(r_2 - r_1)^{1-\beta} + 4K_2(r_2 - r_1)^{\alpha}]} \frac{Vol(B(x, r_1))}{r_1^n}. \tag{13}$$

In above theorems if the function λ is constant, Theorem 2.2 and Theorem 2.7 in [14] can be recovered by our result.

2. Proofs

To prove our main results, we need the following technical proposition.

Proposition 2.1. *If conditions* (1) *and* (2) *hold for an* n*-dimensional Riemannian manifold* (M, g)*, then in the distribution sense we get the following inequality,*

$$\Delta s - \frac{n-1}{s} \leq \frac{1}{s^2} \int_0^s \lambda t^2 dt + \langle V, \nabla s \rangle + \frac{C(\alpha)K}{s^{\alpha}}, \tag{14}$$

where $s = d(y, x)$ *is the distance from any point* y *to some fixed point* x*.*

Proof. Since the complement of the cut locus is star shaped, we can establish inequality (14) in the distribution and so we may just prove the inequality (14) for smooth points of s. By using Bochner formula for s and Cauchy-Schwarz inequality $|\nabla^2 s|^2 \geq \frac{(\Delta s)^2}{n-1}$, we have

$$\frac{1}{s^2} \frac{\partial}{\partial s}(s^2 \Delta s) + \frac{1}{n-1}(\Delta s - \frac{n-1}{s})^2 \leq \frac{n-1}{s^2} - \mathrm{Ric}(\nabla s, \nabla s). \tag{15}$$

We multiply the sides of the inequality by s^2 and then integrating from 0 to s, it yields

$$\Delta s \leq \frac{n-1}{s} - \frac{1}{s^2} \int_0^s t^2 \mathrm{Ric}(\gamma'(t), \gamma'(t)) dt. \tag{16}$$

We can choose an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ with $e_1 = \gamma'(t)$ at any point $\gamma(t)$. Then due to the (1), we get

$$\begin{aligned}
\mathrm{Ric}(\gamma'(t), \gamma'(t)) &= \mathrm{Ric}(e_1, e_1) \\
&\geq -\lambda - \frac{1}{2}\mathcal{L}_V g(e_1, e_1) \\
&= -\lambda - \langle \nabla_{e_1}^V, e_1 \rangle \\
&= -\lambda - e_1 \langle V, e_1 \rangle + \langle V, \nabla_{e_1} e_1 \rangle \\
&= -\lambda - \frac{\partial}{\partial t} \langle V, \gamma'(t) \rangle.
\end{aligned}$$

By this, (16) becomes

$$\Delta s - \frac{n-1}{s} \leq \frac{1}{s^2} \int_0^s \left[t^2 \frac{\partial}{\partial t} \langle V, \gamma'(t) \rangle + \lambda t^2 \right] dt$$

$$\leq \frac{1}{s^2}\int_0^s \lambda t^2 dt + \frac{1}{s^2}\left[t^2\langle V,\gamma^{'}(t)\rangle(\gamma(t)) \mid_0^s -2\int_0^s t\langle V,\gamma^{'}(t)\rangle dt\right]$$

$$\leq \frac{1}{s^2}\int_0^s \lambda t^2 dt + \langle V,\nabla s\rangle - \frac{2}{s^2}\int_0^s t\langle V,\gamma^{'}(t)\rangle dt. \tag{17}$$

We denote $d_0 = d(x,O)$ and to continue proving, we consider two cases.
Case 1: If $s \leq d_0$, we get the following inequality by the fact that $d(\gamma(t),O) \geq d_0 - t$.

$$\begin{aligned}
-\frac{2}{s}\int_0^s t\langle V,\gamma^{'}(t)\rangle dt &\leq \frac{1}{s^2}\int_0^s 2t\cdot\frac{k}{(d_0-t)^\alpha}dt \\
&\leq \frac{C(\alpha)k}{s}[-(d_0-t)^{1-\alpha} \mid_0^s] \\
&= \frac{C(\alpha)k}{s}[d_0^{1-\alpha} - (d_0-s)^{1-\alpha}] \\
&\leq \frac{C(\alpha)k}{s^\alpha}. \qquad (18)
\end{aligned}$$

