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FEKETE-SZEGÖ INEQUALITIES FOR A NEW GENERAL SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING THE (p,q)-DERIVATIVE OPERATOR

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Abstract. In this work, we introduce a new subclass of analytic functions of complex order involving the (p,q)-derivative operator defined in the open unit disc. For this class, several Fekete-Szegö type coefficient inequalities are derived. We obtain the results of Srivastava *et al.* [22] as consequences of the main theorem in this study.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1}$$

which are analytic in the unit disk

$$\mathbb{U}=\left\{z\in\mathbb{C}:|z|<1\right\}.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

For $f\in\mathcal{S}$, Fekete and Szegö [11] proved a noticeable result that the estimate

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}-4\mu+3, & \mu\leq 0,\\ 1+2\exp\left(\frac{-2\mu}{1-\mu}\right), & 0\leq\mu\leq 1,\\ 4\mu-3, & \mu\geq 1,\end{cases} \tag{2}$$

holds. The result is sharp in the sense that for each μ there is a function in the class under consideration for which equality holds.

The coefficient functional

$$\phi_\mu\left(f\right)=a_3-\mu a_2^2=\frac{1}{6}\left(f'''\left(0\right)-\frac{3\mu}{2}\left(f''\left(0\right)\right)^2\right)$$

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on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_\mu \left(e^{-i\theta} f\left(e^{i\theta} z\right)\right) = e^{2i\theta} \phi_\mu (f) \quad (\theta \in \mathbb{R}).$$

Thus it is quite natural to ask about inequalities for ϕ_μ corresponding to subclasses of $\mathcal{S}$. This is called Fekete-Szegö problem. Actually, many authors have considered this problem for typical classes of univalent functions (see, for instance [1, 3, 7–11, 14, 16–19]).

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. The area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [12, 13]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. A comprehensive study on applications of q-calculus in operator theory may be found in [4]. Recently, there is an extension of q-calculus, denoted by (p, q)-calculus which obtained by substituting q by q/p in q-calculus. The (p, q)-integer was considered by Chakrabarti and Jagannathan [6].

For a function $f \in \mathcal{A}$ given by (1) and $0 < q < p \leq 1$, the (p, q)-derivative of function f is defined by (see [2, 15])

$$D_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p-q) z} \qquad (z \neq 0). \tag{3}$$

From (3), we deduce that

$$D_{p,q} f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} \, a_k z^{k-1}, \tag{4}$$

where $[k]_{p,q}$ denotes the (p,q)-integer and is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}. \tag{5}$$

As $p = 1$ and $q \to 1^-$, $[k]_{p,q} \to k$. For a function $g(z) = z^k$, we get

$$D_{p,q}\left(z^k\right) = [k]_{p,q}\, z^{k-1}.$$

We denote by $\mathcal{P}$ the class of all functions φ which are analytic and univalent in $\mathbb{U}$ and for which $\varphi(\mathbb{U})$ is convex with

$$\varphi(0) = 1 \quad \text{and} \quad \Re\{\varphi(z)\} > 0 \quad (z \in \mathbb{U}).$$

By making use of the (p,q)-derivative of a function $f \in \mathcal{A}$ and the principle of subordination, we introduce the following subclass.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the class

$$\mathcal{M}^{\lambda}_{p,q,b}(\varphi) \qquad (0 \leq \lambda \leq 1,\ b \in \mathbb{C}\setminus\{0\},\ \varphi \in \mathcal{P})$$

if it satisfies the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{zD_{p,q}\mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}),$$

where

$$\mathcal{F}_\lambda(z) = \lambda z D_{p,q} f(z) + (1-\lambda) f(z). \tag{6}$$

Remark 1.1. For $p = 1$, the class $\mathcal{M}^{\lambda}_{p,q,b}(\varphi)$ reduces to the class $\mathcal{M}^{\lambda}_{q,b}(\varphi)$ introduced and studied by Bulut [5].

Remark 1.2. (i) If we set $\lambda = 0$ in Definition 1, then we have the class

$$\mathcal{M}^{0}_{p,q,b}(\varphi) = \mathcal{S}^{b}_{p,q}(\varphi)$$

of (p,q)-starlike functions of complex order b which consists of functions satisfying

$$1 + \frac{1}{b}\left(\frac{zD_{p,q}f(z)}{f(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

(ii) If we set $\lambda = 1$ in Definition 1, then we have the class

$$\mathcal{M}^{1}_{p,q,b}(\varphi) = \mathcal{C}^{b}_{p,q}(\varphi)$$

of (p,q)-convex functions of complex order b which consists of functions satisfying

$$1 + \frac{1}{b}\left(\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

The classes $\mathcal{S}^{b}_{p,q}(\varphi)$ and $\mathcal{C}^{b}_{p,q}(\varphi)$ was introduced and studied by Yatkın and Kadıoğlu [23].

