Commun. Korean Math. Soc. **37** (2022), No. 3, pp. 939–955
https://doi.org/10.4134/CKMS.c210261
pISSN: 1225-1763 / eISSN: 2234-3024

A ROBUST NUMERICAL TECHNIQUE FOR SOLVING NON-LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY LAYER

Firat Cakir, Musa Cakir, and Hayriye Guckir Cakir

Abstract. In this paper, we study a first-order non-linear singularly perturbed Volterra integro-differential equation (SPVIDE). We discretize the problem by a uniform difference scheme on a Bakhvalov-Shishkin mesh. The scheme is constructed by the method of integral identities with exponential basis functions and integral terms are handled with interpolating quadrature rules with remainder terms. An effective quasi-linearization technique is employed for the algorithm. We establish the error estimates and demonstrate that the scheme on Bakhvalov-Shishkin mesh is $O(N^{-1})$ uniformly convergent, where N is the mesh parameter. The numerical results on a couple of examples are also provided to confirm the theoretical analysis.

1. Introduction

In this present paper, we consider the following singularly perturbed non-linear Volterra integro-differential equation (SPVIDE)

$$Lu := \varepsilon u' + a(x)u + \lambda \int_0^x K\big(x,t,u(t)\big)dt = f(x), \quad x \in I = [0,\ell], \tag{1.1}$$

subject to

$$u(0) = A, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter. We assume $a(x)$ $(x \in I)$, $f(x)$ $(x \in I)$ and $K(x,t,u)$ $((x,t,u) \in I \times I \times \mathbb{R})$ are sufficiently smooth functions satisfying

$$a(x) \geq \alpha > 0, \quad x \in I,$$
$$\Big|\frac{\partial K}{\partial u}\Big| \leq \bar{K} \leq \infty, \quad (x,t,u) \in I \times I \times \mathbb{R}.$$

The initial layer for the solution $u(x)$ occurs at $x = 0$ for small values of ε.

Received July 31, 2021; Revised December 9, 2021; Accepted December 30, 2021.
2020 *Mathematics Subject Classification.* 65L11, 65L12, 65L20, 65R20, 45G05.
Key words and phrases. Singularly perturbed, VIDE, difference schemes, uniform convergence, error estimates, Bakhvalov-Shishkin mesh.

Singularly perturbed differential equations, which have the highest order derivative term multiplied with a small positive number ε, possess solutions with interior or boundary layers. Boundary layers are regions where rapid changes occur which makes solving such problems more challenging. Since standard schemes fail to give the accurate results for problems with boundary layer for small ε values, many numerical methods have been developed in the literature to solve singularly perturbed differential equations ([7, 9, 10, 14, 19, 20, 22, 25–28, 30]).

Singularly perturbed Volterra integro-differential equations (SPVIDEs) have been of interest to many researchers since these equations are used to model various science problems in engineering, physics, biology and chemistry ([3, 4, 8, 12, 13, 23, 24]). While a problem of nonlinear SPVIDE modelling the elongation ratio of filament is studied and the qualitative properties of the solution is discussed under some physically interesting assumptions in [23], a specific singularly perturbed integro-differential equation which describes the process of filament stretching is studied in [2]. A broad review on the literature of the SPVIDEs can be found in [17]. Numerical solutions of singularly pertur-bed integro differential equations have been also widely studied by many re-searchers. In [15] and [16], implicit Runge-Kutta methods for singularly per-turbed integro-differential and integro-differential-algebraic equations are ana-lyzed. An exponential finite difference method with piecewise-uniform meshes is applied for the inner and the outer layers and a type of implicit Runge-Kutta method is performed for the outer layer of SPVIDEs in [29]. A finite Legendre expansion is constructed to solve singularly perturbed integral equations, first order Volterra integro-differential equations and Volterra delay integro-differen-tial equations in [18]. In [11], tension spline collocation methods are utilized to derive the numerical discretization of singularly perturbed Volterra integro-differential equations and Volterra integral equations. In [31], the authors present different types of exponential schemes to solve SPVIDEs and the stability analysis of the schemes is examined. Fitted difference schemes are also proven to provide accurate results in the solution process of different types of singularly perturbed problems. In [32], a uniform convergent difference scheme is construc-ted on a graded mesh to solve non-linear SPVIDEs. A finite difference scheme is developed to solve a non-linear first order SPVIDE with delay in [34]. Recently, a finite difference scheme is utilized to examine the numerical solutions of a non-linear VIDE in [6]. In [33], a non-linear SPVIDE is discretized by uniform convergent implicit finite difference scheme on an arbitrary non-uniform mesh and a priori and a posteriori estimates are established.

In this present work, we mainly construct a uniform convergent finite difference scheme for the problem (1.1)-(1.2) on a Bakhvalov-Shishkin mesh. Bakhvalov-Shishkin mesh is a mixed version of the Shishkin mesh and Bakhvalov mesh which are known to yield accurate results for singularly perturbed problems with boundary layers. In [21], the author showed that an upwind difference scheme on Bakhvalov-Shishkin mesh for a linear convection-diffusion problem provides more accurate results than upwinding on a standard Shishkin mesh. Further, a finite difference scheme on Bakhvalov-Shishkin mesh is utilized to solve singularly perturbed boundary value problem with a boundary layer in [5].

We organize the rest of the paper in the following order. In Section 2, the stability result of the continuous problem (1.1)-(1.2) is established. In Section 3, we introduce the Bakhvalov-Shishkin mesh points according to the boundary layer conditions of the problem (1.1)-(1.2) and derive a finite difference scheme through the method of integral identities with exponential basis functions and interpolating quadrature rules with remainder terms. In Section 4, we establish the error estimates and show that the scheme demonstrates $O(N^{-1})$ uniform convergence with respect to the perturbation parameter. The numerical results supporting the analytical results are presented in Section 5.

