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A NOTE ON MAXIMAL HYPERSURFACES IN A GENERALIZED ROBERTSON-WALKER SPACETIME

Henrique Fernandes de Lima

Dedicated to the memory of my grandparents, Maria Nila and Francisco Antônio

Abstract. In this note, we apply a maximum principle related to volume growth of a complete noncompact Riemannian manifold, which was recently obtained by Alías, Caminha and do Nascimento in [4], to establish new uniqueness and nonexistence results concerning maximal spacelike hypersurfaces immersed in a generalized Robertson-Walker (GRW) spacetime obeying the timelike convergence condition. A study of entire solutions for the maximal hypersurface equation in GRW spacetimes is also made and, in particular, a new Calabi-Bernstein type result is presented.

1. Introduction

Let $(M^n, \langle\,,\rangle_{M^n})$ be a connected, n-dimensional ($n \geq 2$) oriented Riemannian manifold, I a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$), and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the factors I and M^n, respectively. A particular class of Lorentzian manifolds is the one obtained by furnishing $\overline{M}^{n+1}$ with the metric

$$(1) \qquad \langle v, w\rangle_p = -\langle(\pi_I)_* v, (\pi_I)_* w\rangle_I + (f \circ \pi_I)(p)^2 \langle(\pi_M)_* v, (\pi_M)_* w\rangle_{M^n}$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$. Following the terminology introduced by Alías, Romero and Sánchez in [7], such a space is called a *generalized Robertson-Walker* (GRW) spacetime, f is known as the warping function and we shall write $\overline{M}^{n+1} = -I \times_f M^n$ to denote it. In particular, when the Riemannian fiber M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a

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Robertson-Walker (RW) spacetime, and it is a spatially homogeneous spacetime (for more details, see [16]).

As it was observed in [6], spatial homogeneity, which is reasonable as a first approximation of the large scale structure of the universe, may not be realistic when one considers a more accurate scale. For this reason, GRW spacetimes could be suitable spacetimes to model universes with inhomogeneous spacelike geometry. Besides, small deformations of the metric on the fiber of RW spacetimes fit into the class of GRW spacetimes (see, for instance, [12, 18]).

In this paper, we are interested in the study of *maximal hypersurfaces* (that is, spacelike hypersurfaces with vanishing mean curvature) immersed in a GRW spacetime. Many authors have approached problems in this subject. We may cite, for instance, the works [2, 9–11, 17, 19–21], where the authors have obtained several uniqueness and nonexistence results for maximal hypersurfaces under the assumption that the ambient spacetime obeys either the *timelike convergence condition* or the *null convergence condition*. Let us recall that a spacetime obeys the timelike (null) convergence condition if its Ricci curvature is nonnegative on timelike (null or lightlike) directions.

Here, we deal with complete noncompact maximal hypersurfaces immersed in a GRW spacetime obeying the timelike convergence condition. Considering this setting, in Section 3 we apply a maximum principle related to volume growth of a complete noncompact Riemannian manifold, which was recently obtained by Alías, Caminha and do Nascimento in [4], to establish new uniqueness and nonexistence results concerning these spacelike hypersurfaces. Afterwards, in Section 4 we study entire solutions for the maximal hypersurface equation in such GRW spacetimes and, in particular, we obtain a new Calabi-Bernstein type result. Some preliminary facts related to spacelike hypersurfaces in a GRW spacetime, as well as the notion of the timelike convergence condition, are recalled in Section 2.

2. Preliminaries

In this section, we recall some basic facts concerning spacelike hypersurfaces immersed in a GRW spacetime, as well as we refresh the notion of the timelike convergence condition.

