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SEMI-NEUTRAL GROUPOIDS AND BCK-ALGEBRAS

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Abstract. In this paper, we introduce the notion of a left-almost-zero groupoid, and we generalize two axioms which play important roles in the theory of BCK-algebra using the notion of a projection. Moreover, we investigate a Smarandache disjointness of semi-leftoids.

1. Introduction

The notion of BCK-algebra was formulated by Iséki. The motivation of this notion is based on both set theory and propositional calculus (see [6]). The notion of an implicativity law in BCK-algebra has a very strong condition, and it is known that any bounded implicative BCK-algebra forms a distributive lattice and becomes a Boolean algebra. In Huang's book [4], very simple two axioms lead to an implicative BCK-algebra. We considered this fact, and we found one axiom among the two axioms is necessary to be developed in the theory of groupoid, called a left (right)-almost-zero groupoid.

Since the present authors introduce the notion of $Bin(X)$ in groupoid theory, we apply this notion to $Bin(X)$ story. Using the notion of a projection of trace functions, we generalize two axioms of BCK-algebra which play important role in BCK-algebra.

Finally we investigated the structure of semi-leftoid with group or BE-algebra by using the notion of a Smarandache disjointness.

2. Preliminaries

A *d-algebra* [3] is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

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An algebra $(X, *, 0)$ is said to be a *strong d-algebra* ([3]) if it satisfies (I), (II) and (III*) hold for all $x, y \in X$, where

(III*) $x * y = y * x$ implies $x = y$.

Obviously, every strong d-algebra is a d-algebra, but the converse need not be true.

A BCK-*algebra* [4, 5, 9] is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

A BCK-algebra is said to be *implicative* [9] if $x = x*(y*x)$ for all $x, y \in X$. It is known that a BCK-algebra is implicative if and only if it is both commutative and positive implicative.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$. Given groupoids $(X, *)$ and $(X, \bullet)$, we define a binary operation "$\Box$" on $Bin(X)$ by

$$(X, \Box) := (X, *) \, \Box \, (X, \bullet),$$

where

$$x \, \Box \, y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, Kim and Neggers proved the following theorem.

Theorem 2.1 ([8]). $(Bin(X), \Box)$ *is a semigroup, i.e., the operation "$\Box$" is associative. Furthermore, the left-zero semigroup is the identity for this operation.*

An algebraic structure $(X, *, 0)$ is said to be a BE-*algebra* [7] if (BE1) $x*x = 0$, (BE2) $x * 0 = 0$, (BE3) $0 * x = x$, (BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$. These classes of BE-algebras were introduced as a generalization of BCK-algebras.

Given a non-empty set X, two groupoids $(X, *)$, $(X, \bullet)$ are said to be *Smarandache disjoint* [1, 2] if X has both an $(X, *)$-structure and an $(X, \bullet)$-structure, then $|X| = 1$. The notion of "Smarandache disjoint" means that, given a groupoid $(X, *)$, if we combine another groupoid $(X, \bullet)$ to it, then it can only be a trivial groupoid.

3. Semi-neutral groupoids

K. Iséki and S. Tanaka [6] noted that a BCK-algebra $(X, *, 0)$ is said to be *left* (*right*, resp.)-*almost-zero* if $x \neq y$, then $x * y = x$ ($x * y = y$, resp.) for all $x, y \in X$. We use the notion, "semi-neutral", in the groupoid theory. A groupoid $(X, *, 0)$ is said to be (0-)*semi-neutral* if

(I) $x * x = 0$ for all $x \in X$,
(SN) $x * y = x$ for all $x \neq y$ in X.

A groupoid $(X, *)$ is said to be an *a-semi-neutral* if there exists $a \in X$ such that

(I)$'$ $x * x = a$ for all $x \in X$

and (SN).

Example 3.1. Let X be a non-empty set and let $p \in X$. Define a binary operation "$*$" on X by

$$x * y := \begin{cases} x & \text{if } x \neq y, \\ y & \text{otherwise.} \end{cases}$$

Then $x * p = x$ and $p * x = p$ for all $x \in X$. It is easy to see that $(X, *)$ is a p-semi-neutral groupoid. Moreover, every element of X is a vertex of X, and an edge $x \to y$ can be defined by $x * y = x$ for all $x, y \in X$.