2. Preliminaries

In the following lemma, we study a priori estimates for the asymptotic behavior of the exact solution to the problem (1.1)-(1.2) which is later used in the error analysis.

Lemma 2.1. *Let $a, f \in C(I)$ and $K \in C(I \times I \times \mathbb{R})$. The solution of the problem* (1.1)-(1.2) *satisfies*

$$\|u\|_\infty \leq C. \tag{2.1}$$

In addition, if $a, f \in C^1(I)$ and $K \in C^1(I \times I \times \mathbb{R})$ with

$$\Big|\frac{\partial}{\partial x}K(x,t,u)\Big| \leq \bar{K}_1 < \infty, \tag{2.2}$$

then the solution $u(x)$ satisfies

$$|u'(x)| \leq C\Big(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\Big), \; x \in I. \tag{2.3}$$

Proof. To establish the first estimate given in (2.1) we first linearize the function $K(x,t,u)$ by the Mean Value Theorem for functions with several variables

$$K(x,t,u(t)) = K(x,t,0) + \frac{\partial}{\partial u}K(x,t,\eta u)u(t), \quad 0 < \eta < 1. \tag{2.4}$$

Inserting (2.4) in (1.1) and rearranging (1.1) we have

$$\varepsilon u' + a(x)u = F(x), \tag{2.5}$$

where

$$F(x) = f(x) - \lambda \int_0^x K(x,t,0)dt - \lambda \int_0^x \frac{\partial K}{\partial u}(x,t,\eta u)u(t)dt. \tag{2.6}$$

Solving the equation (2.5) with $u(0) = A$ yields

$$u(x) = Ae^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds} d\xi,$$

and further we calculate

$$|u(x)| \quad \leq |A|e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds} d\xi.$$

Since we have $a(x) \geq \alpha > 0$, it follows

$$\begin{aligned} |u(x)| &\leq |A|e^{-\frac{1}{\varepsilon}\int_0^x \alpha ds} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha ds} d\xi \\ &= |A|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi. \end{aligned} \tag{2.7}$$

Further, letting $\bar{M} = \max_{I\times I}|K(x,t,0)|$ and by the definition of $F(x)$ in (2.6), we get

$$|F(x)| \leq \|f\|_\infty + \lambda\bar{M}\ell + \lambda\bar{K}\int_0^x |u(t)|dt. \tag{2.8}$$

We substitute (2.8) into (2.7) and have

$$\begin{aligned} |u(x)| \leq |A|e^{-\frac{\alpha x}{\varepsilon}} &+ \frac{1}{\varepsilon}\int_0^x \Big(\|f\|_\infty + \lambda\bar{M}\ell + \lambda\bar{K}\int_0^\xi |u(t)|dt\Big)e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi \\ = |A|e^{-\frac{\alpha x}{\varepsilon}} &+ \frac{1}{\varepsilon}(\|f\|_\infty + \lambda\bar{M}\ell)\int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi \\ &+ \frac{\lambda\bar{K}}{\varepsilon}\int_0^x\int_0^\xi |u(t)|dte^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi. \end{aligned}$$

From here, it follows that

$$\begin{aligned} |u(x)| &\leq |A|e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}(\|f\|_\infty + \lambda\bar{M}\ell)\Big(1 - e^{-\frac{\alpha x}{\varepsilon}}\Big) \\ &\quad + \alpha^{-1}\lambda\bar{K}\Big(1 - e^{-\frac{\alpha x}{\varepsilon}}\Big)\int_0^x |u(t)|dt \\ &\leq |A| + \alpha^{-1}(\|f\|_\infty + \lambda\bar{M}\ell) + \alpha^{-1}\lambda\bar{K}\int_0^x |u(t)|dt. \end{aligned} \tag{2.9}$$

Applying the Gronwall's inequality to the inequality (2.9) we get

$$\begin{aligned}|u(x)| &\le \Big(|A| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}\lambda\bar{M}\ell\Big)e^{\alpha^{-1}\lambda\bar{K}x}\\ &\le \Big(|A| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}\lambda\bar{M}\ell\Big)e^{\alpha^{-1}\lambda\bar{K}\ell},\end{aligned}$$

which leads to the desired result in (2.1).

For the next estimate provided in (2.3), we first differentiate the equation (1.1) and get

$$\varepsilon u'' + a'(x)u + a(x)u' + \lambda K(x,x,u(x)) + \lambda\int_0^x \frac{\partial}{\partial x}K(x,t,u(t))dt = f'(x).$$

Then, letting

$$w(x) = u'(x),$$

and

$$\psi(x) = f'(x) - a'(x)u(x) - \lambda K(x,x,u(x)) - \lambda\int_0^x \frac{\partial}{\partial x}K(x,t,u(t))dt, \tag{2.10}$$

we have

$$\varepsilon w' + a(x)w = \psi(x). \tag{2.11}$$

Solving the equation (2.11) provides

$$w(x) = w(0)e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x \psi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi,$$

which can be bounded as the following

$$\begin{aligned}|w(x)| &\le |w(0)|e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x |\psi(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi\\ &\le |w(0)|e^{-\frac{1}{\varepsilon}\int_0^x \alpha ds} + \frac{1}{\varepsilon}\int_0^x |\psi(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha ds}d\xi\\ &\le |w(0)|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^x |\psi(\xi)|e^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi.\end{aligned} \tag{2.12}$$