2.1. Spacelike hypersurfaces in a GRW spacetime

A smooth immersion $\psi : \Sigma^n \to -I \times_f M^n$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted by $\langle\, , \rangle$. In this case, since

$$\partial_t = (\partial/\partial_t)_{(t,x)}\,, \quad (t, x) \in -I \times_f M^n$$

is a unitary timelike vector field globally defined on the ambient spacetime, there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n which is in the same time-orientation of ∂_t. From the inverse Cauchy-Schwarz inequality, we get

$$\langle N, \partial_t \rangle \leq -1 < 0 \quad \text{on } \Sigma^n. \tag{2}$$

In what follows, we will refer to that normal vector field N as the *future-pointing Gauss map* of the spacelike hypersurface Σ^n.

For each $t_0 \in I$, we orient the (spacelike) *slice* $M^n_{t_0} = \{t_0\} \times M^n$ by using its unit normal vector field ∂_t. According to [7], M_{t_0} has constant mean curvature $H = \frac{f'}{f}(t_0)$ with respect to ∂_t.

Now, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$, namely, the (vertical) *height function* $h = (\pi_I)|_\Sigma$ and the *support function* $\langle N, \partial_t \rangle$, where N stands for the future-pointing Gauss map of Σ^n.

Denoting by $\overline{\nabla}$ and ∇ the Levi-Civita connections in $-I \times_f M^n$ and Σ^n, respectively, a simple computation shows that

$$\overline{\nabla}\pi_I = -\langle \overline{\nabla}\pi_I, \partial_t \rangle \partial_t = -\partial_t.$$

Consequently, we obtain

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \langle N, \partial_t \rangle N. \tag{3}$$

Hence, from (3) we get the following relation

$$|\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1, \tag{4}$$

where $|\cdot|$ stands for the norm of a vector field on Σ^n.

We define the *hyperbolic angle* θ of Σ^n as being the smooth function $\theta : \Sigma^n \to [0, +\infty)$ given by

$$\cosh\theta = -\langle N, \partial_t \rangle \geq 1. \tag{5}$$

Therefore, from (4) and (5) we obtain

$$\sinh^2\theta = |\nabla h|^2. \tag{6}$$

2.2. The timelike convergence condition (TCC)

A GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$ obeys the *null convergence condition* (NCC) when its Ricci tensor $\overline{\text{Ric}}$ is such that $\overline{\text{Ric}}(Z, Z) \geq 0$ for all null vector field $Z \in \mathfrak{X}(\overline{M})$. From Corollary 7.43 of [16] we have that

$$\begin{aligned} \overline{\text{Ric}}(Z, W) &= \text{Ric}_M(Z^*, W^*) + (n((\log f)')^2 + (\log f)'')\langle Z, W \rangle \\ &\quad - (n-1)(\log f)''\langle Z, \partial_t \rangle \langle W, \partial_t \rangle, \end{aligned} \tag{7}$$

where Ric_M denotes the Ricci tensor of the Riemannian fiber M^n and $Z^* = Z + \langle Z, \partial_t \rangle \partial_t$ stands for the projection of the vector field Z onto M^n. Consequently, from (7) we have that the NCC holds in $\overline{M}^{n+1}$ if and only if

$$\text{Ric}_M \geq (n-1)\left(f^2(\log f)''\right)\langle\, , \rangle_{M^n}. \tag{8}$$

A more restrictive energy condition is the *timelike converge condition* (TCC), that is,

$$\overline{\mathrm{Ric}}(Z,Z) \geq 0$$

for all timelike vector field $Z \in \mathfrak{X}(\overline{M})$. We note that, by a continuity argument, it turns out that the TCC implies NCC. Moreover, it is not difficult to check that $\overline{M}^{n+1}$ satisfies the TCC if and only if (8) holds and $f'' \leq 0$ (for more details concerning the NCC and the TCC, see [15]).