Proposition 3.2. *Let $(X, *)$ be a left-almost-zero groupoid. If $x * y = y * x$, then $x = y$, i.e., $(X, *)$ is a strong groupoid.*

Proof. Assume $x * y = y * x$ for some $x \neq y$ in X. Since $(X, *)$ is a left-almost-zero groupoid, we obtain $x * y = x, y * x = y$. Hence $x = x * y = y * x = y$, a contradiction. □

We denote the set of all left-almost-zero groupoids defined on a set X by $LAZ(X)$, i.e.,

$$LAZ(X) := \{(X, *) \in Bin(X) \mid (X, *) : \text{left-almost-zero}\}.$$

Similarly, we denote the set of all right-almost-zero groupoids by $RAZ(X)$.

Proposition 3.3. *The collection $LAZ(X)$ forms a subsemigroup of $(Bin(X), \square)$.*

Proof. Given $(X, *), (X, \bullet) \in LAZ(X)$, we let $(X, \square) := (X, *)\square(X, \bullet)$. If we let $x \neq y$, then $x * y = x \bullet y = x$ and $y * x = y \bullet x = y$. It follows that $x \square y = (x * y) \bullet (y \bullet x) = x \bullet y = x$. Similarly, we have $y \square x = y$. Hence $(X, *)\square(X, \bullet) \in LAZ(X)$. Since $(Bin(X), \square)$ is a semigroup, we proved the proposition. □

Proposition 3.4. *Let $(X, *), (X, \bullet)$ be groupoids and let*

$$(X, \square) := (X, *)\square(X, \bullet).$$

Then

(i) *if $(X, *) \in LAZ(X)$ and $(X, \bullet) \in RAZ(X)$, then $(X, \square) \in RAZ(X)$,*
(ii) *if $(X, *) \in RAZ(X)$ and $(X, \bullet) \in LAZ(X)$, then $(X, \square) \in RAZ(X)$,*
(iii) *if $(X, *), (X, \bullet) \in RAZ(X)$, then $(X, \square) \in LAZ(X)$.*

Proof. Straightforward. □

A groupoid $(X, *)$ is said to be *power associative* if $(x * x) * x = x * (x * x)$ for all $x \in X$.

Theorem 3.5. *Let $(X, *)$ be a left-almost-groupoid. If $(X, *)$ is power associative, then $(X, *)$ is a left zero semigroup.*

Proof. It is enough to show that $x * x = x$ for all $x \in X$, since $(X, *)$ is a left-almost-groupoid. Assume there exists $x \in X$ such that $x * x \neq x$ for some $x \in X$. Since $(X, *)$ is a left-zero-groupoid, we obtain $(x * x) * x = x * x$ and $x * (x * x) = x$. It follows from $(X, *)$ is power associative that $x = x * (x * x) = (x * x) * x = x * x$, which is a contradiction. Hence $(X, *)$ is a left zero semigroup. □

Remark 3.6. Every semigroup is power associative, but a power associative groupoid need not be a semigroup in general. See the following example.

Example 3.7. Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := \frac{x+y}{2}$ for all $x, y \in \mathbf{R}$. Then $x * x = \frac{1}{2}(x + x) = x$ and hence $(x * x) * x = x = x * (x * x)$. On the while, $(x * y) * z = \frac{1}{4}(x + y + 2z) \neq \frac{1}{4}(2x + y + z) = x * (y * z)$, i.e., $(\mathbf{R}, *)$ is not associative. Hence $(\mathbf{R}, *)$ is not a left zero semigroup.

Corollary 3.8. *Let $(X, *)$ be a right-almost-groupoid. If $(X, *)$ is power associative, then $(X, *)$ is a right zero semigroup.*

Proof. Similar to Theorem 3.5. □

In Huang's book [4], the following theorem was mentioned without proof. We provide its proof for developing the theory of a semi-neutral groupoid.

Theorem 3.9. *Every semi-neutral groupoid is an implicative BCK-algebra.*

Proof. Let $(X, *, 0)$ be a semi-neutral groupoid. We claim that it is a strong d-algebra. (III)*: Assume $x * y = y * x$ and $x \neq y$. By (SN), we obtain $x = x * y = y * x = y$, which is a contradiction. Hence (III)* holds. (II): If $x := 0$, then $0 * x = 0 * 0 = 0$ by (I). If $x \neq 0$, then, by (SN), $0 * x = 0$. Hence $(X, *, 0)$ is a strong d-algebra. We claim that $x * 0 = x$ for all $x \in X$. If $x = 0$, then $x * 0 = 0 * 0 = 0$ by (I). If $x \neq 0$, then, by (SN), we have $x * 0 = 0$. We claim that $(x * (x * y)) * y = 0$ for all $x, y \in X$. Given $x, y \in X$, if $x = y$, then $(x*(x*y))*y = (x*(x*x))*x = (x*0)*x = x*x = 0$. If $x \neq y$, then $x*y = x$ and hence $x * (x * y) = x * x = 0$. It follows that $(x * (x * y)) * y = 0 * y = 0$. We claim that $[(x * y) * (x * z)] * (z * y) = 0$. We have 3 cases: (i) $x \neq y$, $x \neq z$; (ii) $x = y$; (iii) $x = z$. If (i) $x \neq y$, $x \neq z$, then