Now we get

$$\Delta s - \frac{n-1}{s} \leq \frac{1}{s^2}\int_0^s \lambda t^2 dt + \langle V,\nabla s\rangle + \frac{C(\alpha)K}{s^\alpha}. \tag{19}$$

Case 2: If $s > d_0$, then for any $d_0 \leq t \leq s$, we have

$$d(\gamma(t),O) \geq t - d_0.$$

So, we get the same result for $-\frac{2}{s^2}\int_0^s t\langle V,\gamma^{'}(t)\rangle dt$ and consequently we get

$$\Delta s - \frac{n-1}{s} \leq \int_0^s \lambda t^2 dt + \langle V,\nabla s\rangle + \frac{C(\alpha)K}{s^\alpha}. \tag{20}$$

This completes the proof of the proposition. □

Proof of Theorem 1.1. First, by (14) we have

$$\begin{aligned}
\frac{\partial}{\partial s}\ln(\omega(s,\cdot)) = \frac{\omega^{'}(s,\cdot)}{\omega(s,\cdot)} &= \Delta s \\
&\leq \frac{n-1}{s} + \frac{1}{s^2}\int_0^s \lambda t^2 dt + \langle V,\nabla s\rangle + \frac{C(\alpha)K}{s^\alpha}. \qquad (21)
\end{aligned}$$

Case 1: If $s_1 < s_2 \leq d_0$, by using triangle inequality for assumption (2), (21) changes as

$$\frac{\partial}{\partial s}\ln(\omega(s,\cdot)) \leq \frac{n-1}{s} + \frac{1}{s^2}\int_0^s \lambda t^2 dt + \frac{K}{(d_0-s)^\alpha} + \frac{C(\alpha)K}{s^\alpha}.$$

By integrating from s_1 to s_2 of both sides, we get

$$\begin{aligned}
\ln\frac{\omega(s_2,\cdot)}{\omega(s_1,\cdot)} \leq\ & \ln(\frac{s_2}{s_1})^{n-1} + \int_{s_1}^{s_2}\frac{1}{s^2}\int_0^s \lambda t^2 dt ds + C(\alpha)K(s_2^{1-\alpha} - s_1^{1-\alpha}) \\
&+ C(\alpha)K[(d_0-s_1)^{1-\alpha} - (d_0-s_2)^{1-\alpha}]
\end{aligned}$$

$$\leq \ln(\frac{s_2}{s_1})^{n-1} + \int_{s_1}^{s_2} \frac{1}{s^2} \int_0^s \lambda t^2 dt ds + C(\alpha)K s_2^{1-\alpha} + C(\alpha)K(s_2 - s_1)^{1-\alpha}$$

$$\leq \ln(\frac{s_2}{s_1})^{n-1} + \int_{s_1}^{s_2} \frac{1}{s^2} \int_0^s \lambda t^2 dt ds + C(\alpha)K s_2^{1-\alpha}. \tag{22}$$

Specially for $s_1 \leq d_0$ we get

$$\ln \frac{\omega(d_0, \cdot)}{\omega(s_1, \cdot)} \leq \ln(\frac{d_0}{s_1})^{n-1} + \int_{s_1}^{d_0} \frac{1}{s^2} \int_0^s \lambda t^2 dt ds + C(\alpha)K d_0^{1-\alpha}. \tag{23}$$

Case 2: If $d_0 \leq s_1 < s_2$, then by fact that

$$\langle V, \nabla s \rangle \leq |V|(\gamma(s)) \leq \frac{K}{(s - d_0)^{\alpha}},$$

we get the same results and for $s_2 \geq d_0$, we have

$$\ln \frac{\omega(s_2, \cdot)}{\omega(d_0, \cdot)} \leq \ln(\frac{s_2}{d_0})^{n-1} + \int_{d_0}^{s_2} \frac{1}{s^2} \int_0^s \lambda t^2 dt ds + C(\alpha)K s_2^{1-\alpha}. \tag{24}$$

Case 3: For $s_1 \leq d_0 \leq s_2$, by (23) and (24), it yields

$$\ln \frac{\omega(s_2, \cdot)}{\omega(s_1, \cdot)} \leq \ln(\frac{s_2}{s_1})^{n-1} + \int_{s_1}^{s_2} \frac{1}{s^2} \int_0^s \lambda t^2 dt ds + C(\alpha)K s_2^{1-\alpha}. \tag{25}$$