3. Main results

We start this section, quoting the analytical tool that will be used to prove our results. For this, let Σ^n be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p,r)$ the geodesic ball centered at p and with radius r. Given a polynomial function $\sigma : (0,+\infty) \to (0,+\infty)$, we say that Σ^n has *polynomial volume growth like* σ if there exists $p \in \Sigma^n$ such that

$$\mathrm{vol}(B(p,r)) = \mathcal{O}(\sigma(r)),$$

as $r \to +\infty$, where vol denotes the canonical Riemannian volume of Σ^n. As it was already observed in the beginning of Section 2 in [4], if $p, q \in \Sigma^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,r))}{\sigma(r)} \geq \frac{\mathrm{vol}(B(q,r-d))}{\sigma(r-d)}.\frac{\sigma(r-d)}{\sigma(r)}.$$

Consequently, the choice of p in the notion of volume growth is immaterial, and we will just say that Σ^n has *polynomial volume growth*.

Keeping in mind the previous digression, we have the following lemma which corresponds to a particular case of a maximum principle recently obtained by Alías, Caminha and do Nascimento (see Theorem 2.1 of [4]).

Lemma 3.1. *Let Σ^n be a connected, oriented, complete noncompact Riemannian manifold, and let $\xi \in \mathcal{C}^\infty(\Sigma)$ be a nonnegative smooth function such that $\Delta\xi \geq a\xi$ on Σ^n for some positive constant $a \in \mathbb{R}$. If Σ^n has polynomial volume growth and $|\nabla\xi|$ is bounded on Σ^n, then ξ vanishes identically on Σ^n.*

In Subsections 3.1 and 3.2 we will apply Lemma 3.1 to establish improvements of the results obtained in [10], in the sense that we will replace technical hypotheses like the integrability of $|\nabla h|$, the stochastic completeness of Σ^n and the strong timelike convergence condition (STCC), which appear in Theorems 3.2, 3.7 and 3.9 of [10], by the geometric property that Σ^n has polynomial volume growth.

3.1. Uniqueness of maximal hypersurfaces

In order to establish our results, we recall that a slab

$$[t_1,t_2] \times M^n = \{(t,q) \in -I \times_f M^n : t_1 \leq t \leq t_2\}$$

is called a *timelike bounded region* of the GRW spacetime $-I \times_f M^n$. Now, we are in position to present the following uniqueness result.

Theorem 3.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC and whose Riemannian fiber M^n is complete noncompact. The only complete noncompact maximal hypersurfaces Σ^n with polynomial volume growth, lying in a timelike bounded region of $\overline{M}^{n+1}$, whose hyperbolic angle and second fundamental form are bounded and such that $f''(h) < 0$, are the totally geodesic slices of $\overline{M}^{n+1}$.*

Proof. Since Σ^n is a maximal hypersurface, from Proposition 3.1 of [14] we have

$$\begin{aligned}\frac{1}{2}\Delta \sinh^2\theta \geq\ & n\frac{f'(h)^2}{f(h)^2} + \langle A^2(\nabla h), \nabla h\rangle - 2\frac{f'(h)}{f(h)}\mathrm{Hess}(h)(\nabla h, \nabla h)\\ &+ \cosh^2\theta\,\mathrm{Ric}_M(N^*, N^*) + 2\frac{f'(h)}{f(h)}\cosh\theta\langle A(\nabla h), \nabla h\rangle\\ &+ (2n+1)\frac{f'(h)^2}{f(h)^2}\sinh^2\theta - n\frac{f''(h)}{f(h)}\sinh^2\theta\\ &+ (n+1)\frac{f'(h)^2}{f(h)^2}\sinh^4\theta - n\frac{f''(h)}{f(h)}\sinh^4\theta,\end{aligned} \tag{9}$$

where $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the second fundamental form of Σ^n with respect to its future-pointing Gauss map N.

