Commun. Korean Math. Soc. **37** (2022), No. 3, pp. 735–748
https://doi.org/10.4134/CKMS.c210218
pISSN: 1225-1763 / eISSN: 2234-3024

AN IMPROVED GLOBAL WELL-POSEDNESS RESULT FOR THE MODIFIED ZAKHAROV EQUATIONS IN 1-D

Agus L. Soenjaya

Abstract. The global well-posedness for the fourth-order modified Zakharov equations in 1-D, which is a system of PDE in two variables describing interactions between quantum Langmuir and quantum ion-acoustic waves is studied. In this paper, it is proven that the system is globally well-posed in $(u, n) \in L^2 \times L^2$ by making use of Bourgain restriction norm method and L^2 conservation law in u, and controlling the growth of n via appropriate estimates in the local theory. In particular, this improves on the well-posedness results for this system in [9] to lower regularity.

1. Introduction

It is well-known that an important model in plasma physics is described by Zakharov equations, which describes the interaction between Langmuir waves and ion-acoustic waves. More recently, by taking into account quantum effects, a modified system of Zakharov equations was proposed in [7] via a quantum hydrodynamic approach, and further analyzed in [10,12]. For a more complete account, refer to the monograph [11]. This quantum correction are important for a better description of various models in plasma physics, and are significant especially in high-density scenarios such as small-scale dense plasma systems or compact astrophysics objects.

In this paper, we will study the initial-value problem for 1-D modified fourth-order Zakharov equations (with quantum correction), which reads

$$\begin{cases} i\partial_t u + \partial_x^2 u - h^2 \partial_x^4 u = nu, \\ \partial_t^2 n - \partial_x^2 n + h^2 \partial_x^4 n = \partial_x^2 |u|^2, \\ u(x,0) = u_0(x),\ n(x,0) = n_0(x),\ \partial_t n(x,0) = n_1(x). \end{cases} \tag{1}$$

Here, $u : [0, T^*) \times \mathbb{R} \to \mathbb{C}$ denotes the envelope of the high-frequency electric field and $n : [0, T^*) \times \mathbb{R} \to \mathbb{R}$ denotes the plasma density measured from its

Received June 14, 2021; Revised August 25, 2021; Accepted September 2, 2021.
2010 *Mathematics Subject Classification.* Primary 35Q40, 35G55.
Key words and phrases. Global well-posedness, low regularity, modified Zakharov equations.

equilibrium value, and h is a small quantity measuring the influence of quantum effects. Smooth solutions of (1) satisfy conservation of L^2 norm

$$M[u](t) := \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M[u_0] \tag{2}$$

and conservation of the Hamiltonian

$$\begin{aligned} H[u,n,\partial_t n](t) := \frac{1}{2}\Big(&\|(-\Delta)^{-\frac{1}{2}}\partial_t n(t)\|_{L^2}^2 + \|n(t)\|_{L^2}^2 + \|\partial_x n(t)\|_{L^2}^2 \Big) \\ &+ \|\partial_x u(t)\|_{L^2}^2 + \|\partial_x^2 u(t)\|_{L^2}^2 + \int_{\mathbb{R}} n(t)|u(t)|^2 \,\mathrm{d}x \\ = H&[u_0, n_0, \partial_t n(0)]. \end{aligned}$$

Subsequently, for convenience, let $h := 1$.

Note that when $h = 0$, the system reduces to the classical 1-D Zakharov system

$$\begin{cases} i\partial_t u + \partial_x^2 u = nu, \\ \partial_t^2 n - \partial_x^2 n = \partial_x^2 |u|^2, \\ u(x,0) = u_0(x),\ n(x,0) = n_0(x),\ \partial_t n(x,0) = n_1(x), \end{cases} \tag{3}$$

which has been extensively studied in Physics and Mathematics literatures. In particular, the local well-posedness for (3) is established in [8], and global well-posedness in [5], with $(u_0, n_0) \in L^2(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$. For well-posedness on anisotropic Sobolev spaces, refer to [15]. Local well-posedness for higher dimensional Zakharov system is also studied in [1, 2, 8]. For other aspects of the Zakharov system, such as its limiting behavior when certain parameters are taken to infinity, refer to [14].

On the hand, for the system (1), relatively little is known. Smooth solution of initial boundary value problem for the modified Zakharov equations in various dimensions was considered in [6, 9, 17, 18]. In [9], global well-posedness of (1) is established for $(u_0, n_0) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. Existence of weak solution in some Sobolev spaces for (1) in bounded domain is also studied in [4].

This paper aims to extend the result in [9] to lower regularity by proving global well-posedness of (1) with $(u_0, n_0, n_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$. We will use Bourgain restriction norm method (initiated in [3]) to establish the local theory (following [8]). Then, by controlling the growth of n, we will extend this local solution to a global one.

As a note, by local well-posedness, we mean that the solution exists in a small time interval, and is unique, that the solution has the same regularity as the initial data in that time interval, and the solution depends continuously on the initial data. By global well-posedness, we mean that the above properties hold for all time $t > 0$.

2. Notations

Given $A, B \geq 0$, we write $A \lesssim B$ to mean that for some universal constant k, $A \leq k \cdot B$. We write $A \approx B$ or $A \sim B$ to mean both $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ denotes $B > k{\cdot}A$. We introduce the notation $\langle x\rangle := \sqrt{1+|x|^2}$, and $\langle\nabla\rangle$ for the operator with Fourier multiplier $\langle\xi\rangle$. We will also use the notation $\alpha+$ for $\alpha\in\mathbb{R}$ to mean a number slightly larger than α, i.e., $\alpha+\varepsilon$ for some $\varepsilon>0$.