Here, having the formula of $\psi(x)$ given in (2.10), (2.1), (2.2) and $a, f \in C^1(I)$, $K \in C^1(I \times I \times \mathbb{R})$ we obtain

$$\begin{aligned}|\psi(x)| &\le ||f'||_\infty + ||a'||_\infty|u| + \lambda\bar{M} + \lambda\bar{K}|u| + \lambda\bar{K}_1\int_0^x |u(t)|dt\\ &\le ||f'||_\infty + \lambda\bar{M} + C\Big(||a'||_\infty + \lambda\bar{K} + \lambda\bar{K}_1\ell\Big),\end{aligned} \tag{2.13}$$

which implies $||\psi||_\infty \leq C_*$ for a $C_* \in \mathbb{R}$. Hence, utilizing (2.13) in (2.12) provides

$$\begin{aligned} |w(x)| &\leq |w(0)|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\|\psi\|_\infty \int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi \\ &\leq |w(0)|e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}C_*(1-e^{-\frac{\alpha x}{\varepsilon}}). \end{aligned} \tag{2.14}$$

On the other hand, inserting $x = 0$ in (1.1) and since $a, f \in C^1(I)$ it follows that

$$|w(0)| = |u'(0)| = \frac{1}{\varepsilon}|f(0) - Aa(0)| \leq \frac{C_1}{\varepsilon}.$$

Substituting this into (2.14) yields

$$|w(x)| \leq \frac{C_1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}C_*(1-e^{-\frac{\alpha x}{\varepsilon}}),$$

which provides the desired result. □

3. Difference schemes and mesh

3.1. Notation and Bakhvalov-Shishkin mesh

Let $\bar{\omega}_h = \{0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = \ell\}$ denote a non-uniform mesh on $[0, \ell]$. For each $i = 0, \ldots, N$, let $h_i = x_i - x_{i-1}$ be the step size. For any continuous mesh function v_i defined on ω_h we use the notation

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}$$

for backward difference.

We construct our difference scheme based on Bakhvalov-Shishkin mesh. With the Bakhvalov-Shishkin mesh, the domain is split into two subintervals $[0, \sigma]$ and $[\sigma, \ell]$, where σ is the transition parameter. For an even discretization parameter $N > 0$, we fix the transition parameter

$$\sigma = \min\left\{\frac{\ell}{2}, \varepsilon\alpha^{-1}\ln N\right\}. \tag{3.1}$$

We assume $\varepsilon \ll N^{-1}$ as it is used in practice. We introduce a set of the mesh points in the following

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon\ln[1-2(1-N^{-1})\frac{i}{N}], & x_i \in [0,\sigma],\ i = 0, 1, \ldots, \frac{N}{2}, \\ \sigma + \left(i - \frac{N}{2}\right)h, \quad h = \frac{2(\ell-\sigma)}{N}, & x_i \in [\sigma, \ell],\ i = \frac{N}{2}+1, \ldots, N. \end{cases} \tag{3.2}$$

3.2. Discretization

For the difference approximation we utilize the following integral identity

$$\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} Lu(x)\varphi_i(x)dx = \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx, \tag{3.3}$$

with the exponential basis function

$$\varphi_i(x) = e^{-\frac{a_i}{\varepsilon}(x_i - x)}, \quad i = 1, \ldots, N,$$

where

$$\chi_i = {h_i}^{-1}\int_{x_{i-1}}^{x_i} \varphi_i(x)dx = \frac{1-e^{-a_i\rho_i}}{a_i\rho_i}, \quad \rho_i = \frac{h_i}{\varepsilon}.$$

We remark that φ_i solves the equation

$$\begin{aligned} &-\varepsilon\varphi(x) + a_i\varphi(x) = 0, \quad x_{i-1} \le x \le x_i \\ &\varphi(x_i) = 1. \end{aligned} \tag{3.4}$$

We handle (3.3) evaluating each term separately through the quadrature rules with weight functions and obtain the remainder terms as provided in [1]. In the following, we deal with the differential term on the left-hand side of (3.3),

$$\begin{aligned} &\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} \Big[\varepsilon u'(x) + a(x)u(x)\Big]\varphi_i(x)dx \\ &= \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} [\varepsilon u'(x) + a_i u(x)]\varphi_i(x)dx \\ &\quad + \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} [a(x) - a_i]u(x)\varphi_i(x)dx \\ &= \varepsilon\theta_i u_{\bar{x},i} + a_i u_i + R_i^{(1)}, \end{aligned} \tag{3.5}$$

where

$$\theta_i = \frac{a_i\rho_i e^{-a_i\rho_i}}{1-e^{-a_i\rho_i}}, \tag{3.6}$$

and

$$R_i^{(1)} = \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i} [a(x) - a_i]u(x)\varphi_i(x)dx. \tag{3.7}$$

For the integral term in (3.3), we use the first quadrature rules provided in [1] which turns it into

$$\chi_i^{-1}h_i^{-1}\lambda\int_{x_{i-1}}^{x_i} \varphi_i(x)\int_0^x K(x,t,u(t))dtdx = \lambda\int_0^{x_i} K(x_i,t,u(t))dt + R_i^{(2)}, \tag{3.8}$$

where

$$R_i^{(2)} = \lambda\int_{x_{i-1}}^{x_i} \frac{\partial}{\partial\xi}\Big(\int_0^{\xi} K(\xi,t,u(t))dt\Big)\Big[T_0(x-\xi) - h_i^{-1}(x - x_{i-1})\Big]d\xi, \tag{3.9}$$

and $T_0(\lambda)=1$ for $\lambda\geq 0$ and $T_0(\lambda)=0$ for $\lambda<0$. Here, we apply the composite right-side rectangle rule to the integral term in the right-hand side of (3.8) and get

$$\lambda\int_0^{x_i}K(x_i,t,u(t))dt=\lambda\sum_{j=1}^{i}h_jK(x_i,x_j,u_j)+R_i^{(3)}, \tag{3.10}$$

where

$$R_i^{(3)}=-\lambda\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}(\xi-x_{j-1})\frac{\partial}{\partial\xi}\Big(K\big(x_i,\xi,u(\xi)\big)\Big)d\xi. \tag{3.11}$$