Remark 1.3. In Remark 1.2, letting $b = 1$, we get the classes $\mathcal{S}^{1}_{p,q}(\varphi) = \mathcal{S}^{*}_{p,q}(\varphi)$ and $\mathcal{C}^{1}_{p,q}(\varphi) = \mathcal{C}_{p,q}(\varphi)$ of (p,q)-starlike functions and (p,q)-convex functions, respectively. These classes was introduced by Srivastava *et al.* [22].

Remark 1.4. In Remark 1.2, letting $p = 1$, we get the classes $\mathcal{S}_{1,q}^{b}(\varphi) = \mathcal{S}_{q,b}(\varphi)$ and $\mathcal{C}_{1,q}^{b}(\varphi) = \mathcal{C}_{q,b}(\varphi)$ of q-starlike functions of complex order b and q-convex functions of complex order b, respectively. These classes was introduced by Seoudy and Aouf [21].

We shall require the following lemmas.

Lemma 1.5 ([20])**.** *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then for any complex number ν*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |2\nu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 1.6 ([19])**.** *If $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2, & \nu \leq 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, then the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $\nu = 1$, then the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case when $\nu = 0$.

Although the above upper bound is sharp, in the case when $0 < \nu < 1$, it can be further improved as follows:

$$\left|c_2 - \nu c_1^2\right| + \nu |c_1|^2 \leq 2 \qquad \left(0 < \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) |c_1|^2 \leq 2 \qquad \left(\frac{1}{2} \leq \nu \leq 1\right).$$

2. Fekete-Szegö problem for the function class $\mathcal{M}_{p,q,b}^{\lambda}(\varphi)$

Unless otherwise mentioned, we assume throughout this paper that

$$0 \leq \lambda \leq 1, \quad 0 < q < p \leq 1, \quad b \in \mathbb{C}\backslash\{0\}, \quad \varphi \in \mathcal{P},$$

$[k]_{p,q}$ is given by (5) and $z \in \mathbb{U}$.

Theorem 2.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ *with* $B_1 \neq 0$. *If* $f(z)$ *given by* (1) *belongs to the function class* $\mathcal{M}^{\lambda}_{p,q,b}(\varphi)$, *then for any complex number* μ

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1 b|}{\left([3]_{p,q} - 1\right)\left(1 - \lambda + [3]_{p,q}\lambda\right)}$$

$$\times \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_{p,q} - 1}\left(1 - \frac{\left([3]_{p,q} - 1\right)\left(1 - \lambda + [3]_{p,q}\lambda\right)}{\left([2]_{p,q} - 1\right)\left(1 - \lambda + [2]_{p,q}\lambda\right)^2}\mu\right)\right|\right\}. \tag{7}$$

The result is sharp.

Proof. If $f \in \mathcal{M}^{\lambda}_{p,q,b}(\varphi)$, then we have

$$h(z) \prec \varphi(z),$$

where

$$h(z) = 1 + \frac{1}{b}\left(\frac{zD_{p,q}\mathcal{F}_{\lambda}(z)}{\mathcal{F}_{\lambda}(z)} - 1\right) = 1 + h_1 z + h_2 z^2 + \cdots. \tag{8}$$

By the definition of the function $\mathcal{F}_{\lambda}$ given by (6) and by the (p,q)-derivative defined by (4), we obtain

$$\mathcal{F}_{\lambda}(z) = \lambda z D_{p,q} f(z) + (1 - \lambda) f(z) = z + \sum_{k=2}^{\infty}\left(1 - \lambda + [k]_{p,q}\lambda\right) a_k z^k. \tag{9}$$

Therefore from (8) and (9), we have

$$h_1 = \frac{1}{b}\left([2]_{p,q} - 1\right)\left(1 - \lambda + [2]_{p,q}\lambda\right) a_2, \tag{10}$$

$$h_2 = \frac{1}{b}\left[\left([3]_{p,q} - 1\right)\left(1 - \lambda + [3]_{p,q}\lambda\right) a_3 - \left([2]_{p,q} - 1\right)\left(1 - \lambda + [2]_{p,q}\lambda\right)^2 a_2^2\right]. \tag{11}$$