Given Lebesgue space exponents q, r and a function $f(x,t)$ on $\mathbb{R}\times\mathbb{R}$, we define the mixed (space-time) Lebesgue norm

$$\|f\|_{L_t^q L_x^r} := \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x,t)|^r\,\mathrm{d}x\right)^{\frac{q}{r}}\mathrm{d}t\right)^{\frac{1}{q}}.$$

Let H^s be the usual Sobolev spaces equipped with the norm

$$\|f\|_{H^s} = \left(\int_\xi (1+|\xi|^2)^s|\widehat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac12} =: \|\langle\xi\rangle^s\widehat{f}\|_{L^2_\xi},$$

where $\widehat{f}$ is the Fourier transform of f. Denote also by $f^\vee$ the inverse Fourier transform of f. Also, introduce the norm

$$\|(n_0,n_1)\|_{\mathcal{W}} := \left(\|n_0\|^2_{L^2_x}+\|n_1\|^2_{H_x^{-1}}\right)^{\frac12},$$

and the shorthand notation $\|n(t)\|_{\mathcal{W}} := \left\|\left(n(t),(-\Delta)^{-\frac12}\partial_t n(t)\right)\right\|_{\mathcal{W}}$. We will also need the following Bourgain spaces (see [16] for more details). Define the spaces $X^S_{0,\alpha}$ and $X^W_{0,\alpha}$ for $\alpha\in\mathbb{R}$, respectively, equipped with the following norms

$$\begin{aligned}
\|z\|_{X^S_{0,\alpha}} &:= \left\|\langle\tau+\xi^2+\xi^4\rangle^\alpha\,\widehat{z}\right\|_{L^2_\xi L^2_\tau}\\
&= \left(\iint_{\xi,\tau}\langle\tau+\xi^2+\xi^4\rangle^{2\alpha}|\widehat{z}(\xi,\tau)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\right)^{\frac12},\\
\|z\|_{X^W_{0,\alpha}} &:= \left\|\left\langle|\tau|-\sqrt{\xi^2+\xi^4}\right\rangle^\alpha\widehat{z}\right\|_{L^2_\xi L^2_\tau}\\
&= \left(\iint_{\xi,\tau}\left\langle|\tau|-\sqrt{\xi^2+\xi^4}\right\rangle^{2\alpha}|\widehat{z}(\xi,\tau)|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\right)^{\frac12},
\end{aligned}$$

where here $\widehat{z}(\xi,\tau)$ denotes the space-time Fourier transform of z.

Also, let $\psi\in C_0^\infty(\mathbb{R})$ satisfy $\psi(t)=1$ on $[-1,1]$ and $\psi(t)=0$ outside of $[-2,2]$. Let $\psi_T(t)=\psi(t/T)$, which will serve as a time cutoff for some local estimates. For clarity, we write $\psi_1(t)=\psi(t)$.

3. Estimates for the group term

Let $U(t)$ denote the group generated by the linear part of the fourth-order Schrödinger equation (first equation of (1)), i.e.,

$$U(t)u_0 := \left(e^{-it(\xi^2+\xi^4)}\widehat{u_0}(\xi)\right)^\vee,$$

and let $W(t)$ denote the group generated by the linear part of the fourth-order wave equation (second equation of (1)), i.e.,

$$W(t)(n_0, \partial_x n_1) := W_0(t)n_0 + W_1(t)\partial_x n_1,$$

where

$$W_0(t)n_0 := \Big(\cos(t\sqrt{\xi^2+\xi^4})\widehat{n_0}(\xi)\Big)^{\vee},$$

$$W_1(t)(\partial_x n_1) := \left(\frac{\sin(t\sqrt{\xi^2+\xi^4})}{\sqrt{\xi^2+\xi^4}}\widehat{\partial_x n_1}(\xi)\right)^{\vee}.$$

We have the following estimates for the group terms which will be used subsequently.

Proposition 3.1. *Let $0 \le b_1 \le \frac{1}{2}$ and $0 < T \le 1$. Then*

(a) $\|\psi_T(t)U(t)u_0\|_{C(\mathbb{R};L^2)} = \|u_0\|_{L^2}$.

(b) $\|\psi_T(t)U(t)u_0\|_{X^S_{0,b_1}} \lesssim T^{\frac{1}{2}-b_1}\|u_0\|_{L^2}$.

(c) (*Strichartz estimates*) $\|D_x^{\zeta\eta\theta/2}U(t)u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2}$, *where* $0 \le \zeta, \eta, \theta \le 1$ *and admissible pair* $(q,r) = \left(\frac{8}{\zeta(\eta+1)}, \frac{2}{1-\zeta}\right)$.

Proof. Proof of (a) is immediate by Plancherel's theorem. For (b), note that we have $[\psi_T(t)U(t)u_0]\widehat{\ }(\xi,\tau) = \widehat{\psi_T}(\tau+\xi^2+\xi^4)\widehat{u_0}(\xi)$, and so

$$\|\psi_T(t)U(t)u_0\|_{X^S_{0,b_1}} \le C\|\psi_T\|_{H^{b_1}}\|u_0\|_{L^2}.$$

To complete the proof, we note that

$$\|\psi_T\|_{H^{b_1}} \le \|\psi_T\|_{L^2} + \|\psi_T\|_{\dot{H}^{b_1}} = T^{\frac{1}{2}}\|\psi_1\|_{L^2} + T^{\frac{1}{2}-b_1}\|\psi_1\|_{\dot{H}^{b_1}}$$

by scaling, and thus (b) follows.

For (c), let $\zeta \in [0,1]$ and $(q,r) = \left(\frac{4}{\zeta}, \frac{2}{1-\zeta}\right)$. Then by an oscillatory integral estimate in [13, Theorem 2.1], we have

$$\left\| \int_{\mathbb{R}} e^{ix\xi - it(\xi^2+\xi^4)} |2+12\xi^2|^{\zeta/4}\widehat{u_0}(\xi)\,\mathrm{d}\xi \right\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2}. \tag{4}$$

Now, note that we have the inequalities $\max(1,|\xi|^2) \lesssim |1+6\xi^2|$ for $\xi \in \mathbb{R}$, which then implies

$$\|U(t)u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2} \quad \text{and} \quad \|D_x^{\zeta/2}U(t)u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2}.$$

Interpolating these, we have that for any $\theta \in [0,1]$,

$$\|D_x^{\zeta\theta/2}U(t)u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2}. \tag{5}$$