Then, inserting (3.10) in (3.8) provides

$$\begin{aligned}&\chi_i^{-1}h_i^{-1}\lambda\int_{x_{i-1}}^{x_i}\varphi_i(x)\int_0^{x}K(x,t,u(t))dtdx\\&=\lambda\sum_{j=1}^{i}h_jK(x_i,x_j,u_j)+R_i^{(2)}+R_i^{(3)}.\end{aligned} \tag{3.12}$$

On the other hand, the right-hand side of (3.3) gets the in the form

$$\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}f(x)\varphi_i(x)dx=f_i+R_i^{(4)}, \tag{3.13}$$

where

$$R_i^{(4)}=\chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}[f(x)-f(x_i)]\varphi_i(x)dx. \tag{3.14}$$

Inserting the relations (3.5), (3.12) and (3.13) in (3.3), we obtain the difference problem for the problem (1.1)-(1.2) as

$$\begin{aligned}&\varepsilon\theta_iu_{\bar{x},i}+a_iu_i+\lambda\sum_{j=1}^{i}h_jK(x_i,x_j,u_j)=f_i-R_i,\quad i=1,2,\ldots,N,\\&u_0=A,\end{aligned} \tag{3.15}$$

where

$$R_i=R_i^{(1)}+R_i^{(2)}+R_i^{(3)}-R_i^{(4)}. \tag{3.16}$$

Neglecting the error term R_i in (3.15) provides the difference scheme

$$L_Ny_i:=\varepsilon\theta_iy_{\bar{x},i}+a_iy_i+\lambda\sum_{j=1}^{i}h_jK(x_i,x_j,y_j)=f_i,\quad i=1,2,\ldots,N, \tag{3.17}$$

$$y_0=A, \tag{3.18}$$

where θ_i defined by (3.6).

4. Error estimates and convergence results

In this section, we establish the error estimates of the approximate solution y and analyze the convergence of the difference scheme given in (3.17)-(3.18).

The error of the difference problem, $z_i = y_i - u_i$ for $1 \leq i \leq N$, is the solution to the discrete problem

$$\varepsilon\theta_i z_{\bar{x},i} + a_i z_i + \lambda \sum_{j=1}^{i} h_j \big[K(x_i, x_j, y_j) - K(x_i, x_j, u_j)\big] = R_i, \quad i = 1, 2, \ldots, N, \tag{4.1}$$

$$z_0 = 0. \tag{4.2}$$

Lemma 4.1. *Consider the following difference problem*

$$\ell_N v_i := \varepsilon\theta_i v_{\bar{x},i} + a_i v_i = F_i, \quad 1 \leq i \leq N, \tag{4.3}$$

$$v_0 = A. \tag{4.4}$$

Let $|F_i| \leq \mathcal{F}_i$ and $\mathcal{F}_i$ be a non-decreasing function. Then, the solution of the problem (4.3)-(4.4) *satisfies*

$$|v_i| \leq |A| + \alpha^{-1}\mathcal{F}_i, \quad 1 \leq i \leq N. \tag{4.5}$$

Proof. The proof follows from the maximum principle for difference operators. Details can be found in [6]. □

Lemma 4.2. *Let z_i be the solution of* (4.1)-(4.2)*. Then, z_i satisfies the estimate*

$$\|z\|_\infty \leq C\|R\|_\infty. \tag{4.6}$$

Proof. The difference scheme equation given in (4.1) can be rewritten in the form

$$\varepsilon\theta_i z_{\bar{x},i} + a_i z_i = F_i, \tag{4.7}$$

where

$$F_i = R_i - \lambda \sum_{j=1}^{i} h_j \big[K(x_i, x_j, y_j) - K(x_i, x_j, u_j)\big]. \tag{4.8}$$

By the Mean Value Theorem for functions with several variables we linearize the non-linear kernel K as

$$K(x_i, x_j, y_j) = K(x_i, x_j, 0) + \frac{\partial}{\partial u} K(x_i, x_j, \eta y_j) y_j, \quad 0 \leq \eta \leq 1, \tag{4.9}$$

and

$$K(x_i, x_j, u_j) = K(x_i, x_j, 0) + \frac{\partial}{\partial u} K(x_i, x_j, \zeta u_j) u_j, \quad 0 \leq \zeta \leq 1. \tag{4.10}$$

Inserting the linearizations (4.9) and (4.10) into (4.8) we have the estimate

$$|F_i| \leq \|R\|_\infty + \lambda\bar{K}\sum_{j=1}^{i} h_j|z_j|. \tag{4.11}$$

Then, applying Lemma 4.1 to (4.7) and utilizing the estimate (4.11) provide

$$|z_i| \leq \alpha^{-1}\|R\|_\infty + \alpha^{-1}\lambda\bar{K}\sum_{j=1}^{i} h_j|z_j|. \tag{4.12}$$

Further, applying the difference analogue of the Gronwall's inequality to (4.12) we have

$$|z_i| \leq \alpha^{-1}e^{\alpha^{-1}\lambda\bar{K}\ell}\|R\|_\infty,$$

which yields the result in (4.6). □

Lemma 4.3. *Let $a, f \in C^1(I)$ and $K \in C^1(I \times I \times \mathbb{R})$ with*

$$\bar{M} = \max_{I\times I}|K(x,t,0)|, \tag{4.13}$$

$$\Big|\frac{\partial}{\partial x}K(x,t,u(t))\Big| \leq \bar{K}_1 < \infty, \tag{4.14}$$

and

$$\Big|\frac{\partial}{\partial t}K(x,t,u(t))\Big| \leq \bar{K}_2 < \infty. \tag{4.15}$$