Since $\varphi(z)$ is univalent and $h(z) \prec \varphi(z)$, the function

$$p_1(z) = \frac{1 + \varphi^{-1}(h(z))}{1 - \varphi^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

is analytic and has a positive real part in $\mathbb{U}$. Also we have

$$h(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right)$$

$$= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots. \tag{12}$$

Thus by (8)-(12) we get

$$a_2 = \frac{B_1 c_1 b}{2\left([2]_{p,q} - 1\right)\left(1 - \lambda + [2]_{p,q}\lambda\right)}, \tag{13}$$

(14) $$a_3 = \frac{B_1 b}{2\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}\left[c_2 - \frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{B_1 b}{[2]_{p,q}-1}\right)c_1^2\right].$$

Taking into account (13) and (14) , we obtain

(15) $$a_3 - \mu a_2^2 = \frac{B_1 b}{2\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}\left(c_2 - \delta c_1^2\right),$$

where

(16) $$\delta = \frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1 b}{[2]_{p,q}-1}\left(1-\frac{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2}\mu\right)\right].$$

Our result now follows by an application of Lemma 1.5. The result is sharp for the functions

$$1+\frac{1}{b}\left(\frac{zD_{p,q}\mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)}-1\right)=\varphi\left(z^2\right) \quad \text{and} \quad 1+\frac{1}{b}\left(\frac{zD_{p,q}\mathcal{F}_\lambda(z)}{\mathcal{F}_\lambda(z)}-1\right)=\varphi(z).$$

This completes the proof of Theorem 2.1. □

Corollary 2.2. *Taking* $\lambda = 0$ *and* $\lambda = 1$ *in Theorem* 2.1*, we get* [23, Theorem 4] *and* [23, Theorem 5]*, respectively.*

Corollary 2.3. *Taking* $\lambda = 0$ *and* $\lambda = 1$ *with* $b = 1$ *in Theorem* 2.1*, we get* [22, Theorem 2.1] *and* [22, Theorem 2.2]*, respectively.*

Corollary 2.4. *Taking* $\lambda = 0$ *and* $\lambda = 1$ *with* $p = 1$ *in Theorem* 2.1*, we get* [21, Theorem 1] *and* [21, Theorem 2]*, respectively.*

Theorem 2.5. *Let* $\varphi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots$ *with* $B_1 > 0$ *and* $B_2 \geq 0$*. If* $f(z)$ *given by* (1) *belongs to the function class* $\mathcal{M}_{p,q,b}^{\lambda}(\varphi)$ *with* $b > 0$*, then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{B_2 b}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}+\frac{B_1^2b^2}{\left([2]_{p,q}-1\right)}\left[\frac{1}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}-\frac{\mu}{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2}\right], \\ \text{if } \mu \leq \sigma_1 \\ \frac{B_1 b}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}, \\ \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{B_2 b}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}-\frac{B_1^2b^2}{\left([2]_{p,q}-1\right)}\left[\frac{1}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}-\frac{\mu}{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2}\right], \\ \text{if } \mu \geq \sigma_2, \end{cases}$$

where

(17) $$\sigma_1 = \frac{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2\left[B_1^2 b+\left([2]_{p,q}-1\right)(B_2-B_1)\right]}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)B_1^2 b},$$

$$(18) \qquad \sigma_2 = \frac{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2\left[B_1^2 b+\left([2]_{p,q}-1\right)(B_2+B_1)\right]}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)B_1^2 b},$$

$$(19) \qquad \sigma_3 = \frac{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2\left[B_1^2 b+\left([2]_{p,q}-1\right)B_2\right]}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)B_1^2 b}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} &\left|a_3-\mu a_2^2\right| + \frac{\left([2]_{p,q}-1\right)^2\left(1-\lambda+[2]_{p,q}\lambda\right)^2}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)B_1^2 b} \\ &\times\left\{B_1-B_2-\frac{B_1^2 b}{\left([2]_{p,q}-1\right)}\left(1-\frac{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2}\mu\right)\right\}|a_2|^2 \\ \leq\ &\frac{B_1 b}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}. \end{aligned}$$

Furthermore, if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\begin{aligned} &\left|a_3-\mu a_2^2\right| + \frac{\left([2]_{p,q}-1\right)^2\left(1-\lambda+[2]_{p,q}\lambda\right)^2}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)B_1^2 b} \\ &\times\left\{B_1+B_2+\frac{B_1^2 b}{\left([2]_{p,q}-1\right)}\left(1-\frac{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}{\left([2]_{p,q}-1\right)\left(1-\lambda+[2]_{p,q}\lambda\right)^2}\mu\right)\right\}|a_2|^2 \\ \leq\ &\frac{B_1 b}{\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}. \end{aligned}$$

Each of these results is sharp.