By Sobolev embedding $W^{\zeta/8,4/\zeta}(\mathbb{R}) \hookrightarrow L^{8/\zeta}(\mathbb{R})$, elementary inequality $|1+\xi^2| \lesssim |1+6\xi^2|$ and the estimate (4),

$$\|U(t)u_0\|_{L^{8/\zeta}_tL^r_x} \lesssim \|D_t^{\zeta/8}\widehat{u_0}\|_{L^{4/\zeta}_tL^r_x}$$

$$\approx \left\| \int_{\mathbb{R}} e^{ix\xi - it(\xi^2+\xi^4)} |\xi|^{\zeta/4} |1+\xi^2|^{\zeta/8} \widehat{u_0}(\xi)\,\mathrm{d}\xi \right\|_{L_t^{4/\zeta} L_x^r}$$
$$\lesssim \|u_0\|_{L^2}.$$

Finally, interpolating the above and (5) gives the result for any $\eta \in [0,1]$. □

Proposition 3.2. *Let* $0 \le b \le \frac{1}{2}$ *and* $0 < T \le 1$. *Then*

(a) $\|\psi_T(t)W(t)(n_0, \partial_x n_1)\|_{C(\mathbb{R};L^2)} \le \|n_0\|_{L^2} + \|n_1\|_{H^{-1}}$.
(b) $\|\psi_T(t)W(t)(n_0, \partial_x n_1)\|_{X^W_{0,b}} \lesssim T^{\frac{1}{2}-b} \|(n_0,n_1)\|_{\mathcal{W}}$.
(c) $\|\psi_T(t)W(t)(n_0, \partial_x n_1)\|_{C(\mathbb{R};\mathcal{W})} = \|(n_0,n_1)\|_{\mathcal{W}}$.

Proof. Part (a) follows immediately by Plancherel's theorem and definitions of $W(t)$. For (b), note that letting $\phi(\xi) := \sqrt{\xi^2+\xi^4}$, we have

$$[\psi_T(t)W(t)(n_0,\partial_x n_1)]\widehat{\ }(\xi,\tau) = \frac{\widehat{\psi_T}(\tau-\phi(\xi))}{2} h_1(\xi) + \frac{\widehat{\psi_T}(\tau+\phi(\xi))}{2} h_2(\xi),$$

where

$$h_1(\xi) := \widehat{n_0}(\xi) + \frac{i\xi\widehat{n_1}(\xi)}{\phi(\xi)} \quad \text{and} \quad h_2(\xi) := \widehat{n_0}(\xi) - \frac{i\xi\widehat{n_1}(\xi)}{\phi(\xi)}.$$

Therefore,

$$\begin{aligned}
&\|\psi_T(t)W(t)(n_0,\partial_x n_1)\|^2_{X^W_{0,b}} \\
\le{}& \int_\xi |h_1(\xi)|^2 \left(\int_\tau \langle |\tau| - \phi(\xi)\rangle^{2b} \left| \frac{\widehat{\psi_T}(\tau-\phi(\xi)) + \widehat{\psi_T}(\tau+\phi(\xi))}{2} \right|^2 \mathrm{d}\tau \right) \mathrm{d}\xi \\
&+ \int_\xi |h_2(\xi)|^2 \left(\int_\tau \langle |\tau| - \phi(\xi)\rangle^{2b} \left| \frac{\widehat{\psi_T}(\tau-\phi(\xi)) + \widehat{\psi_T}(\tau+\phi(\xi))}{2} \right|^2 \mathrm{d}\tau \right) \mathrm{d}\xi.
\end{aligned}$$

Noting that $||\tau| - \phi(\xi)| \le \min(|\tau-\phi(\xi)|, |\tau+\phi(\xi)|)$, we have

$$\begin{aligned}
\|\psi_T(t)W(t)(n_0,\partial_x n_1)\|^2_{X^W_{0,b}} &\lesssim \left(\|h_1\|^2_{L^2} + \|h_2\|^2_{L^2}\right) \|\psi_T\|^2_{H^b} \\
&\lesssim \left(\|n_0\|^2_{L^2} + \|n_1\|^2_{H^{-1}}\right) \|\psi_T\|^2_{H^b},
\end{aligned}$$

and

$$\|\psi_T\|_{H^b} \le \|\psi_T\|_{L^2} + \|\psi_T\|_{\dot{H}^b} = T^{\frac{1}{2}} \|\psi_1\|_{L^2} + T^{\frac{1}{2}-b} \|\psi_1\|_{\dot{H}^b}$$

by scaling. Thus (b) follows.

For (c), let $n(x,t)$ solve the linear fourth order wave equation

$$\partial_t^2 n - \partial_x^2 n + \partial_x^4 n = 0, \tag{6}$$

with initial data $f(x,0) = n_0(x)$, $\partial_t f(x,0) = \partial_x n_1$. We apply the operator $\langle\nabla\rangle^{-1}(-\Delta)^{-1}$ to (6), then multiply by $\langle\nabla\rangle^{-1}\partial_t n$ and integrate over x. Then on the Fourier side, we have

$$\int_\xi \widehat{n}(\xi)\,\partial_t\widehat{n}(\xi) + \langle\xi\rangle^{-2}|\xi|^{-2}\partial_t\widehat{n}(\xi)\,\partial_t^2\widehat{n}(\xi)\,\mathrm{d}\xi = 0$$

or equivalently the following conservation identity

$$\partial_t\left(\|n\|_{L^2}^2 + \left\|(-\Delta)^{-\frac{1}{2}}\partial_t n\right\|_{H^{-1}}^2\right) = 0,$$

which implies (c). □

Here we note that it is essential for estimate (a) and (c) in Proposition 3.2 to not have implicit constant multiples on the right-hand side, as these estimates will be used to control the growth of the norm of the solution in the iteration argument.

4. Estimates for the Duhamel term

Throughout this section, let $U(t)$ and $W(t)$ be as in the previous section. We have the following estimates for the integral term.

Proposition 4.1. *Let* $0 < T \leq 1$.