On the other hand, since $N = N^* + \cosh\theta\partial_t$, from (1) we have that

$$\sinh^2\theta = f(h)^2\langle N^*, N^*\rangle_{M^n}. \tag{10}$$

Thus, using inequality (8) and equation (10) into (9), we obtain

$$\begin{aligned}\frac{1}{2}\Delta \sinh^2\theta \geq\ & 2\frac{f'(h)}{f(h)}\Big(\cosh\theta\langle A(\nabla h), \nabla h\rangle - \mathrm{Hess}(h)(\nabla h, \nabla h)\Big)\\ &+ (n-1)\cosh^2\theta\sinh^2\theta\left(\frac{f''(h)}{f(h)} - \frac{f'(h)^2}{f(h)^2}\right)\\ &+ (2n+1)\frac{f'(h)^2}{f(h)^2}\sinh^2\theta - n\frac{f''(h)}{f(h)}\sinh^2\theta\\ &+ (n+1)\frac{f'(h)^2}{f(h)^2}\sinh^4\theta - n\frac{f''(h)}{f(h)}\sinh^4\theta.\end{aligned} \tag{11}$$

On the other hand, since $f(t)\partial_t$ is a conformal vector field globally defined on $\overline{M}^{n+1}$, we have that

$$\overline{\nabla}_X f(t)\partial_t = f'(t)X \tag{12}$$

for all vector field $X \in \mathfrak{X}(\overline{M}^{n+1})$. Thus, taking into account Gauss formula $AX = -\overline{\nabla}_X N$ and using (12), from (5) we obtain the following equation

$$\nabla \cosh\theta = A(\nabla h) + \cosh\theta \frac{f'(h)}{f(h)} \nabla h. \tag{13}$$

Consequently, from (6) and (13) we get

$$\begin{aligned} \mathrm{Hess}(h)(\nabla h, \nabla h) &= \langle \nabla_{\nabla h} \nabla h, \nabla h \rangle = \frac{1}{2} \nabla h(\sinh^2\theta) \\ &= \frac{1}{2} \nabla h(\cosh^2\theta) = \cosh\theta \langle \nabla \cosh\theta, \nabla h \rangle \\ &= \cosh\theta \langle A(\nabla h), \nabla h \rangle + \cosh^2\theta \sinh^2\theta \frac{f'(h)}{f(h)}. \end{aligned} \tag{14}$$

Inserting (14) into (11), it is not difficult to verify that several terms will be canceled, and we deduce that

$$\frac{1}{2}\Delta \sinh^2\theta \geq n \left(\frac{f'(h)}{f(h)}\right)^2 \sinh^2\theta - \frac{f''(h)}{f(h)} \sinh^2\theta - \frac{f''(h)}{f(h)} \sinh^4\theta. \tag{15}$$

Thus, using the hypothesis that $f''(h) < 0$, from (15) we reach at

$$\frac{1}{2}\Delta \sinh^2\theta \geq n \left(\frac{f'(h)}{f(h)}\right)^2 \sinh^2\theta - \frac{f''(h)}{f(h)} \sinh^2\theta. \tag{16}$$

So, from inequality (16) we obtain

$$\Delta \sinh^2\theta \geq -2\frac{f''(h)}{f(h)} \sinh^2\theta. \tag{17}$$

Hence, since we are assuming that Σ^n lies in a bounded timelike region of $\overline{M}^{n+1}$ and using once more that $f''(h) < 0$, from (17) we conclude that there exists a positive constant $a \in \mathbb{R}$ such that

$$\Delta \sinh^2\theta \geq a \sinh^2\theta.$$

But, from (13) we also have

$$\nabla \sinh^2\theta = \nabla \cosh^2\theta = 2\cosh\theta \left(A + \frac{f'(h)}{f(h)} \cosh\theta Id\right) \nabla h. \tag{18}$$

Consequently, since we are also supposing that A and θ are bounded and using once more that Σ^n lies in a timelike bounded region of $\overline{M}^{n+1}$, from (18) we get that $|\nabla \sinh^2\theta|$ is bounded on Σ^n.