Proof. Applying Lemma 1.6 to (15) and (16), we can get our results. On the other hand, using (15) for the values of $\sigma_1 \leq \mu \leq \sigma_3$, we have

$$\begin{aligned} \left|a_3-\mu a_2^2\right| + (\mu-\sigma_1)|a_2|^2 = &\frac{B_1 b}{2\left([3]_{p,q}-1\right)\left(1-\lambda+[3]_{p,q}\lambda\right)}\left|c_2-\delta c_1^2\right| \\ &+(\mu-\sigma_1)\frac{B_1^2 b^2 |c_1|^2}{4\left([2]_{p,q}-1\right)^2\left(1-\lambda+[2]_{p,q}\lambda\right)^2} \end{aligned}$$

$$= \frac{B_1 b \left\{ \left| c_2 - \delta c_1^2 \right| + \delta \left| c_1 \right|^2 \right\}}{2 \left([3]_{p,q} - 1 \right) \left(1 - \lambda + [3]_{p,q} \lambda \right)}$$

$$\leq \frac{B_1 b}{\left([3]_{p,q} - 1 \right) \left(1 - \lambda + [3]_{p,q} \lambda \right)}.$$

Similarly, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we get

$$\left| a_3 - \mu a_2^2 \right| + (\sigma_2 - \mu) \left| a_2 \right|^2 = \frac{B_1 b}{2 \left([3]_{p,q} - 1 \right) \left(1 - \lambda + [3]_{p,q} \lambda \right)} \left| c_2 - \delta c_1^2 \right|$$

$$+ (\sigma_2 - \mu) \frac{B_1^2 b^2 \left| c_1 \right|^2}{4 \left([2]_{p,q} - 1 \right)^2 \left(1 - \lambda + [2]_{p,q} \lambda \right)^2}$$

$$= \frac{B_1 b \left\{ \left| c_2 - \delta c_1^2 \right| + (1 - \delta) \left| c_1 \right|^2 \right\}}{2 \left([3]_{p,q} - 1 \right) \left(1 - \lambda + [3]_{p,q} \lambda \right)}$$

$$\leq \frac{B_1 b}{\left([3]_{p,q} - 1 \right) \left(1 - \lambda + [3]_{p,q} \lambda \right)}.$$

To show that the bounds asserted by Theorem 2.5 are sharp, we define the following functions:

$$K_{\varphi_n}(z) \qquad (n = 2, 3, \ldots),$$

with

$$K_{\varphi_n}(0) = 0 = K'_{\varphi_n}(0) - 1,$$

by

$$1 + \frac{1}{b} \left(\frac{z D_{p,q} K_{\varphi_n}(z)}{K_{\varphi_n}(z)} - 1 \right) = \varphi \left(z^{n-1} \right),$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \leq \eta \leq 1)$, with

$$F_\eta(0) = 0 = F'_\eta(0) - 1 \quad \text{and} \quad G_\eta(0) = 0 = G'_\eta(0) - 1,$$

by

$$1 + \frac{1}{b} \left(\frac{z D_{p,q} F_\eta(z)}{F_\eta(z)} - 1 \right) = \varphi \left(\frac{z (z + \eta)}{1 + \eta z} \right)$$

and

$$1 + \frac{1}{b} \left(\frac{z D_{p,q} G_\eta(z)}{G_\eta(z)} - 1 \right) = \varphi \left(-\frac{z (z + \eta)}{1 + \eta z} \right),$$

respectively. Then, clearly, the functions $K_{\varphi_n}, F_\eta, G_\eta \in \mathcal{M}_{p,q,b}^{\lambda}(\varphi)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality in Theorem 2.5 holds if and only if f is K_{φ_2} or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is K_{φ_3} or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_η or one of its rotations. □

Taking $\lambda = 0$ in Theorem 2.5, we get following consequence.

Corollary 2.6. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1) belongs to the function class $\mathcal{S}^b_{p,q}(\varphi)$ with $b > 0$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_2 b}{[3]_{p,q}-1} + \frac{B_1^2 b^2}{[2]_{p,q}-1}\left[\frac{1}{[3]_{p,q}-1} - \frac{\mu}{[2]_{p,q}-1}\right], & \mu \leq \sigma_1, \\ \frac{B_1 b}{[3]_{p,q}-1}, & \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{B_2 b}{[3]_{p,q}-1} - \frac{B_1^2 b^2}{[2]_{p,q}-1}\left[\frac{1}{[3]_{p,q}-1} - \frac{\mu}{[2]_{p,q}-1}\right], & \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{\left([2]_{p,q} - 1\right)\left[B_1^2 b + \left([2]_{p,q} - 1\right)(B_2 - B_1)\right]}{\left([3]_{p,q} - 1\right) B_1^2 b},$$