(a) *If* $0 \leq c_1 < \frac{1}{2}$, *then*

$$\left\|\int_0^t U(t-t')f(x,t')\,\mathrm{d}t'\right\|_{C([0,T];L^2)} \lesssim T^{\frac{1}{2}-c_1}\|f\|_{X^S_{0,-c_1}}.$$

(b) *If* $0 \leq c_1 < \frac{1}{2}$ *and* $0 \leq b_1, b_1 + c_1 \leq 1$, *then*

$$\left\|\psi_T(t)\int_0^t U(t-t')f(x,t')\,\mathrm{d}t'\right\|_{X^S_{0,b_1}} \lesssim T^{1-b_1-c_1}\|f\|_{X^S_{0,-c_1}}.$$

Proof. For (a), we sketch the proof essentially given in [5, Lemma 2.3]. First, the estimate

$$\left\|\psi_T(t)\int_0^t f(t')\,\mathrm{d}t'\right\|_{L^\infty_t} \lesssim T^{\frac{1}{2}-c_1}\|f\|_{H^{-c_1}} \tag{7}$$

is established for a function $f(t)$ of t-variable alone (by breaking $f(t) = f_+(t) + f_-(t)$, where $\widehat{f}_-(\tau) = \chi_{|\tau|<1/T}\widehat{f}(\tau)$ and $\widehat{f}_+(\tau) = \chi_{|\tau|>1/T}\widehat{f}(\tau)$ as in the paper). Then this is used to establish the estimate

$$\left\|\psi_T(t)\int_0^t U(t-t')f(x,t')\,\mathrm{d}t'\right\|_{L^\infty_t L^2_x} \lesssim T^{\frac{1}{2}-c_1}\|f\|_{X^S_{0,-c_1}},$$

as done in [5], but now with auxiliary function $f_\xi(t) = e^{it(\xi^2+\xi^4)}\widehat{f}(\xi,t)$ instead. The remaining steps (including the statement of continuity) are identical.

The proof of (b) is given under a general framework in [8, Lemma 2.1(ii)]. □

Proposition 4.2. *Let* $0 < T \leq 1$.

(a) *If* $0 \le c < \frac{1}{2}$, *then*

$$\left\| \int_0^t W_1(t-t')\, \partial_x^2 f(x,t')\, \mathrm{d}t' \right\|_{C([0,T];\mathcal{W})} \lesssim T^{\frac{1}{2}-c} \|f\|_{X^W_{0,-c}}.$$

(b) *If* $0 \le c < \frac{1}{2}$ *and* $0 \le b, b+c \le 1$, *then*

$$\left\| \psi_T(t) \int_0^t W_1(t-t')\, \partial_x^2 f(x,t')\, \mathrm{d}t' \right\|_{X^W_{0,b}} \lesssim T^{1-b-c} \|f\|_{X^W_{0,-c}}.$$

Proof. For (a), there are two components of the norm, so we begin by showing

$$\left\| \psi_T(t) \int_0^t W_1(t-t')\, \partial_x^2 f(x,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_x} \lesssim T^{\frac{1}{2}-c} \|f\|_{X^W_{0,-c}}. \tag{8}$$

Letting $\phi(\xi) := \sqrt{\xi^2 + \xi^4}$ as before, we have by definition,

$$\begin{aligned} &\left\| \psi_T(t) \int_0^t W_1(t-t')\, \partial_x^2 f(x,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_x} \\ &\lesssim \left\| \psi_T(t) \int_0^t \sin\left(t'\phi(\xi)\right) \widehat{f}(\xi,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_\xi} \end{aligned}$$

and so writing $\sin(t'\phi(\xi))$ as sums of complex exponentials,

$$\begin{aligned} &\text{LHS of (8)} \\ &\lesssim \left\| \psi_T(t) \int_0^t e^{it'\phi(\xi)} \widehat{f}(\xi,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_\xi} + \left\| \psi_T(t) \int_0^t e^{-it'\phi(\xi)} \widehat{f}(\xi,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_\xi} \\ &\lesssim T^{\frac{1}{2}-c} \left(\left\| \langle \tau + \phi(\xi) \rangle^{-c} \widehat{f}(\xi,\tau) \right\|_{L^2_\xi L^2_\tau} + \left\| \langle \tau - \phi(\xi) \rangle^{-c} \widehat{f}(\xi,\tau) \right\|_{L^2_\xi L^2_\tau} \right) \\ &\lesssim T^{\frac{1}{2}-c} \|f\|_{X^W_{0,-c}}, \end{aligned}$$

where in the penultimate step we used Minkowski's integral inequality and inequality (7), and in the last step we used elementary inequality $||\tau| - \phi(\xi)| \le \min(|\tau - \phi(\xi)|, |\tau + \phi(\xi)|)$.

Next, we will show

$$\left\| \psi_T(t) (-\Delta)^{-\frac{1}{2}} \partial_t \int_0^t W_1(t-t')\, \partial_x^2 f(x,t')\, \mathrm{d}t' \right\|_{L^\infty_t H^{-1}_x} \lesssim T^{\frac{1}{2}-c} \|f\|_{X^W_{0,-c}}. \tag{9}$$

To this end, note that by definition,

$$\begin{aligned} \text{LHS of (9)} &= \left\| |\xi|^{-1} \langle \xi \rangle^{-1} \psi_T(t) \int_0^t \cos\left((t-t')\phi(\xi)\right) |\xi|^2 \widehat{f}(\xi,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_\xi} \\ &\lesssim \left\| \psi_T(t) \int_0^t e^{it'\phi(\xi)} \widehat{f}(\xi,t')\, \mathrm{d}t' \right\|_{L^\infty_t L^2_\xi} \end{aligned}$$

$$+\left\|\psi_T(t)\int_0^t e^{-it'\phi(\xi)}\widehat{f}(\xi,t')\,\mathrm{d}t'\right\|_{L_t^\infty L_\xi^2}$$
$$\lesssim T^{\frac{1}{2}-c}\|f\|_{X^W_{0,-c}},$$

as in the proof of (8). It now remains to show continuity. We will only prove this for (8) since the other will be similar, i.e., we show that for a fixed $f \in X^W_{0,-c}$ and each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, f) > 0$ such that if $|t_2 - t_1| < \delta$, then

$$\left\|\int_0^{t_2} W_1(t_2-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t' - \int_0^{t_1} W_1(t_1-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\right\|_{L_x^2} < \varepsilon. \tag{10}$$

By density, it suffices to show this for $f \in \mathcal{S}(\mathbb{R}^2) \subset X^W_{0,-c}$. Note that we have

$$\partial_t \int_0^t W_1(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t' = \int_0^t W_0(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'.$$

Also, with identical proof as above, we have the estimate

$$\left\|\psi_T(t)\int_0^t W_0(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\right\|_{L_t^\infty L_x^2} \lesssim T^{\frac{1}{2}-c}\|\partial_x^2 f\|_{X^W_{0,-c}}. \tag{11}$$

By Fundamental Theorem of Calculus and the above estimate, we have that

$$\begin{aligned}\text{LHS of (10)} &= \left\|\int_{t_1}^{t_2}\int_0^t W_0(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\,\mathrm{d}t\right\|_{L_x^2}\\ &\lesssim (t_2-t_1)\left\|\psi_T(t)\int_0^t W_0(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\right\|_{L_t^\infty L_x^2}\\ &\lesssim (t_2-t_1)\|\partial_x^2 f\|_{X^W_{0,-c}},\end{aligned}$$

which proves the continuity, and so completes the proof of (a).