This finished the proof of the theorem. □

In the following we consider the volume comparison Theorem 1.1 by non-negative function λ that bounded above with K_1.

lemma 2.2. *Let the following volume noncollapsing condition hold*

$$Vol(B(x, r)) \geq \rho, \ \forall x \in M, \tag{26}$$

for some constant $\rho > 0$. If the conditions (1) *and* (2) *hold, then for any $q \in (0, \frac{n}{\alpha})$ we get*

$$\sup_{x \in M, 0 < r \leq 1} r^{\alpha} \|V\|^*_{q, B(x,r)} \leq C(n, K, K_1, \alpha, \rho, q)K, \tag{27}$$

where

$$\|V\|^*_{q,B(x,r)} = \left(\fint_{B(x,r)} |V|^q dg \right)^{\frac{1}{q}} = \left(\frac{1}{Vol(B(x,r))} \int_{B(x,r)} |V|^q dg \right)^{\frac{1}{q}}.$$

Proof. By assumption (2), for any $0 < r \leq 1$, we obtain

$$\|V\|^*_{q,B(x,r)} \leq \left(\frac{1}{Vol(B(x,r))} \int_{B(x,r)} \frac{K^q}{d(y, O)^{\alpha q}} dg(y) \right)^{\frac{1}{q}}. \tag{28}$$

Case 1: For $d(x, O) \leq 2r$, we know $B(x, r) \subset B(O, 3r)$, then for any $0 < q < \frac{n}{\alpha}$, we conclude that

$$\int_{B(x,r)} \frac{1}{d(y, O)^{\alpha q}} dg(y) \leq \int_{B(O,3r)} \frac{1}{d(y, O)^{\alpha q}} dg(y)$$

$$\leq C(n,\alpha,q)e^{[C(\alpha)Kr^{1-\alpha}+K_1r]}r^{n-\alpha q}. \tag{29}$$

In fact, from (4), for any $\gamma < n$, we get

$$\int_{B(O,r)} \frac{1}{d(y,O)^\gamma} dg(y) \leq C(n\cdot\gamma)e^{[C(\alpha)Kr^{1-\alpha}+K_1r]}r^{n-\gamma}. \tag{30}$$

So, by (4) and (26) for $r \leq 1$ it follows

$$\begin{aligned} Vol(B(x,r)) &= \int_{S^{n-1}}\int_0^r \omega(s,\theta)dsd\theta \\ &= r\int_{S^{n-1}}\int_0^1 \omega(rt,\theta)dtd\theta \\ &\geq r\int_{S^{n-1}}\int_0^1 \frac{1}{e^{C(\alpha)K+K_1}}r^{n-1}\omega(t,\theta)dtd\theta \\ &= \frac{\rho}{e^{C(\alpha)K+K_1}}r^n. \end{aligned} \tag{31}$$

By this and (29), we get

$$\begin{aligned} \frac{1}{Vol(B(x,r))}\int_{B(O,r)} \frac{1}{d(y,O)^{\alpha q}} dg(y) \leq C(n,\alpha,q)&e^{[C(\alpha)Kr^{1-\alpha}+K_1r]} \\ &\times \frac{e^{C(\alpha)K+K_1}}{\rho}r^{-\alpha q}. \end{aligned}$$

Case 2: For $d(x,O) > 2r$, by triangle inequality we have $d(y,O) \geq r$ for all $y \in B(x,r)$. Then

$$\int_{B(x,r)} \frac{1}{d(y,O)^{\alpha q}} dg(y) \leq r^{-\alpha q} Vol(B(x,r)), \tag{32}$$

so, in each case we get that

$$\|V\|^*_{q,B(x,r)} \leq C(n,K,K_1,\alpha,\rho,q)Kr^{-\alpha}. \qquad \square$$

Proof of Theorem 1.2. Let $\psi = (\Delta s - \frac{n-1}{s})_+$. By (14), we get

$$\psi \leq K_1 s + \frac{C(\alpha)K}{s^\alpha} + |V|. \tag{33}$$

We deduce from (27) that

$$\fint_{B(x,r)} |V|dg \leq \frac{C(n,K,K_1,\alpha,\rho,q)K}{r^\alpha}. \tag{34}$$