$$\begin{aligned}[(x * y) * (x * z)] * (z * y) &= (x * x) * (z * y) \\ &= 0 * (z * y) \\ &= 0.\end{aligned}$$

If (ii) $x = y$, then

$$\begin{aligned}[(x * y) * (x * z)] * (z * y) &= [0 * (x * z)] * (z * x) \\ &= 0 * (z * x)\end{aligned}$$

$$= 0.$$

If (ii) $x = z$, then

$$\begin{aligned}[(x * y) * (x * z)] * (z * y) &= [(x * y) * 0] * (x * y) \\ &= (x * y) * (x * y) \\ &= 0.\end{aligned}$$

Hence $(X, *, 0)$ is a BCK-algebra. We claim that $(x * y) * y = x * y$ for all $x, y \in X$. Given $x, y \in X$, if $x = y$, then $x * y = x * x = 0$ and hence $(x * y) * y = 0 * y = 0 = x * y$. If $x \neq y$, then, by (SN), we obtain $x * y = x$ and hence $(x * y) * y = x * y$. We claim that $x * (x * y) = y * (y * x)$ for all $x, y \in X$. Given $x, y \in X$, if $x = y$, it holds trivially. If $x \neq y$, then $x * y = x$ and $y * x = y$, and hence $x * (x * y) = x * x = 0$ and $y * (y * x) = y * y = 0$. This proves that $(X, *, 0)$ is an implicative BCK-algebra. □

Example 3.10. Let $X := \{0, 1, 2, 3\}$ with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then it is easy to see that $(X, *, 0)$ is both a semi-neutral groupoid and an implicative BCK-algebra.

Remark 3.11. The converse of Theorem 3.9 need not be true in general.

Example 3.12. Let $X := \{0, 1, 2, 3\}$ with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then it is easy to see that $(X, *, 0)$ is an implicative BCK-algebra, but not a semi-neutral groupoid, since $3 * 2 = 1 \neq 3$ and $3 * 1 = 2 \neq 3$.

Theorem 3.13. *Let* $(X, *, 0), (X, \bullet, \widehat{0})$ *be semi-neural groupoids, and let*

$$(X, \Box) := (X, *)\Box(X, \bullet).$$

Then $(X, \Box, \widehat{0})$ *is a semi-neutral groupoid.*

Proof. Given $x \in X$, we have $x \Box x = (x * x) \bullet (x * x) = 0 \bullet 0 = \widehat{0}$. Let $x \neq y$ in X. Then $x \Box y = (x * y) \bullet (y * x) = x \bullet y = x$. Hence $(X, \Box, \widehat{0})$ is a semi-neutral groupoid. □

Theorem 3.13 shows that *the semi-neutral groupoids with respect to* $\Box$ *forms a subsemigroup of* $(Bin(X), \Box)$.

We generalize Theorem 3.13 by using the notion of a semi-leftoid. A groupoid $(X, *, 0)$ is said to be a *semi-leftoid* over f if $x, y \in X$, (I) $x * x = 0$ for all $x \in X$; (SL) $x * y = f(x)$ for all $x, y \in X$, where $f : X \to X$ is a map.

Proposition 3.14. *Let* $(X, *, 0)$ *be a semi-leftoid over* f *and let* $(X, \bullet, \widehat{0})$ *be a semi-leftoid over* g*, where* $f : X \to X$ *is an injective function. If we define* $(X, \Box) := (X, *)\Box(X, \bullet)$*, then* $(X, \Box, \widehat{0})$ *is a semi-leftoid over* $g \circ f$.

Proof. Given x, y in X, we have $x\Box x = (x * x) \bullet (x * x) = 0 \bullet 0 = \widehat{0}$. Let $x \neq y$ in X. Since f is injective, we have $f(x) \neq f(y)$. It follows that $x\Box y = (x * y) \bullet (y * x) = f(x) \bullet f(y) = g(f(x)) = (g \circ f)(x)$. □

Proposition 3.14 shows that *the collection of all semi-leftoids* $(X, *, 0)$ *over* f*, where* $f : X \to X$ *is an injective function, forms a subsemigroup of* $(Bin(X), \Box)$.