For (b), note that by direct calculation, we have

$$\left(\psi_T(t)\int_0^t W_1(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\right)\widehat{\ }(\xi,t) = e^{it\phi(\xi)}\widehat{h_1}(\xi,t) - e^{-it\phi(\xi)}\widehat{h_2}(\xi,t),$$

where

$$\begin{aligned}\widehat{h_1}(\xi,t) &:= \psi_T(t)\int_0^t \frac{e^{-it'\phi(\xi)}|\xi|^2\widehat{f}(\xi,t')}{2i\phi(\xi)} \quad\text{and}\\ \widehat{h_2}(\xi,t) &:= \psi_T(t)\int_0^t \frac{e^{it'\phi(\xi)}|\xi|^2\widehat{f}(\xi,t')}{2i\phi(\xi)},\end{aligned}$$

and thus

$$\left(\psi_T(t)\int_0^t W_1(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\right)\widehat{\ }(\xi,\tau)=\widehat{h_1}\left(\xi,\tau-\phi(\xi)\right) - \widehat{h_2}\left(\xi,\tau+\phi(\xi)\right).$$

By definition of the space $X^W_{0,b}$,

$$\left\|\psi_T(t)\int_0^t W_1(t-t')\,\partial_x^2 f(x,t')\,\mathrm{d}t'\right\|^2_{X^W_{0,b}} \leq \left\|\langle|\tau+\phi(\xi)|-\phi(\xi)\rangle^b\,\widehat{h_1}\right\|_{L^2_\xi L^2_\tau} + \left\|\langle|\tau-\phi(\xi)|-\phi(\xi)\rangle^b\,\widehat{h_2}\right\|_{L^2_\xi L^2_\tau}. \tag{12}$$

Now, note that we have

$$\max\left(||\tau+\phi(\xi)|-\phi(\xi)|\,,||\tau+\phi(\xi)|-\phi(\xi)|\right)\leq|\tau|,$$

and the following estimate (for $0\leq c<\frac{1}{2}$ and $0\leq b,b+c\leq 1$) shown in [8, Lemma 2.1]

$$\left\|\psi_T(t)\int_0^t g(t')\,\mathrm{d}t'\right\|_{H^b_t}\leq T^{1-b-c}\|g\|_{H^{-c}_t}. \tag{13}$$

Therefore, applying these, we finally get

$$\begin{aligned}
&\text{RHS of (12)}\\
&\lesssim \|\widehat{h_1}(\xi,t)\|^2_{L^2_\xi H^b_t}+\|\widehat{h_2}(\xi,t)\|^2_{L^2_\xi H^b_t}\\
&\lesssim T^{1-b-c}\|\widehat{h_1}(\xi,t)\|^2_{L^2_\xi H^{-c}_t}+\|\widehat{h_2}(\xi,t)\|^2_{L^2_\xi H^{-c}_t}\\
&= T^{1-b-c}\left(\left\|\langle\tau-\phi(\xi)\rangle^{-c}\,\widehat{f}(\xi,\tau)\right\|_{L^2_\xi L^2_\tau}+\left\|\langle\tau+\phi(\xi)\rangle^{-c}\,\widehat{f}(\xi,\tau)\right\|_{L^2_\xi L^2_\tau}\right)\\
&\leq T^{1-b-c}\|f\|_{X^W_{0,-c}},
\end{aligned}$$

where we used the fact that $||\tau|-\phi(\xi)|\leq\min(|\tau-\phi(\xi)|,|\tau+\phi(\xi)|)$ in the last step. □

5. Bilinear estimates and global well-posedness

Before proving the main result, we need the following lemma and bilinear estimates.

Lemma 5.1. *Let $X^+_{0,\alpha}$ be a Bourgain space equipped with the norm*

$$\|f\|_{X^+_{0,\alpha}}:=\left\|\langle\tau+\xi^2\rangle^\alpha\widehat{f}(\xi,\tau)\right\|_{L^2_\xi L^2_\tau},$$

and $X^-_{0,\alpha}$ be another Bourgain space with the norm

$$\|f\|_{X^-_{0,\alpha}}:=\left\|\langle\tau-\xi^2\rangle^\alpha\widehat{f}(\xi,\tau)\right\|_{L^2_\xi L^2_\tau}.$$

Then

$$\|u\|_{L^3_t L^3_x}\lesssim\min\left(\|u\|_{X^+_{0,\frac{1}{4}+}},\|u\|_{X^-_{0,\frac{1}{4}+}},\|u\|_{X^S_{0,\frac{1}{4}+}}\right).$$

Proof. By [8, Lemma 2.3] (taking into account Strichartz estimates Proposition 3.1(c)), and also [8, Lemma 2.4], we have

$$\|u\|_{L_t^6 L_x^6} \lesssim \min\left(\|u\|_{X^+_{0,\frac{1}{2}+}}, \|u\|_{X^-_{0,\frac{1}{2}+}}, \|u\|_{X^S_{0,\frac{1}{2}+}} \right). \tag{14}$$

By definition,

$$\|u\|_{L_t^2 L_x^2} = \|u\|_{X^+_{0,0}} = \|u\|_{X^-_{0,0}} = \|u\|_{X^S_{0,0}}. \tag{15}$$

Therefore, interpolating between (14) and (15) (where here we use interpolation results between Bourgain spaces - see [16]), we prove the statement. $\square$

Lemma 5.2. *Let* $\frac{1}{4} < b, b_1, c, c_1 < \frac{1}{2}$. *Then*

(a) $\|nu\|_{X^S_{0,-c_1}} \lesssim \|n\|_{X^W_{0,b}} \|u\|_{X^S_{0,b_1}}$.