Letting $Q(r) = \frac{Vol(B(x,r))}{r^n}$, we get, from (33) and (34),

$$\begin{aligned} \frac{d}{dr}Q(r) &\leq \frac{1}{r^{n+1}}\int_{S^{n-1}}\int_0^r s(K_1 s + \frac{C(\alpha)K}{s^\alpha} + |V|)\omega(s,\theta)dsd\theta \\ &\leq [K_1 r + \frac{C(n,K,K_1,\alpha,\rho,q)K}{r^\alpha}]Q(r). \end{aligned}$$

By integrating of both sides the proof is complete. $\square$

At the end of the proof of part (b) you can get the results by setting $V = \nabla L$ in (1).

Remark 2.3. Note that in particular, we can improve the Volume comparison theorems without assumption (26).

Corollary 2.4. *Assume that* (1) *holds and moreover*

$$|V| \leq K. \tag{35}$$

Then we get the following conclusions.

(a) *In distribution sense we have*

$$\Delta s - \frac{n-1}{s} \leq K_1 s + K. \tag{36}$$

(b) *For volume element in geodesic polar coordinate with any* $0 < s_1 < s_2$ *we get*

$$\frac{\omega(s_2, \cdot)}{s_2^{n-1}} \leq e^{2Ks_2 + K_1 s_2^2} \frac{\omega(s_1, \cdot)}{s_1^{n-1}}. \tag{37}$$

In general, by letting $s_1 \to 0$*, we have*

$$\omega(s, \cdot) \leq e^{2Ks + K_1 s^2} s^{n-1}, \quad \forall s > 0, \tag{38}$$

and then

$$Vol(B(x, r)) \leq e^{2Kr + K_1 r^2} \frac{Vol(S^{n-1})}{n} r^n, \quad \forall r > 0. \tag{39}$$

(c) *According to the* (39)*, for any* $0 < r_1 < r_2$ *we have*

$$\frac{Vol(B(x, r_2))}{r_2^n} \leq e^{[2K(r_2 - r_1) + K_1(r_2^2 - r_1^2)]} \frac{Vol(B(x, r_1))}{r_1^n}. \tag{40}$$

Proof. Since $|V|$ is bounded, Lemma 2.2 holds without any other condition for $\alpha = 0$ and $q = \infty$. The proof of this corollary is similar to the proof of Proposition 2.1, Theorem 1.1 and Theorem 1.2 as $\alpha = 0$. □

Proof of Theorem 1.3. (a) We just prove the inequalities at smooth points s. First by (16) we have

$$\Delta s \leq \frac{n-1}{s} - \frac{1}{s^2} \int_0^s t^2 \mathrm{Ric}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) ds. \tag{41}$$

From (9), along the geodesic $\gamma(t)$, we get

$$\mathrm{Ric}(\partial_t, \partial_t) \geq -\lambda - \mathrm{Hess}L(\partial_t, \partial_t) = -\lambda - \frac{d^2}{dt^2} L(\gamma(t)). \tag{42}$$

By (42) and integration of two sides, we rewrite (41) as following

$$\begin{aligned} \Delta s - \frac{n-1}{s} &\leq \frac{1}{s^2} \int_0^s \left[t^2 \frac{d^2}{dt^2} L(\gamma(y)) + \lambda t^2 \right] dt \\ &\leq K_1 s + \frac{1}{s^2} \left[t^2 \frac{d}{dt} L(\gamma(t)) \mid_0^s -2 \int_0^s t \frac{d}{dt} L(\gamma(t)) dt \right] \end{aligned}$$

$$
\begin{aligned}
&\leq K_1 s + \langle \nabla L, \nabla s \rangle - \frac{2}{s^2}\int_0^s t\frac{d}{dt}L(\gamma(t))dt \\
&= K_1 s + \langle \nabla L, \nabla s \rangle - \frac{2}{s}L(\gamma(s)) \mid_0^s + \frac{2}{s^2}\int_0^s L(\gamma(t))dt \\
&= K_1 s + \langle \nabla L, \nabla s \rangle - \frac{2}{s}[L(\gamma(s)) - L(\gamma(0))] \\
&\quad + \frac{2}{s^2}\int_0^s [L(\gamma(t)) - L(\gamma(0))]dt \\
&\leq K_1 s + \langle \nabla L, \nabla s \rangle + \frac{4K_2}{s^{1-\alpha}}.
\end{aligned}
$$

From this together with ($\partial_s \omega = \Delta s \omega$), we can obtain (12).