Let $(X, *, 0)$ be a semi-leftoid for f. If f is a bijection, then we have a question: Is $(X, *, 0)$ a semi-leftoid for f^{-1}? The answer is no. See the following example.

Example 3.15. Let $X := \mathbb{R}$ and let $x * y := 2x + 1$ for all $x, y \in \mathbb{R}$. If we define a map $f : X \to X$ by $f(x) := 2x + 1$, then $(X, *, 0)$ is a semi-leftoid for f. If we assume $(X, *, 0)$ is a semi-leftoid for f^{-1}, then $f(2) = 2 * 3 = f^{-1}(2)$. It follows that $5 = \frac{1}{2}$, a contradiction.

Proposition 3.16. *Let* $f : X \to X$ *be a map. Let* $(X, *, 0)$ *and* $(X, \bullet, \widehat{0})$ *be groupoids with*

(i) $x * x = 0 \neq \widehat{0} = x \bullet x$ *for all* $x \in X$,

(ii) *if* $x \neq y$ *in* X*, then* $x * y = f(x) = x \bullet y$.

Then $(X, *, 0)$ *and* $(X, \bullet, \widehat{0})$ *are isomorphic as groupoids, while they are distinct elements of* $Bin(X)$.

Proof. Define a map $\varphi : (X, *, 0) \to (X, \bullet, \widehat{0})$ by $\varphi(x) := x$. Then it is a bijective function. Given $x, y \in X$, we have $\varphi(x * y) = \varphi(f(x)) = f(x) = x \bullet y = \varphi(x) \bullet \varphi(y)$. This proves the proposition. □

4. Trace functions and left-almost-zero groupoids

Given a groupoid $(X, *)$, we define a map $T(X, *) : X \to X$ by $T(X, *)(x) := x * x$ for all $x \in X$. We call such a map $T(X, *)$ a *trace function* of $(X, *)$.

Proposition 4.1. *If* $(X, *), (X, \bullet)$ *are groupoids, then* $T((X, *)\Box(X, \bullet)) = T(X, \bullet) \circ T(X, *)$.

Proof. Let $(X, \Box) := (X, *)\Box(X, \bullet)$. Then, for any $x \in X$, we have $T((X, *) \Box(X, \bullet))(x) = T(X, \Box)(x) = x\Box x = (x*x)\bullet(x*x) = T(X, *)(x)\bullet T(X, *)(x) = T(X, \bullet)(T(X, *)(x)) = [T(X, \bullet) \circ T(X, *)](x)$. □

Given a groupoid $(X,*)$, we define a map $\iota : X \to X$ by $\iota(x) := x$ for all $x \in X$. For any trace function T, we define a subset $Ker(T)$ of $Bin(X)$ by

$$Ker(T) := \{(X,*) \in Bin(X) \mid T(X,*) = \iota\}.$$

Proposition 4.2. *Let $(X,*)$ be a groupoid and let $(X,\bullet) \in Ker(T)$. Then*

$$T((X,*)\Box(X,\bullet)) = T(X,*) = T((X,\bullet)\Box(X,*)).$$

Proof. Given $x \in X$, since $(X,\bullet) \in Ker(T)$ and Proposition 4.1, we have

$$\begin{aligned} T((X,*)\Box(X,\bullet))(x) &= [T(X,\bullet) \circ T(X,*)](x) \\ &= T(X,\bullet)(T(X,*)(x)) \\ &= T(X,\bullet)(x*x) \\ &= x*x \\ &= T(X,*)(x). \end{aligned}$$

Similarly, we obtain $T((X,\bullet)\Box(X,*)) = T(X,*)$, proving the proposition. $\Box$

Let $(X,*)$ be a groupoid. A map $\varphi : X \to X$ is said to be a *projection* on X if $\varphi \circ \varphi = \varphi$, i.e., $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in X$.

Theorem 4.3. *Let $(X,*)$ be a left-almost-zero groupoid and let $\varphi := T(X,*)$. If φ is a projection on $(X,*)$, then*

$$(x*(x*y))*y = \varphi(x)$$

for all $x, y \in X$.