(b) $\|u_1 \overline{u_2}\|_{X^W_{0,-c}} \lesssim \|u_1\|_{X^S_{0,b_1}} \|u_2\|_{X^S_{0,b_1}}$.

Proof. For (a), letting $\phi(\xi) := \sqrt{\xi^2+\xi^4}$ as usual, define

$$\widehat{f}_1(\xi,\tau) := \langle |\tau| - \phi(\xi) \rangle^b \, \widehat{v}(\xi,\tau) \quad \text{and} \quad \widehat{f}_2(\xi,\tau) := \langle \tau + \xi^2 + \xi^4 \rangle^{b_1} \, \widehat{u}(\xi,\tau).$$

By duality, the estimate is equivalent to

$$|S(\widehat{f}_1, \widehat{f}_2, \widehat{f})| \lesssim \|f_1\|_{L^2_\xi L^2_\tau} \|f_2\|_{L^2_\xi L^2_\tau} \|f\|_{L^2_\xi L^2_\tau},$$

where

$$S(\widehat{f}_1, \widehat{f}_2, \widehat{f}) := \int_* \frac{\widehat{f}_1(\xi_1,\tau_1)\widehat{f}_2(\xi_2,\tau_2)\widehat{f}(\xi,\tau)}{\langle\sigma\rangle^{c_1}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^{b_1}} \, \mathrm{d}\xi_1 \, \mathrm{d}\tau_1 \, \mathrm{d}\xi \, \mathrm{d}\tau,$$

with $\sigma = \tau + \xi^2 + \xi^4$, $\sigma_1 = |\tau_1| - \phi(\xi_1)$, and $\sigma_2 = \tau_2 + \xi_2^2 + \xi_2^4$, and $*$ indicates the restriction $\xi_2 = \xi_1 - \xi$ and $\tau_2 = \tau_1 - \tau$. Without loss of generality, we can assume $\widehat{f}_1, \widehat{f}_2, \widehat{f}$ are real-valued and nonnegative. Now, by the following simple inequality

$$\frac{1}{c} \le \sup_{x,y\ge 0} \frac{1+|x-y|}{1+\left|x - \sqrt{y^2+y}\right|} \le c \quad \text{for some } c > 0,$$

we have

$$\begin{aligned} S(\widehat{f}_1, \widehat{f}_2, \widehat{f}) \lesssim & \int_* \frac{\widehat{f}_1(\xi_1,\tau_1)\widehat{f}_2(\xi_2,\tau_2)\widehat{f}(\xi,\tau)}{\langle\sigma\rangle^{c_1}\langle\tau_1+\xi_1^2\rangle^b\langle\sigma_2\rangle^{b_1}} \\ & + \int_* \frac{\widehat{f}_1(\xi_1,\tau_1)\widehat{f}_2(\xi_2,\tau_2)\widehat{f}(\xi,\tau)}{\langle\sigma\rangle^{c_1}\langle\tau_1-\xi_1^2\rangle^b\langle\sigma_2\rangle^{b_1}} \, \mathrm{d}\xi_1 \, \mathrm{d}\tau_1 \, \mathrm{d}\xi \, \mathrm{d}\tau \\ & =: S_1(\widehat{f}_1, \widehat{f}_2, \widehat{f}) + S_2(\widehat{f}_1, \widehat{f}_2, \widehat{f}). \end{aligned}$$

For S_1, we have (using Plancherel's theorem and then Hölder's inequality),

$$S_1(\widehat{f}_1, \widehat{f}_2, \widehat{f}) = \int_{\xi_1,\tau_1} \left(\frac{\widehat{f}_1(\xi_1,\tau_1)}{\langle\tau_1+\xi_1^2\rangle^b} \right) \left(\int_* \frac{\widehat{f}(\xi,\tau)\widehat{f}_2(\xi_2,\tau_2)}{\langle\sigma\rangle^{c_1}\langle\sigma_2\rangle^{b_1}} \, \mathrm{d}\xi \, \mathrm{d}\tau \right) \mathrm{d}\xi_1 \, \mathrm{d}\tau_1$$

$$
\begin{aligned}
&= \int_{x_1,t_1} \left(\frac{\widehat{f_1}(\xi_1,\tau_1)}{\langle \tau_1+\xi_1^2\rangle^b} \right)^\vee \left(\int_* \frac{\widehat{f}(\xi,\tau)\widehat{f_2}(\xi_2,\tau_2)}{\langle\sigma\rangle^{c_1}\langle\sigma_2\rangle^{b_1}}\,\mathrm{d}\xi\,\mathrm{d}\tau \right)^\vee \mathrm{d}x_1\,\mathrm{d}t_1 \\
&= \int_{x,t} \left(\frac{\widehat{f_1}(\xi_1,\tau_1)}{\langle \tau_1+\xi_1^2\rangle^b} \right)^\vee \left(\frac{\widehat{f}(\xi,\tau)}{\langle\sigma\rangle^{c_1}} \right)^\vee \left(\frac{\widehat{f_2}(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b_1}} \right)^\vee \mathrm{d}x\,\mathrm{d}t \\
&\le \left\| \left(\frac{\widehat{f_1}}{\langle \tau_1+\xi_1^2\rangle^b} \right)^\vee \right\|_{L_t^3L_x^3} \left\| \left(\frac{\widehat{f_2}}{\langle\sigma_2\rangle^{b_1}} \right)^\vee \right\|_{L_t^3L_x^3} \left\| \left(\frac{\widehat{f}}{\langle\sigma\rangle^{c_1}} \right)^\vee \right\|_{L_t^3L_x^3}.
\end{aligned}
$$

Now, applying Lemma 5.1, and noting that $b, b_1, c_1 > \frac{1}{4}$, we have the result. Estimate on S_2 is proven similarly, and so the proof is completed.