Then, the truncation error R_i satisfies the estimate

$$\|R\|_\infty \leq CN^{-1}. \tag{4.16}$$

Proof. To obtain the estimate (4.16), we proceed by bounding each error term R_1, R_2, R_3 and R_4 separately. For $R_i^{(1)}$, we have

$$|R_i^{(1)}| \leq \chi_i^{-1}h_i^{-1}\int_{x_{i-1}}^{x_i}|(a'(s)(x-x_i))u(x)|\varphi_i(x)dx,$$

where $s \in [x, x_i]$ comes from the Mean Value Theorem. Then, since $a \in C^1(I)$ and from (2.1) we get

$$|R_i^{(1)}| \leq C_1 h_i. \tag{4.17}$$

For $R_i^{(2)}$ we take into account of (4.13) and $|T_0(\lambda)| \leq 1$, so

$$\begin{aligned}
|R_i^{(2)}| &\leq \lambda\int_{x_{i-1}}^{x_i}\Big|\big(1+h_i^{-1}(x-x_i)\big)\frac{\partial}{\partial\xi}\Big(\int_0^\xi K(\xi,t,u(t))dt\Big)\Big|d\xi \\
&\leq 2\lambda\int_{x_{i-1}}^{x_i}\Big|\frac{\partial}{\partial\xi}\Big(\int_0^\xi K(\xi,t,u(t))dt\Big)\Big|d\xi \\
&\leq 2\lambda\int_{x_{i-1}}^{x_i}\Big|\frac{\partial}{\partial\xi}\Big(\int_0^\xi K(\xi,t,0)+\frac{\partial}{\partial u}K(\xi,t,\eta u)u(t)dt\Big)\Big|d\xi,
\end{aligned} \tag{4.18}$$

where ηu is the intermediate value for $0 < \eta < 1$ coming from the Mean Value Theorem. Then, from (4.13) and (2.1) we have

$$|R_i^{(2)}| \leq 2\lambda(\bar{M} + C\bar{K})h_i. \tag{4.19}$$

On the other hand, from (4.15) and (2.3) we have

$$\begin{aligned} |R_i^{(3)}| &\leq \lambda \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} \left(\left|\frac{\partial}{\partial \xi} K(x_i, \xi, u(\xi))\right| + \left|\frac{\partial}{\partial u} K(x_i, \xi, u(\xi))\right| |u'(\xi)| \right) d\xi \\ &\leq \lambda \sum_{j=1}^{i} \left(\bar{K}_2 h_j + \bar{K} \int_{x_{j-1}}^{x_j} \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha \xi}{\varepsilon}}\right) \right) d\xi \\ &= \lambda \sum_{j=1}^{i} \left(\bar{K}_2 h_j + \bar{K} h_j + \alpha^{-1} \bar{K} \left(e^{-\frac{\alpha x_{j-1}}{\varepsilon}} - e^{-\frac{\alpha x_j}{\varepsilon}} \right) \right). \end{aligned} \tag{4.20}$$

Then, by the Mean Value Theorem applied to the exponential term in (4.20) with $s \in [x_{j-1}, x_j]$ it follows that

$$\begin{aligned} |R_i^{(3)}| &\leq \lambda \sum_{j=1}^{i} \left(\bar{K}_2 h_j + \bar{K} h_j + \alpha^{-1} \bar{K} h_j e^{-\frac{\alpha s}{\varepsilon}} \right) \\ &\leq C_3^* |h^*|, \end{aligned} \tag{4.21}$$

where $h^* = \max\limits_{1 \leq j \leq i} h_j$. Lastly, for $R_i^{(4)}$, similarly to the work above and since $f \in C^1(I)$ we have

$$\begin{aligned} |R_i^{(4)}| &\leq \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} |f'(s)(x - x_i)| \varphi_i(x) dx \\ &\leq C_4 h_i, \end{aligned} \tag{4.22}$$

where $s \in [x_{i-1}, x_i]$ by the Mean Value Theorem.

Further in the proof, we need to evaluate each estimate above on the sub-intervals $[0, \sigma]$ and $[\sigma, \ell]$. For this, we first establish the bounds on the step-size h_i on each interval. In the first sub-interval $[0, \sigma]$ with $\sigma \leq \frac{\ell}{2}$,

$$x_i = -\alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i}{N}], \quad i = 1, \ldots, N/2$$

and hence,

$$h_i = -\alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i}{N}] + \alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i-1}{N}].$$

Then, we apply the Mean Value Theorem to h_i with $i_* \in [i - 1, i]$ and get

$$h_i \leq \alpha^{-1} \varepsilon \frac{2(1 - N^{-1}) N^{-1}}{1 - 2 i_* (1 - N^{-1}) N^{-1}} \leq C N^{-1}. \tag{4.23}$$