$$\sigma_2 = \frac{\left([2]_{p,q} - 1\right)\left[B_1^2 b + \left([2]_{p,q} - 1\right)(B_2 + B_1)\right]}{\left([3]_{p,q} - 1\right) B_1^2 b},$$

$$\sigma_3 = \frac{\left([2]_{p,q} - 1\right)\left[B_1^2 b + \left([2]_{p,q} - 1\right) B_2\right]}{\left([3]_{p,q} - 1\right) B_1^2 b}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{\left([2]_{p,q} - 1\right)^2}{\left([3]_{p,q} - 1\right) B_1^2 b} \times \left\{B_1 - B_2 - \frac{B_1^2 b}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right\} |a_2|^2 \leq \frac{B_1 b}{[3]_{p,q} - 1}.$$

Furthermore, if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{\left([2]_{p,q} - 1\right)^2}{\left([3]_{p,q} - 1\right) B_1^2 b} \times \left\{B_1 + B_2 + \frac{B_1^2 b}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right\} |a_2|^2 \leq \frac{B_1 b}{[3]_{p,q} - 1}.$$

Each of these results is sharp.

Remark 2.7. Letting $b = 1$ in Corollary 2.6, we get [22, Theorem 3.1].

Remark 2.8. Letting $p = 1$ in Corollary 2.6, we get [21, Theorem 3].

Taking $\lambda = 1$ in Theorem 2.5, we get following consequence.

Corollary 2.9. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1) belongs to the function class $\mathcal{C}_{p,q}^{b}(\varphi)$ with $b > 0$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_2 b}{[3]_{p,q}\left([3]_{p,q}-1\right)} + \frac{B_1^2 b^2}{[2]_{p,q}-1}\left[\frac{1}{[3]_{p,q}\left([3]_{p,q}-1\right)} - \frac{\mu}{[2]_{p,q}^2\left([2]_{p,q}-1\right)}\right], \\ \mu \leq \sigma_1, \\ \frac{B_1 b}{[3]_{p,q}\left([3]_{p,q}-1\right)}, \\ \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{B_2 b}{[3]_{p,q}\left([3]_{p,q}-1\right)} - \frac{B_1^2 b^2}{[2]_{p,q}-1}\left[\frac{1}{[3]_{p,q}\left([3]_{p,q}-1\right)} - \frac{\mu}{[2]_{p,q}^2\left([2]_{p,q}-1\right)}\right], \\ \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{[2]_{p,q}^2\left([2]_{p,q}-1\right)\left[B_1^2 b + \left([2]_{p,q}-1\right)(B_2 - B_1)\right]}{[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2 b},$$

$$\sigma_2 = \frac{[2]_{p,q}^2\left([2]_{p,q}-1\right)^2\left[B_1^2 b + \left([2]_{p,q}-1\right)(B_2 + B_1)\right]}{[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2 b},$$

$$\sigma_3 = \frac{[2]_{p,q}^2\left([2]_{p,q}-1\right)\left[B_1^2 b + \left([2]_{p,q}-1\right)B_2\right]}{[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2 b}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| + \frac{[2]_{p,q}^2\left([2]_{p,q}-1\right)^2}{[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2 b} \\ &\times \left\{B_1 - B_2 - \frac{B_1^2 b}{[2]_{p,q}-1}\left(1 - \frac{[3]_{p,q}\left([3]_{p,q}-1\right)}{[2]_{p,q}^2\left([2]_{p,q}-1\right)}\mu\right)\right\}|a_2|^2 \\ \leq\; &\frac{B_1 b}{[3]_{p,q}\left([3]_{p,q}-1\right)}. \end{aligned}$$

Furthermore, if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{[2]_{p,q}^2\left([2]_{p,q}-1\right)^2}{[3]_{p,q}\left([3]_{p,q}-1\right)B_1^2 b}$$

$$\times \left\{ B_1 + B_2 + \frac{B_1^2 b}{[2]_{p,q} - 1} \left(1 - \frac{[3]_{p,q} \left([3]_{p,q} - 1 \right)}{[2]_{p,q}^2 \left([2]_{p,q} - 1 \right)} \mu \right) \right\} |a_2|^2$$
$$\leq \frac{B_1 b}{[3]_{p,q} \left([3]_{p,q} - 1 \right)}.$$

Each of these results is sharp.

Remark 2.10. Letting $b = 1$ in Corollary 2.9, we get [22, Theorem 3.2].

Remark 2.11. Letting $p = 1$ in Corollary 2.9, we get [21, Theorem 4].

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