For (b), define

$$\widehat{f_1}(\xi,\tau) := \langle \tau+\xi^2\rangle^{b_1}\widehat{u_1}(\xi,\tau) \quad\text{and}\quad \widehat{f_2}(\xi,\tau) := \langle \tau+\xi^2\rangle^{b_1}\widehat{u_2}(\xi,\tau).$$

By duality, it suffices to show

$$|T(\widehat{f_1},\widehat{f_2},\widehat{f})| \lesssim \|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f\|_{L_\xi^2L_\tau^2},$$

where

$$T(\widehat{f_1},\widehat{f_2},\widehat{f}) := \int_* \frac{\widehat{f_1}(\xi_1,\tau_1)\widehat{f_2}(\xi_2,\tau_2)\widehat{f}(\xi,\tau)}{\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_1}\langle\sigma\rangle^{c}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

with $\sigma = |\tau| - \phi(\xi)$, $\sigma_1 = \tau_1+\xi_1^2$, $\sigma_2 = \tau_2+\xi_2^2$, and $*$ indicates the restriction $\xi_2 = \xi_1 - \xi$ and $\tau_2 = \tau_1 - \tau$.

This estimate is identical to that in (a) proven above, and so we are done. $\square$

Finally, we can state the following global well-posedness result.

Theorem 5.3. *The modified fourth-order Zakharov equations* (1) *is globally well-posed for* $(u_0,n_0,n_1) \in L^2(\mathbb{R})\times L^2(\mathbb{R})\times H^{-1}(\mathbb{R})$. *Moreover, the solution* (u,n) *satisfies*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \tag{16}$$

and

$$
\begin{aligned}
&\|n(t)\|_{L^2} + \left\|(-\Delta)^{-\frac{1}{2}}\partial_t n(t)\right\|_{H^{-1}} \\
&\quad\le \exp\left(c|t|\|u_0\|_{L^2}^2\right)\max\left(\|u_0\|_{L^2}^2, \|(n_0,n_1)\|_{\mathcal{W}}^2\right).
\end{aligned} \tag{17}
$$

Proof. Let $(u_0,n_0,n_1) \in L^2(\mathbb{R})\times L^2(\mathbb{R})\times H^{-1}(\mathbb{R})$ and fix $0 < T \le 1$. Consider the Duhamel maps Λ_S and Λ_W,

$$\Lambda_S(u,n)(t) := \psi_T(t)U(t)u_0 - i\psi_T(t)\int_0^t U(t-t')\,(nu)(x,t')\,\mathrm{d}t', \tag{18}$$

$$
\begin{aligned}
\Lambda_W(u)(t) :=\ &\psi_T(t)\left(W_0(t)n_0 + W_1(t)\,\partial_x n_1\right) \\
&- \psi_T(t)\int_0^t W_1(t-t')\,\partial_x^2(|u|^2)(x,t')\,\mathrm{d}t'.
\end{aligned} \tag{19}
$$

We seek a fixed point $(u(t), n(t)) = (\Lambda_S(u,n), \Lambda_W(u))$. Estimating (18) in X^S_{0,b_1} by applying Proposition 3.1 and Proposition 4.1 followed by Lemma 5.2(a); and then estimating (19) in $X^W_{0,b}$ by applying Proposition 3.2 and Proposition 4.2 followed by Lemma 5.2(b), we obtain

$$\|\Lambda_S(u,n)\|_{X^S_{0,b_1}} \leq kT^{\frac{1}{2}-b_1}\|u_0\|_{L^2} + kT^{1-b_1-c_1}\|n\|_{X^W_{0,b}}\|u\|_{X^S_{0,b_1}},$$
$$\|\Lambda_W(u)\|_{X^W_{0,b}} \leq kT^{\frac{1}{2}-b}\|(n_0,n_1)\|_{\mathcal{W}} + kT^{1-b-c}\|u\|^2_{X^S_{0,b_1}},$$

and also

$$\begin{aligned}&\|\Lambda_S(u_1,n_1) - \Lambda_S(u_2,n_2)\|_{X^S_{0,b_1}}\\ &\leq kT^{1-b_1-c_1}\left(\|n\|_{X^W_{0,b}}\|u_1-u_2\|_{X^S_{0,b_1}} + \|n_1-n_2\|_{X^W_{0,b}}\|u_2\|_{X^S_{0,b_1}}\right),\end{aligned}$$

and

$$\|\Lambda_W(u_1) - \Lambda_W(u_2)\|_{X^W_{0,b}} \leq kT^{1-b-c}\left(\|u_1\|_{X^S_{0,b_1}} + \|u_2\|_{X^S_{0,b_1}}\right)\|u_1-u_2\|_{X^S_{0,b_1}}.$$

Now, define the (complete metric) space $B\left(X^S\right) \times B\left(X^W\right)$ to be

$$\left\{(u,n) : \|u\|_{X^S_{0,b_1}} \leq 2kT^{\frac{1}{2}-b_1}\|u_0\|_{L^2},\ \|n\|_{X^W_{0,b}} \leq 2kT^{\frac{1}{2}-b}\|(n_0,n_1)\|_{\mathcal{W}}\right\}.$$

By taking T such that

$$T^{\frac{3}{2}-2b_1-c_1}\|u_0\|_{L^2} \lesssim 1,$$
$$T^{\frac{3}{2}-b-b_1-c}\|u_0\|_{L^2} \lesssim 1,$$
$$T^{\frac{3}{2}-b-b_1-c_1}\|(n_0,n_1)\|_{\mathcal{W}} \lesssim 1, \tag{20}$$
$$T^{\frac{3}{2}-2b_1-c}\|u_0\|^2_{L^2} \lesssim \|(n_0,n_1)\|_{\mathcal{W}}, \tag{21}$$

we have sufficient conditions for (Λ_S, Λ_W) to be a contraction on $B\left(X^S\right) \times B\left(X^W\right)$, yielding the existence of a fixed point $u \in X^S_{0,b_1}$ and $n \in X^W_{0,b}$ of (18)-(19) such that

$$\|u\|_{X^S_{0,b_1}} \lesssim T^{\frac{1}{2}-b_1}\|u_0\|_{L^2} \text{ and } \|n\|_{X^W_{0,b}} \lesssim T^{\frac{1}{2}-b}\|(n_0,n_1)\|_{\mathcal{W}}. \tag{22}$$