Therefore, we can apply Lemma 3.1 to obtain that $\sinh^2\theta$ is identically zero, which means that Σ^n must be a totally geodesic slice of $\overline{M}^{n+1}$. □

A spacetime $\overline{M}^{n+1}$ obeys the *ubiquitous energy condition* (UEC) when, for all timelike vector field $Z \in \mathfrak{X}(\overline{M})$, its Ricci curvature satisfies $\overline{\mathrm{Ric}}(Z, Z) > 0$. This last energy condition is stronger than the TCC and roughly means a real presence of matter at any point of the spacetime. Furthermore, it is not difficult to verify that, if $\overline{M}^{n+1} = -I \times_f M^n$ is a GRW spacetime obeying the UEC, then $f^{''} < 0$.

On the other hand, as it was showed in Example 4.3 of [15], we can model the anti-de Sitter space $\mathbb{H}_1^{n+1}$ as the following GRW spacetime

$$\mathbb{H}_1^{n+1} = -(-\pi/2, \pi/2) \times_{\cos t} \mathbb{H}^n. \tag{19}$$

Consequently, we have that the anti-the Sitter space (19), as the so-called Einstein-de Sitter cosmological model $-(0,\infty) \times_{t^{2/3}} \mathbb{R}^3$ and certain big bang cosmological models (see, for instance, Chapter 12 of [16], Chapter 5 of [8] or Chapter 5 of [12]), constitute examples of GRW spacetimes obeying the UEC. In this case, since the hypothesis $f''(h) < 0$ in Theorem 3.2 is automatically satisfied, we obtain the following consequence:

Corollary 3.3. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the UEC and whose Riemannian fiber M^n is complete noncompact. The only complete noncompact maximal hypersurfaces Σ^n with polynomial volume growth, lying in a timelike bounded region of $\overline{M}^{n+1}$, whose hyperbolic angle and second fundamental form are bounded, are the totally geodesic slices of $\overline{M}^{n+1}$.*

3.2. Nonexistence of maximal hypersurfaces

Proceeding with the context of the previous subsection, we get the following nonexistence result.

Theorem 3.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC and whose Riemannian fiber M^n is complete noncompact. There are no complete noncompact maximal hypersurfaces with polynomial volume growth, lying in a timelike bounded region of $\overline{M}^{n+1}$, whose hyperbolic angle and second fundamental form are bounded and such that $f'(h) \neq 0$.*

Proof. Let us suppose by contradiction the existence of such a maximal hypersurface Σ^n. Since our ambient spacetime obeys the TCC, from (16) we obtain

$$\frac{1}{2}\Delta \sinh^2\theta \geq n\left(\frac{f'(h)}{f(h)}\right)^2 \sinh^2\theta. \tag{20}$$

Thus, since Σ^n is contained in a timelike bounded region of $\overline{M}^{n+1}$ and assuming that $f'(h) \neq 0$, from (20) we obtain that

$$\Delta \sinh^2\theta \geq a \sinh^2\theta$$

for some positive constant $a \in \mathbb{R}$. At this point, we can reason as in the last part of the proof of Theorem 3.2 to conclude that Σ^n must be a totally geodesic slice of $\overline{M}^{n+1}$, which corresponds to a contradiction with the hypothesis that $f'(h) \neq 0$. □

Ishihara proved that an n-dimensional complete maximal hypersurface immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}$ must have the squared norm of the second fundamental form bounded from above by n, and that this bound is reached only by the maximal hyperbolic cylinders $\mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$, with $1 \leq m \leq n-1$ (see Theorems 1.2 and 1.3 of [13]).

Taking into account once more the GRW model (19) of the anti-de Sitter space $\mathbb{H}_1^{n+1}$ and adopting the terminology established by Aledo, Alías and Romero in [3], the open regions given by $-(0, \pi/2) \times_{\cos t} \mathbb{H}^n$ and $-(-\pi/2, 0) \times_{\cos t} \mathbb{H}^n$ are called, respectively, the *chronological future* and the *chronological past* of $\mathbb{H}_1^{n+1}$. Considering this context and using Ishihara's result, we obtain the following consequence of Theorem 3.4.