Proof. Given $x, y \in X$, if $x = y$, then $(x*(x*y))*y = (x*(x*x))*x = (x*\varphi(x))*x$. If $x = \varphi(x)$, then $(x*\varphi(x))*x = (x*x)*x = \varphi(x)*x = x*x = \varphi(x)$. If $x \neq \varphi(x)$, then $(x*\varphi(x))*x = x*x = \varphi(x)$. Hence $(x*(x*y))*y = \varphi(x)$. If $x \neq \varphi(x)$, then $x*y = x$, since $(X,*)$ is a left-almost-zero groupoid. It follows that

$$(x*(x*y))*y = (x*x)*y = \varphi(x)*y.$$

If $\varphi(x) \neq y$, then $\varphi(x)*y = \varphi(x)$. If $\varphi(x) = y$, then $\varphi(x)*y = \varphi(\varphi(x)) = \varphi(x)$, since φ is a projection on X. Hence we obtain $(x*(x*y))*y = \varphi(x)$, proving the theorem. $\Box$

Corollary 4.4. *Let $(X,*)$ be a left-almost-zero groupoid and let $\varphi := T(X,*)$. If there exists $0 \in X$ such that $\varphi(x) = 0$ for all $x \in X$, then*

$$(x*(x*y))*y = 0$$

for all $x, y \in X$.

Proof. Assume that there exists $0 \in X$ such that $\varphi(x) = 0$ for all $x \in X$. Then $(\varphi \circ \varphi)(x) = \varphi(\varphi(x)) = \varphi(0) = 0 = \varphi(x)$ for all $x \in X$, i.e., φ is a projection on X. By Theorem 4.3, we prove that $(x*(x*y))*y = \varphi(x) = 0$ for all $x, y \in X$. $\Box$

Theorem 4.5. *Let* $(X, *)$ *be a left-almost-zero groupoid and let* $\varphi := T(X, *)$*. If* φ *is a projection on* $(X, *)$*, then*

$$((x * y) * (x * z)) * (z * y) = \varphi(x)$$

for all $x, y, z \in X$*.*

Proof. Given $x, y, z \in X$, we have 4 cases: (i) $x \neq y, x \neq z$; (ii) $x = y, x \neq z$; (iii) $x \neq y, x = z$; (iv) $x = y, x = z$.

(i) Assume $x \neq y, x \neq z$. Then $x * y = x, x * z = x$. It follows that

$$\begin{aligned}((x * y) * (x * z)) * (z * y) &= (x * x) * (z * y)\\ &= (x * x) * (z * y)\\ &= \varphi(x) * (z * y)\\ &= \begin{cases} \varphi(x) * \varphi(x) & \text{if } \varphi(x) = z * y,\\ \varphi(x) & \text{if } \varphi(x) \neq z * y\end{cases}\\ &= \begin{cases} \varphi(\varphi(x)) & \text{if } \varphi(x) = z * y,\\ \varphi(x) & \text{if } \varphi(x) \neq z * y\end{cases}\\ &= \varphi(x),\end{aligned}$$

since φ is a projection on X.

(ii) Assume $x = y, x \neq z$. Then $x * y = x * x = \varphi(x), x * z = x$ and $z * x = z$. It follows that

$$\begin{aligned}((x * y) * (x * z)) * (z * y) &= (\varphi(x) * x) * z\\ &= \begin{cases} (\varphi(x) * \varphi(x)) * z & \text{if } \varphi(x) = x,\\ \varphi(x) * z & \text{if } \varphi(x) \neq x\end{cases}\\ &= \begin{cases} \varphi(\varphi(x)) * z & \text{if } \varphi(x) = z * y,\\ \varphi(x) * z & \text{if } \varphi(x) \neq z * y\end{cases}\\ &= \varphi(x) * z,\end{aligned}$$

since φ is a projection on X. We claim that $\varphi(x) * z = \varphi(x)$. In fact, if $\varphi(x) \neq z$, then $\varphi(x) * z = \varphi(x)$, since $(X, *)$ is a left-almost-zero groupoid. If $\varphi(x) = z$, then $\varphi(x) * z = \varphi(x) * \varphi(x) = \varphi(\varphi(x)) = \varphi(x)$, since φ is a projection on X. Hence we obtain $((x * y) * (x * z)) * (z * y) = \varphi(x) * z = \varphi(x)$.