In the second sub-interval $[\sigma, \ell]$, we have

$$x_i = \sigma + \big(i - \frac{N}{2}\big)h, \quad i = N/2+1, \dots, N,$$

where $\sigma \le \frac{\ell}{2}$ and

$$h_i = \frac{2(\ell-\sigma)}{N} \le CN^{-1}. \tag{4.24}$$

Inserting the bounds (4.23) and (4.24) in (4.17), (4.19), (4.21) and (4.22), we have

$$|R_i^{(k)}| \le CN^{-1}, \quad k = 1, 2, 3, 4,$$

which implies the desired result (4.16). □

Theorem 4.4. *Let u be the solution of* (1.1)-(1.2) *and y be the solution of* (3.17)-(3.18). *If the assumptions on the functions a, f and K provided in Lemma 4.3 hold, then*

$$\|y-u\|_\infty \le CN^{-1}.$$

Proof. This statement is a result of Lemma 4.2 and Lemma 4.3. □

5. Algorithm and numerical results

In this section, we present the numerical results on two examples, one with an exact solution and one with no known solution. We provide the graphs of the approximate solutions, error estimates and the convergence values of the approximate solution to the exact solution. Since the scheme given in (3.17)-(3.18) is a non-linear problem, we first apply the quasi-linearization technique to the difference scheme

$$\varepsilon\theta_i y_{\bar{x},i}^{(n)} + a_i y_i^{(n)} + \lambda \sum_{j=1}^{i} h_j \Big[K(x_i, x_j, y_j^{(n-1)}) + \frac{\partial}{\partial y}K(x_i, x_j, y_j^{(n-1)})\big(y_j^{(n)} - y_j^{(n-1)}\big)\Big] = f_i.$$

Further, we use the elimination method

$$y_i^{(n)} = \frac{f_i + A_i y_{i-1}^{(n)} + B_i y_i^{(n-1)} - C_i - D_i}{\frac{\varepsilon\theta_i}{h_i} + a_i + \lambda h_i \frac{\partial}{\partial y}K(x_i, x_i, y_i^{(n-1)})},$$

$$y_0^{(n)} = A,$$

where

$$A_i = \frac{\varepsilon\theta_i}{h_i},$$

$$B_i = \lambda h_i \frac{\partial}{\partial y}K(x_i, x_i, y_i^{(n-1)}),$$

$$C_i = \lambda h_i K(x_i, x_i, y_i^{(n-1)}),$$

$$D_i = \lambda \sum_{j=1}^{i-1} h_j \Big[K(x_i, x_j, y_j^{(n-1)}) \Big].$$

Example 5.1. Consider the following initial value problem

$$\begin{aligned} &\varepsilon u'(x) + e^{-x}u(x) - \int_0^x u^2(t)dt \\ &= \varepsilon x e^{-x} + e^{\frac{-x}{\varepsilon}}(e^{-x} + 2\varepsilon^2 x + 2\varepsilon^3 - 1) \\ &\quad + \frac{\varepsilon}{2} e^{\frac{-2x}{\varepsilon}} - \varepsilon^2(\frac{x^3}{3} - 1) - 2\varepsilon^3 - \frac{\varepsilon}{2}, \quad 0 \le x \le 1, \\ &u(0) = 1. \end{aligned}$$

The exact solution to this problem is

$$u(x) = \varepsilon x + e^{-\frac{x}{\varepsilon}}.$$

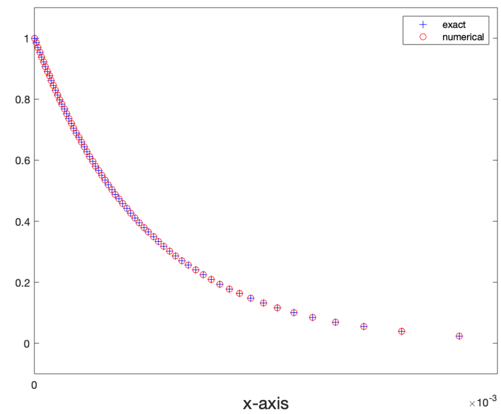

FIGURE 5.1. The figure represents the graphs for the exact solution and the approximate solution for $\varepsilon = 2^{-12}$ and $N = 64$.

We calculate the exact error through the formula

$$e_\varepsilon^N = \|y^N - u\|_\infty,$$

where y^N is the numerical approximation of u for different N and ε values. The convergence rate is obtained by

$$r^N = \frac{\ln\left(e^N / e^{2N}\right)}{\ln 2}.$$

In Table 5.1, we provide the errors e^N, e^{2N} and the convergence rates of the approximate solution for various N and $\varepsilon = 2^{-i}$ values.

TABLE 5.1. Errors e^N, e^{2N}, and rate of convergence r for Example 5.1.

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	3.662×10^{-5}	1.924×10^{-5}	9.835×10^{-6}	4.921×10^{-6}	2.405×10^{-6}
2^{-12}	1.924×10^{-5}	9.835×10^{-6}	4.921×10^{-6}	2.405×10^{-6}	1.153×10^{-6}
	0.929	0.968	0.999	1.033	1.060
	5.745×10^{-7}	3.030×10^{-7}	1.561×10^{-7}	7.935×10^{-8}	4.002×10^{-8}
2^{-18}	3.030×10^{-7}	1.561×10^{-7}	7.935×10^{-8}	4.002×10^{-8}	2.009×10^{-8}
	0.923	0.957	0.977	0.987	0.994
	8.977×10^{-9}	4.735×10^{-9}	2.440×10^{-9}	1.240×10^{-9}	6.260×10^{-10}
2^{-24}	4.735×10^{-9}	2.440×10^{-9}	1.240×10^{-9}	6.260×10^{-10}	3.146×10^{-10}
	0.923	0.957	0.976	0.987	0.993

Example 5.2. For the second test problem

$$\varepsilon u' + (x+1)u + \int_0^x \left[(x-t)^2 + x^2 - u^2(t) + xe^{(u(t))^2}\right]dt$$
$$= e^{\frac{-2x}{\varepsilon}} + x^2 + \tanh^2(x) + x, \quad 0 \le x \le 1,$$
$$u(0) = 1.$$

The exact solution to this problem is not known. Therefore, we compute the approximate solution y^N and use the double mesh principle to estimate the errors and find the convergence rate. In the double mesh principle, the error is taken as the difference between the approximate solution on mesh size N and the approximate solution computed on double mesh $2N$, namely

$$e_\varepsilon^N = \|y^N - y^{2N}\|_\infty,$$

where y^N is the approximate solution on mesh N and y^{2N} is the approximate solution on mesh $2N$. The convergence rate is calculated as it is in Example 5.1.