Corollary 3.5. *There are no complete noncompact maximal hypersurfaces with polynomial volume growth, lying in a timelike bounded region of the chronological future* (*past*) *of* $\mathbb{H}_1^{n+1}$ *and whose hyperbolic angle is bounded.*

4. Entire solutions for the maximal hypersurface equation

In this last section, we will apply our previous uniqueness and nonexistence results on maximal hypersurfaces in order to study entire solutions of a suitable maximal hypersurface equation in GRW spacetimes obeying the TCC.

Let $\Omega \subseteq M^n$ be a connected domain of the complete noncompact Riemannian fiber $(M^n, \langle\, , \rangle_M)$. For every $u \in \mathcal{C}^\infty(\Omega)$ such that $|Du|_M < f(u)$, where $|Du|_M$ stands for the norm of the gradient Du of u on the metric $\langle\, , \rangle_M$, we will consider the (vertical) graph over Ω determined by a smooth function $u \in \mathcal{C}^\infty(\Omega)$, which is given by

$$\Sigma(u) = \{(u(x), x);\, x \in \Omega\} \subset -I \times_f M^n. \tag{21}$$

The metric induced on Ω from the Lorentzian metric (1) via $\Sigma(u)$ is

$$\langle\, , \rangle = -du^2 + f^2(u)\langle, \rangle_{M^n}. \tag{22}$$

The graph is said to be entire if $\Omega = M^n$. From (22), we conclude that a graph $\Sigma(u)$ is a spacelike hypersurface if and only if $|Du|_M < f(u)$.

When M^n is a simply connected manifold, from Lemma 3.1 of [7] we have that every complete spacelike hypersurface Σ^n in $-I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph in this GRW spacetime. In particular, this happens for complete spacelike hypersurfaces bounded away from the infinity of $-I \times_f M^n$. However, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph in a Lorentzian spacetime is not necessarily complete, in the sense that the induced Riemannian metric (22) is not necessarily complete on M^n. For instance, Albujer [1] has obtained explicit examples of noncomplete entire maximal graphs in the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^2$.

It is not difficult to verify that the future-pointing Gauss map of $\Sigma(u)$ is given by

$$N = \frac{f(u)}{\sqrt{f^2(u) - |Du|_M^2}} \left(\partial_t + \frac{1}{f^2(u)} Du \right). \tag{23}$$

Moreover, the second fundamental form A of $\Sigma(u)$ with respect to its orientation (23) is given by

$$\begin{aligned} AX = & -\frac{1}{f(u)\sqrt{f^2(u) - |Du|_M^2}} D_X Du - \frac{f'(u)}{\sqrt{f^2(u) - |Du|_M^2}} X \\ & + \left(\frac{-\langle D_X Du, Du\rangle_M}{f(u)\left(f^2(u) - |Du|_M^2\right)^{3/2}} + \frac{f'(u)\langle Du, X\rangle}{\left(f^2(u) - |Du|_M^2\right)^{3/2}} \right) Du \end{aligned} \tag{24}$$

for any tangent vector field X. Consequently, denoting by div the divergence operator on $\Sigma(u)$, the mean curvature function $H(u)$ associated to A is given by

$$H(u) = -\mathrm{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|_M^2}} \right) - \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|_M^2}} \left(n + \frac{|Du|_M^2}{f(u)^2} \right).$$

The differential equation $H(u) = 0$ with the constraint $|Du|_M < f(u)$ is called the *maximal hypersurface equation* in the GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$, and its solutions provide maximal graphs in $\overline{M}^{n+1}$.