(b) According to (11), we have

$$
\psi \leq K_1 s + \frac{4K_2}{s^{1-\alpha}} + |\nabla L|,
$$

and by (10), we get

$$
\oint_{B(x,r)} |\nabla L| dV \leq \left(\oint_{B(x,r)} |\nabla L|^q dV \right)^{\frac{1}{q}} \leq \frac{K_3}{r^\beta}. \tag{43}
$$

By considering $Q(r) = \frac{Vol(B(x,r))}{r^n}$ and (43) we have

$$
\begin{aligned}
\frac{d}{dr}Q(r) &\leq \frac{1}{r^{n+1}}\int_{s^{n-1}}\int_0^r s(K_1 s + \frac{4K_2}{s^{1-\alpha}} + |\nabla L|)\omega(s,\theta)dsd\theta \\
&\leq (K_1 r + \frac{4K_2}{r^{1-\alpha}} + \frac{K_3}{r^\beta})Q(r).
\end{aligned}
$$

The proof of (b) finishes by integrating both sides of the above inequality. □

For $V = \nabla L$ similar to Lemma 2.8 in [14] we have the following lemma due to bounded λ, which cause the assumption (10) be a weaker condition than (8) and (26).

lemma 2.5. *Assume that* (9)*,* (8) *and* (26) *hold. Then for* $q \in (n, \frac{n}{\alpha})$*, we have*

$$
\sup_{x\in M, 0<r\leq 1} r^\alpha \|\nabla L\|^*_{q,B(x,r)} \leq C(n, K, K_1, \alpha, \rho)K, \tag{44}
$$

by the fact that

$$
|L(y) - L(z)| \leq \frac{2K}{1-\alpha}d(y,z)^{1-\alpha} \tag{45}
$$

for any $y, z \in M$*.*

Corollary 2.6 (Gradient Almost Ricci soliton)**.** *Let* (M, g_{ij}) *be an almost Ricci soliton satisfying in following*

$$
R + |\nabla L|^2 - 2\lambda L = 0 \tag{46}
$$

for positive bounded function λ. Moreover assume that

$$|L| \leq K, \ \ in\, B(O, 2\delta) \tag{47}$$

for some fixed point $O \in M$ and radius δ.

Let $\Lambda(n, \lambda, K)$ be the following constant

$$\Lambda(n, \lambda, K) = \sqrt{2K_1K}.$$

Then the following statements are valid.

(a) *In distribution sense given that $|\nabla L| \leq \Lambda(n, \lambda, K)$, we have*

$$\Delta s - \frac{n-1}{s} \leq K_1 s + 2\sqrt{2K_1K}. \tag{48}$$

(b) *Let $\omega(s, \cdot)$ be the volume element of the metric g under geodesic polar coordinates. Then we get*

$$\frac{\omega(s_2, \cdot)}{s_2^{n-1}} \leq e^{2\sqrt{2K_1K}s_2 + K_1 s_2^2} \frac{\omega(s_1, \cdot)}{s_1^{n-1}} \tag{49}$$

for any $0 < s_1 < s_2 < d(x, \partial B(O, \delta))$, specially by letting $s_1 \to 0$, for any $0 < s < d(x, \partial B(O, \delta))$, we have

$$\omega(s, \cdot) \leq e^{2\sqrt{2K_1K}s + K_1 s^2} s^{n-1}, \tag{50}$$

and hence

$$Vol(B(x, r)) \leq C(n) e^{2\sqrt{2K_1K}r + K_1 r^2} r^n \tag{51}$$

for any $0 < r < d(x, \partial B(O, \delta))$.

(c) *For $0 < r_1 < r_2 < d(x, \partial B(O, \delta))$ and any point $x \in B(O, \delta)$, we have*

$$\frac{Vol(B(x, r_2))}{r_2^n} \leq e^{[2\sqrt{2K_1K}(r_2 - r_1) + K_1(r_2^2 - r_1^2)]} \frac{Vol(B(x, r_1))}{r_1^n}. \tag{52}$$

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