In Table 5.2, the errors and the convergence rates of the approximate solution for various N and $\varepsilon = 2^{-i}$ values are presented.

6. Conclusion

In this study, a finite difference scheme on a Bakhvalov-Shishkin mesh is constructed to obtain the numerical solution of an initial value problem for a quasi-linear first-order singularly perturbed Volterra integro-differential equation with a boundary layer. It is shown that the method is first-order uniformly convergent with respect to the perturbation parameter. The numerical results provided in Tables 5.1 and 5.2 also agree with the analytical results on the error estimates and convergence order. So, it is confirmed that the convergence of the scheme is first order. We also suggest that this difference scheme method on Bakhvalov-Shishkin mesh can be applied to the singularly perturbed linear or non-linear problems with delay to obtain accurate numerical solutions.

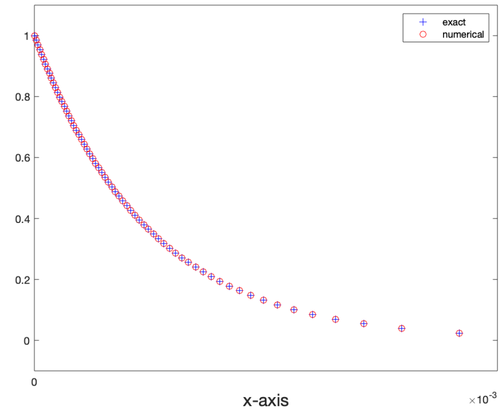

FIGURE 5.2. The figure shows the graphs of the approximate solutions for $\varepsilon = 2^{-12}$ and $N = 64$, $N = 128$ and $N = 256$.

TABLE 5.2. Errors e^N, e^{2N}, and rate of convergence r for Example 5.2.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
	2.671×10^{-2}	1.409×10^{-2}	7.227×10^{-3}	3.610×10^{-3}	1.812×10^{-3}
2^{-12}	1.409×10^{-2}	7.227×10^{-3}	3.610×10^{-3}	1.812×10^{-3}	9.244×10^{-4}
	0.923	0.963	1.001	0.995	0.992
	2.269×10^{-2}	1.428×10^{-2}	7.410×10^{-3}	3.787×10^{-3}	1.917×10^{-3}
2^{-18}	1.428×10^{-2}	7.410×10^{-3}	3.787×10^{-3}	1.917×10^{-3}	9.646×10^{-4}
	0.914	0.947	0.968	0.982	0.991
	2.691×10^{-2}	1.428×10^{-2}	7.413×10^{-3}	3.790×10^{-3}	1.920×10^{-3}
2^{-24}	1.428×10^{-2}	7.413×10^{-3}	3.790×10^{-3}	1.920×10^{-3}	9.673×10^{-4}
	0.914	0.946	0.967	0.981	0.989

References

[1] G. M. Amiraliyev and Ya. D. Mamedov, *Difference schemes on the uniform mesh for singular perturbed pseudo-parabolic equations*, Turkish J. Math. **19** (1995), no. 3, 207–222.

[2] J. S. Angell and W. E. Olmstead, *Singular perturbation analysis of an integro-differential equation modelling filament stretching*, Z. Angew. Math. Phys. **36** (1985), no. 3, 487–490. https://doi.org/10.1007/BF00944639

[3] J. S. Angell and W. E. Olmstead, *Singularly perturbed Volterra integral equations. II*, SIAM J. Appl. Math. **47** (1987), no. 6, 1150–1162. https://doi.org/10.1137/0147077

[4] J. S. Angell and W. E. Olmstead, *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math. **47** (1987), no. 1, 1–14. https://doi.org/10.1137/0147001

[5] D. Arslan, *A uniformly convergent numerical study on bakhvalov-shishkin mesh for singularly perturbed problem*, Commun. Math. Appl. **11** (2020), no. 1, 161–171.

[6] M. Cakir, B. Gunes, and H. Duru, *A novel computational method for solving nonlinear Volterra integro-differential equation*, Kuwait J. Sci. **48** (2021), no. 1, 1–9.

[7] Z. Cen, A. Le, and A. Xu, *Parameter-uniform hybrid difference scheme for solutions and derivatives in singularly perturbed initial value problems*, J. Comput. Appl. Math. **320** (2017), 176–192. https://doi.org/10.1016/j.cam.2017.02.009

[8] A. De Gaetano and O. Arino, *Mathematical modelling of the intravenous glucose tolerance test*, J. Math. Biol. **40** (2000), no. 2, 136–168. https://doi.org/10.1007/s002850050007

[9] E. P. Doolan, J. J. H. Miller, and W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole Press, Dún Laoghaire, 1980.

[10] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Robust computational techniques for boundary layers*, Applied Mathematics (Boca Raton), 16, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[11] V. Horvat and M. Rogina, *Tension spline collocation methods for singularly perturbed Volterra integro-differential and Volterra integral equations*, J. Comput. Appl. Math. **140** (2002), no. 1-2, 381–402. https://doi.org/10.1016/S0377-0427(01)00517-9

[12] G. S. Jordan, *A nonlinear singularly perturbed Volterra integro-differential equation of nonconvolution type*, Proc. Roy. Soc. Edinburgh Sect. A **80** (1978), no. 3-4, 235–247. https://doi.org/10.1017/S030821050001026X

[13] G. S. Jordan, *Some nonlinear singularly perturbed Volterra integro-differential equations*, in Volterra equations (Proc. Helsinki Sympos. Integral Equations, Otaniemi, 1978), 107–119, Lecture Notes in Math., 737, Springer, Berlin, 1979.