(iii) Assume $x \neq y, x = z$. Then $x * y = x, x * z = x * x = \varphi(x)$ and $z * y = x * y = x$. It follows that

$$\begin{aligned}((x * y) * (x * z)) * (z * y) &= (x * \varphi(x)) * x\\ &= \begin{cases} (\varphi(x) * \varphi(x)) * x & \text{if } \varphi(x) = x,\\ x * x & \text{if } \varphi(x) \neq x\end{cases}\\ &= \begin{cases} \varphi(\varphi(x)) * x & \text{if } \varphi(x) = x,\\ \varphi(x) & \text{if } \varphi(x) \neq x\end{cases}\\ &= \varphi(x),\end{aligned}$$

since φ is a projection on X.

(iv) Assume $x = y, x = z$. Then $((x*y)*(x*z))*(z*y) = ((x*x)*(x*x))*(x*x) = (\varphi(x)*\varphi(x))*\varphi(x) = \varphi(\varphi(x))*\varphi(x) = \varphi(x)*\varphi(x) = \varphi(\varphi(x)) = \varphi(x)$. This proves the theorem. □

Corollary 4.6. *Let $(X, *)$ be a left-almost-zero groupoid and let $\varphi := T(X, *)$. If there exists $0 \in X$ such that $\varphi(x) = 0$ for all $x \in X$, then*

$$((x * y) * (x * z)) * (z * y) = 0$$

for all $x, y, z \in X$.

Proof. As we have seen in Theorem 4.3, φ is a projection and $\varphi(x) = 0$ for all $x \in X$. By Theorem 4.5, we obtain $((x * y) * (x * z)) * (z * y) = 0$ for all $x, y, z \in X$. □

Theorem 4.7. *Let $(X, *)$ be a left-almost-zero groupoid and let $\varphi := T(X, *)$. If there exists $0 \in X$ such that $\varphi(x) = 0$ for all $x \in X$, then $(X, *, 0)$ is a BCK-algebra.*

Proof. Given $x \in X$, we have $x * x = \varphi(x) = 0$ and $0 * x = 0$, since $(X, *)$ is a left-almost-zero groupoid. Assume there exist x, y in X with $x \neq y$ such that $x * y = 0 = y * x$. Since $(X, *)$ is a left-almost-zero groupoid, we have $x * y = x$ and $y * x = y$, and hence $x = 0 = y$, which is a contradiction. Hence $(X, *, 0)$ is a d-algebra. By applying Theorem 4.3 and Corollary 4.6, we prove that $(X, *, 0)$ is a BCK-algebra. □

A groupoid $(X, *)$ is said to be *positive implicative* if $(x * y) * y = x * y$ for all $x, y \in X$. Every positive implicative BCK-algebra is a positive implicative groupoid. The following proposition shows the another way to find positive implicative groupoids.

Proposition 4.8. *Let $(X, *)$ be a left-almost-zero groupoid and let $\varphi := T(X, *)$. Then $(X, *)$ is positive implicative.*

Proof. Given $x, y \in X$, if $x \neq y$, then $x * y = x$ and hence $(x * y) * y = x * y$. If $x = y$, then $(x * y) * y = (x * x) * x = \varphi(x) * x$. We claim that $\varphi(x) * x = \varphi(x)$. In fact, if $\varphi(x) \neq x$, then $\varphi(x) * x = \varphi(x)$, since $(X, *)$ is a left-almost-zero groupoid. If $\varphi(x) = x$, then $\varphi(x) * x = x * x = \varphi(x)$. Hence $(x * y) * y = (x * x) * x = \varphi(x) * x = \varphi(x) = x * x = x * y$, proving the proposition. □

5. Smarandache disjointness

The notion of the Smarandache disjointness is very important to investigate the structures of several general algebraic structures. It gives to test some collision between axioms which consisting two algebraic structures. If we find two algebraic structures are Smarandache disjoint, then two algebraic structures have different branches.

In this section we investigate some relations between the semi-leftoid and other two algebraic structures as follow:

Theorem 5.1. *The class of groups and the class of a semi-leftoids are Smarandache disjoint.*

Proof. Let $(X,*,0)$ be both a group and a semi-leftoid for f. If we take y_1 and y_2 with $y_1 \neq y_2$, then $x*y_1 = f(x) = x*y_2$ for all $x \in X$. Since X is a group, by cancellative laws, we obtain $y_1 = y_2$, a contradiction. □

Theorem 5.2. *The class of BE-algebras and the class of a semi-leftoids are Smarandache disjoint.*

Proof. Let $(X,*,0)$ be both a BE-algebra and a semi-leftoid for f. Then we have $f(0) = 0*x = x$ for all $x \in X$. If $|X| \geq 2$, then the mapping f has two values, which is a contradiction. Hence $|X| = 1$, proving the theorem. □

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