Estimating (18) in $C([0,T]; L^2_x)$ by similarly applying Proposition 3.1 and Proposition 4.1, as well as Proposition 3.2 and Proposition 4.2, we have in fact $u, n \in C([0,T]; L^2_x)$. We may thus invoke the L^2 conservation law (2) to conclude $\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}$, and so we are only concerned with the possibility of growth in $\|n(t)\|_{\mathcal{W}}$ from one time step to the next. Suppose that after some number of iterations, we reach a time when $\|n(t)\|_{\mathcal{W}} \gg \|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}$. Take this time position as the initial time $t = 0$ so that $\|u_0\|^2_{L^2} \ll \|(n_0,n_1)\|_{\mathcal{W}}$. Then (21) is automatically satisfied and by (20), we may select a time increment of size

$$T \sim \|(n_0,n_1)\|_{\mathcal{W}}^{-1/\left(\frac{3}{2}-b-b_1-c_1\right)}. \tag{23}$$

Now, applying Proposition 3.2 and Proposition 4.2, followed by (22) to (19), we get

$$\|n(T)\|_{\mathcal{W}} \leq \|(n_0, n_1)\| + kT^{\frac{3}{2}-2b_1-c}\|u_0\|_{L^2}^2.$$

From this, we see that we can perform m iterations on time intervals, each of length (23), where

$$m \sim \frac{\|(n_0, n_1)\|_{\mathcal{W}}}{T^{\frac{3}{2}-2b_1-c}\|u_0\|_{L^2}^2} \tag{24}$$

before the quantity $\|n(t)\|_{\mathcal{W}}$ doubles. By (23) and (24), the total time we advance after these m iterations is

$$mT \sim \frac{\|(n_0, n_1)\|_{\mathcal{W}}}{\|(n_0, n_1)\|_{\mathcal{W}}^{\beta}\|u_0\|_{L^2}^2}, \quad \text{where } \beta := -\frac{\frac{1}{2} - 2b_1 - c}{\frac{3}{2} - b - b_1 - c_1}.$$

Now, taking $b = b_1 = c = c_1 = \frac{1}{3}$ such that $\beta = 1$, we have that mT depends only on $\|u_0\|_{L^2}$ (and is independent of $\|n(t)\|_{\mathcal{W}}$). We can now repeat this entire procedure, each time advancing a time of length $\|u_0\|_{L^2}^{-2}$. Upon each iteration, the size of $\|n(t)\|_{\mathcal{W}}$ will be at most doubled, giving the exponential-in-time upper bound stated in (17). This completes the proof. □

6. Conclusion

Global well-posedness for the 1-D fourth-order Zakharov equations (modified to include quantum correction) was proven for initial data $(u, n) \in L^2 \times L^2$. This extends the well-posedness result in [9] to lower regularity. Future research direction may include extending the arguments here to higher dimensions. Another interesting question to explore would be whether the global solution corresponding to the parameter h in (1) would converge to the global solution of the classical 1-D Zakharov system as $h \to 0$. This would be the subject of future research.

References

[1] I. Bejenaru and S. Herr, *Convolutions of singular measures and applications to the Zakharov system*, J. Funct. Anal. **261** (2011), no. 2, 478–506. `https://doi.org/10.1016/j.jfa.2011.03.015`

[2] I. Bejenaru, S. Herr, J. Holmer, and D. Tataru, *On the 2D Zakharov system with L^2-Schrödinger data*, Nonlinearity **22** (2009), no. 5, 1063–1089. `https://doi.org/10.1088/0951-7715/22/5/007`

[3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156. `https://doi.org/10.1007/BF01896020`

[4] I. Chueshov, *Quantum Zakharov model in a bounded domain*, Z. Angew. Math. Phys. **64** (2013), no. 4, 967–989. `https://doi.org/10.1007/s00033-012-0278-9`

[5] J. Colliander, J. Holmer, and N. Tzirakis, *Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems*, Trans. Amer. Math. Soc. **360** (2008), no. 9, 4619–4638. `https://doi.org/10.1090/S0002-9947-08-04295-5`

[6] S. Fang, L. Jin, and B. Guo, *Existence of weak solution for quantum Zakharov equations for plasmas model*, Appl. Math. Mech. (English Ed.) **32** (2011), no. 10, 1339–1344. https://doi.org/10.1007/s10483-011-1504-7

[7] L. G. Garcia, F. Haas, L. P. L. de Oliveira, and J. Goedert, *Modified Zakharov equations for plasmas with a quantum correction*, Phys. Plasmas **12**, 012302, 2005.

[8] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), no. 2, 384–436. https://doi.org/10.1006/jfan.1997.3148

[9] Y. Guo, J. Zhang, and B. Guo, *Global well-posedness and the classical limit of the solution for the quantum Zakharov system*, Z. Angew. Math. Phys. **64** (2013), no. 1, 53–68. https://doi.org/10.1007/s00033-012-0215-y

[10] F. Haas, *Variational approach for the quantum Zakharov system*, Phys. Plasmas **14** (2007), 042309.

[11] F. Haas, *Quantum plasmas*, Springer Series on Atomic, Optical, and Plasma Physics, 65, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-8201-8

[12] F. Haas and P. K. Shukla, *Quantum and classical dynamics of Langmuir wave packets*, Phys. Rev. E **79** (2009), 066402.

[13] C. E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), no. 1, 33–69. https://doi.org/10.1512/iumj.1991.40.40003

[14] N. Masmoudi and K. Nakanishi, *Energy convergence for singular limits of Zakharov type systems*, Invent. Math. **172** (2008), no. 3, 535–583. https://doi.org/10.1007/s00222-008-0110-5

[15] H. Pecher, *An improved local well-posedness result for the one-dimensional Zakharov system*, J. Math. Anal. Appl. **342** (2008), no. 2, 1440–1454. https://doi.org/10.1016/j.jmaa.2008.01.035

[16] T. Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. https://doi.org/10.1090/cbms/106

[17] S. You and B. Guo, *Global solutions for quantum Zakharov equations*, Appl. Math. Mech. (English Ed.) **38** (2017), no. 4, 603–616. https://doi.org/10.1007/s10483-017-2181-6

[18] S. You, B. Guo, and X. Ning, *Initial boundary value problem for modified Zakharov equations*, Acta Math. Sci. Ser. B (Engl. Ed.) **32** (2012), no. 4, 1455–1466. https://doi.org/10.1016/S0252-9602(12)60115-5

Agus L. Soenjaya
Pre-University Mathematics Department
Merlion School
Surabaya 60189, Indonesia
Email address: agus.leonards16@gmail.com