[14] M. K. Kadalbajoo and V. Gupta, *A brief survey on numerical methods for solving singularly perturbed problems*, Appl. Math. Comput. **217** (2010), no. 8, 3641–3716. https://doi.org/10.1016/j.amc.2010.09.059

[15] J.-P. Kauthen, *Implicit Runge-Kutta methods for some integrodifferential-algebraic equations*, Appl. Numer. Math. **13** (1993), no. 1-3, 125–134. https://doi.org/10.1016/0168-9274(93)90136-F

[16] J.-P. Kauthen, *Implicit Runge-Kutta methods for singularly perturbed integro-differential systems*, Appl. Numer. Math. **18** (1995), no. 1-3, 201–210. https://doi.org/10.1016/0168-9274(95)00053-W

[17] J.-P. Kauthen, *A survey of singularly perturbed Volterra equations*, Appl. Numer. Math. **24** (1997), no. 2-3, 95–114. https://doi.org/10.1016/S0168-9274(97)00014-7

[18] A. H. Khater, A. B. Shamardan, D. K. Callebaut, and M. R. A. Sakran, *Numerical solutions of integral and integro-differential equations using Legendre polynomials*, Numer. Algorithms **46** (2007), no. 3, 195–218. https://doi.org/10.1007/s11075-007-9130-2

[19] M. Kumar, P. Singh, and H. K. Mishra, *A recent survey on computational techniques for solving singularly perturbed boundary value problems*, Int. J. Comput. Math. **84** (2007), no. 10, 1439–1463. https://doi.org/10.1080/00207160701295712

[20] V. Kumar and B. Srinivasan, *An adaptive mesh strategy for singularly perturbed convection diffusion problems*, Appl. Math. Model. **39** (2015), no. 7, 2081–2091. https://doi.org/10.1016/j.apm.2014.10.019

[21] T. Linß, *Analysis of a Galerkin finite element method on a Bakhvalov-Shishkin mesh for a linear convection-diffusion problem*, IMA J. Numer. Anal. **20** (2000), no. 4, 621–632. https://doi.org/10.1093/imanum/20.4.621

[22] T. Linß, *Layer-adapted meshes for convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg. **192** (2003), no. 9-10, 1061–1105. https://doi.org/10.1016/S0045-7825(02)00630-8

[23] A. S. Lodge, J. B. McLeod, and J. A. Nohel, *A nonlinear singularly perturbed Volterra integro-differential equation occurring in polymer rheology*, Proc. Roy. Soc. Edinburgh Sect. A **80** (1978), no. 1-2, 99–137. https://doi.org/10.1017/S0308210500010167

[24] S. Marino, E. Beretta, and D. E. Kirschner, *The role of delays in innate and adaptive immunity to intracellular bacterial infection*, Math. Biosci. Eng. **4** (2007), no. 2, 261–286. https://doi.org/10.3934/mbe.2007.4.261

[25] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, revised edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. https://doi.org/10.1142/9789814390743

[26] H. K. Mishra and S. Saini, *Various numerical methods for singularly perturbed boundary value problems*, Am. J. Appl. Math. Statist. **2** (2014), 129–142.

[27] A. H. Nayfeh, *Introduction to Perturbation Techniques*, A Wiley-Interscience Publication, Wiley-Interscience, New York, 1981.

[28] R. E. O'Malley, Jr., *Singular perturbation methods for ordinary differential equations*, Applied Mathematical Sciences, 89, Springer-Verlag, New York, 1991. https://doi.org/10.1007/978-1-4612-0977-5

[29] J. I. Ramos, *Exponential techniques and implicit Runge-Kutta methods for singularly-perturbed Volterra integro-differential equations*, Neural Parallel Sci. Comput. **16** (2008), no. 3, 387–404.

[30] H.-G. Roos, M. Stynes, and L. Tobiska, *Numerical methods for singularly perturbed differential equations*, Springer Series in Computational Mathematics, 24, Springer-Verlag, Berlin, 1996. https://doi.org/10.1007/978-3-662-03206-0

[31] A. A. Salama and S. A. Bakr, *Difference schemes of exponential type for singularly perturbed volterra integro-differential problems*, Appl. Math. Model. **31** (2007), 866–879.

[32] S. Şevgin, *Numerical solution of a singularly perturbed Volterra integro-differential equation*, Adv. Difference Equ. **2014** (2014), 171, 15 pp. https://doi.org/10.1186/1687-1847-2014-171

[33] Sumit, S. Kumar, and J. Vigo-Aguiar, *Analysis of a nonlinear singularly perturbed Volterra integro-differential equation*, J. Comput. Appl. Math. **404** (2022), Paper No. 113410. https://doi.org/10.1016/j.cam.2021.113410

[34] Ö. Yapman, G. M. Amiraliyev, and I. Amirali, *Convergence analysis of fitted numerical method for a singularly perturbed nonlinear Volterra integro-differential equation with delay*, J. Comput. Appl. Math. **355** (2019), 301–309. https://doi.org/10.1016/j.cam.2019.01.026

Firat Cakir
Department of Mathematics
Batman University
Batman, Turkiye
Email address: firat.cakir@hotmail.com

Musa Cakir
Department of Mathematics
Van Yuzuncu Yil University
Van, Turkiye
Email address: cakirmusa@hotmail.com

Hayriye Guckir Cakir
Department of Mathematics
Adiyaman University
Adiyaman, Turkiye
Email address: hcakir@adiyaman.edu.tr