Motivated by this previous digression, we will consider the following maximal hypersurface equation

$$(\mathbf{E}) \quad \begin{cases} \mathrm{div}\left(\dfrac{Du}{f(u)\sqrt{f(u)^2 - |Du|_M^2}} \right) = -\dfrac{f'(u)}{\sqrt{f(u)^2 - |Du|_M^2}} \left(n + \dfrac{|Du|_M^2}{f(u)^2} \right) \\ \\ |Du|_M \leq \alpha f(u), \end{cases}$$

where $0 < \alpha < 1$ is constant. We observe that $(\mathbf{E})$ is uniformly elliptic and that the constraint on $|Du|_{M^n}$ assures the boundedness of the hyperbolic angle θ of $\Sigma(u)$. Indeed, from (23) we obtain that

$$|\nabla h|^2 = \frac{|Du|_M^2}{f^2(u) - |Du|_M^2}. \tag{25}$$

Hence, using (6) and (25) we see that $|Du|_M \leq \alpha f(u)$ implies $\cosh\theta \leq \dfrac{1}{\sqrt{1-\alpha^2}}$.

In order to study equation $(\mathbf{E})$, we also recall that

$$|u|_{\mathcal{C}^2(M)} = \max_{|\gamma|\leq 2} |D^\gamma u|_{L^\infty(M)}.$$

According to this setting, from Theorem 3.2 we get the following Calabi-Bernstein type result.

Corollary 4.1. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC and whose Riemannian fiber M^n is complete noncompact with polynomial volume growth. The only entire solutions of* (**E**) *such that $|u|_{\mathcal{C}^2(M)} < +\infty$ and $f''(u) < 0$ are the constant functions $u = c$, with $f'(c) = 0$.*

Proof. We observe first that, under the assumptions of Corollary 4.1, the entire graph $\Sigma(u)$ is a complete spacelike hypersurface. Indeed, from (22) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \langle X, X\rangle &= -\langle Du, X^*\rangle^2_{M^n} + f^2(u)\langle X^*, X^*\rangle_{M^n} \\ &\geq (f^2(u) - |Du|^2_{M^n})\langle X^*, X^*\rangle_{M^n} \end{aligned} \tag{26}$$

for every tangent vector field X on $\Sigma(u)$, where (as before) X^* denotes the projection of X onto the Riemannian fiber M^n. Hence, from the hypothesis $|Du|_M \leq \alpha f(u)$ for some constant $0 < \alpha < 1$, jointly with (26) we get that

$$\langle X, X\rangle \geq \delta\langle X^*, X^*\rangle_{M^n}, \tag{27}$$

where $\delta = (1-\alpha^2)\inf_M f^2(u) > 0$. So, (27) implies that $L = \sqrt{\delta} L_{M^n}$, where L and L_{M^n} denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics $\langle\,,\rangle$ and $\langle\,,\rangle_{M^n}$, respectively. As a consequence, since we are always assuming that M^n is complete, the induced metric (22) must be also complete.

Moreover, since we are supposing that $\Sigma(u)$ is such that $|u|_{\mathcal{C}^2(M)} < +\infty$, from (24) and using once more that $|Du|_M \leq \alpha f(u)$ for some constant $0 < \alpha < 1$, we obtain that $|A|$ must be bounded on $\Sigma(u)$.

On the other hand, from equation (5.9) of [5] we have that

$$d\Sigma = f^{n-1}(u)\sqrt{f^2(u) - |Du|^2_{M^n}}\,dM,$$

where $d\Sigma$ and dM denote the Riemannian volume elements of $(\Sigma(u), \langle\,,\rangle)$ and $(M^n, \langle\,,\rangle_{M^n})$, respectively. Consequently, assuming that M^n has polynomial volume growth, the same will hold for $\Sigma(u)$.

Therefore, following the same procedure of the proof of Corollary 5.1 in [5], we can apply Theorem 3.2 to conclude the proof. □

Reasoning as in the proof of Corollary 4.1, we also obtain the following nonparametric version of Theorem 3.4.

Corollary 4.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the TCC and whose Riemannian fiber M^n is complete noncompact with polynomial volume growth. There are no entire solutions u of* (**E**) *such that $|u|_{\mathcal{C}^2(M)} < +\infty$ and $f'(u) \neq 0$.*

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Henrique Fernandes de Lima
Departamento de Matemática
Universidade Federal de Campina Grande
Campina Grande 58.429-970, Paraíba, Brazil
Email address: henriquedelima74@gmail.com