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A NEW CHARACTERIZATION OF TYPE (A) AND RULED REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

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Abstract. In this paper, we obtain an inequality involving the squared norm of the covariant differentiation of the shape operator for a real hypersurface in nonflat complex space forms. It is proved that the equality holds for non-Hopf case if and only if the hypersurface is ruled and the equality holds for Hopf case if and only if the hypersurface is of type (A).

1. Introduction

Let $\mathbb{C}M^n(c)$, for an integer $n \geq 2$, be a complex space form which is defined as a Kählerian manifold of complex dimension n with constant holomorphic sectional curvature c. A complete and simply connected complex space form is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$;
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$;
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$.

Let M be a real hypersurface in a nonflat complex space form $\mathbb{C}M^n(c)$ whose Kähler metric and complex structure are denoted by $\overline{g}$ and J, respectively. On M there is an almost contact metric structure (ϕ, ξ, η, g) induced from $\overline{g}$ and J, respectively; where ξ is called the structure or Reeb vector field. Let A be the shape operator of M in $\mathbb{C}M^n(c)$. If the structure vector field ξ is principal for the shape operator at each point, i.e., $A\xi = \alpha\xi$ with $\alpha = \eta(A\xi)$, M is called a Hopf hypersurface and α is said to be the Hopf principal curvature. The Hopf hypersurfaces can be classified completely under additional assumptions. Among others, the following two theorems are well-known.

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Theorem 1.1 ([3]). *A connected Hopf hypersurface of $\mathbb{C}P^n(c)$ has constant principal curvatures if and only if it is locally congruent to one of the following*:

(A_1) *a geodesic sphere of radius r with $0 < r < \pi/\sqrt{c}$*;
(A_2) *a tube of radius r around a totally geodesic $\mathbb{C}P^k(c)$ $(1 \leq k \leq n-2)$ with $0 < r < \pi/\sqrt{c}$*;
(B) *a tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$ with $0 < r < \pi/(2\sqrt{c})$*;
(C) *a tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{\frac{n-1}{2}}(c)$ and $n \geq 5$ is odd with $0 < r < \pi/(2\sqrt{c})$*;
(D) *a tube of radius r around a complex Grassmannian $\mathbb{C}G_{2.5}$ and $n = 9$ with $0 < r < \pi/(2\sqrt{c})$*;
(E) *a tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$ and $n = 15$ with $0 < r < \pi/(2\sqrt{c})$.*

When the ambient space is the complex hyperbolic space $\mathbb{C}H^n(c)$, the corresponding version of above theorem is given as follows.

Theorem 1.2 ([1]). *A connected Hopf hypersurface of $\mathbb{C}H^n(c)$ has constant principal curvatures if and only if it is locally congruent to one of the following*:

(A_0) *a self-tube, that is, a horosphere*;
($A_{1,0}$) *a geodesic hypersphere of radius r with $0 < r < \infty$*;
($A_{1,1}$) *a tube of radius r around a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}(c)$ with $0 < r < \infty$*;
(A_2) *a tube of radius r around a totally geodesic $\mathbb{C}H^k(c)$ $(1 \leq k \leq n-2)$ with $0 < r < \infty$*;
(B) *a tube of radius r around a totally real totally geodesic hyperbolic space $\mathbb{R}H^n(c/4)$ with $0 < r < \infty$.*

In literature, a real hypersurface is said to be of type (A) if it is of type (A_1) or (A_2) in $\mathbb{C}P^n$ or of type (A_0), ($A_{1,0}$), or ($A_{1,1}$) in $\mathbb{C}H^n$. The characterizations for this kind of real hypersurfaces are very rich (see many references in [2,8,11]) because they can be viewed as one of the most fundamental models in geometry of real hypersurfaces. Among others, their characterizations by means of some inequalities (related to some important geometric quantities) are interesting. For example, according to Theorem 1.11 and Corollary 4.4 of [8], type (A) real hypersurfaces have been characterized as the following.

Theorem 1.3 ([8]). *On a real hypersurface in nonflat complex space forms $\mathbb{C}M^n(c)$ there holds*

$$|\nabla A|^2 \geq \frac{1}{4}(n-1)c^2,$$

and the equality occurs if and only if the hypersurface is of type (A).

A real hypersurface in nonflat complex space forms is said to be ruled if the holomorphic distribution $\{\xi\}^\perp$ is integrable and its leaves are totally geodesic. The ruled hypersurface is one of the most important models for non-Hopf hypersurfaces. Thus, in view of Theorem 1.3, a natural question can be proposed as the following: is there an equality involving the squared norm of the covariant differentiation of the shape operator for characterizing ruled hypersurfaces? In this paper, by applying those results in [10, 12], we aim to answer this question and present the following result.

Theorem 1.4. *On a real hypersurface in nonflat complex space forms* $\mathbb{C}M^n(c)$ *there holds*

$$|\nabla A|^2 \geq \frac{1}{4}(n-1)c^2 + 2|\nabla_\xi A|^2 + 2g(\phi(\nabla_\xi A)\phi A\xi, \phi A\xi) - |(\nabla_\xi A)\xi|^2 - 2|\phi A\xi|^4,$$

and the equality occurs for a non-Hopf hypersurface if and only if the hypersurface is ruled.

It is well-known that for any Hopf hypersurface, the Hopf principal curvature is a constant (cf. [8, Corollary 2.6]). Thus, by an application of this, Theorem 1.4 can be extended as follows.

Theorem 1.5. *On a real hypersurface in nonflat complex space forms* $\mathbb{C}M^n(c)$ *there holds*

$$|\nabla A|^2 \geq \frac{1}{4}(n-1)c^2 + 2|\nabla_\xi A|^2,$$

and the equality occurs for a Hopf hypersurface if and only if the hypersurface is of type (A).

The combination of the above two theorems gives the following corollary.

Corollary 1.6. *On a real hypersurface in nonflat complex space forms* $\mathbb{C}M^n(c)$ *there holds*

$$|\nabla A|^2 \geq \frac{1}{4}(n-1)c^2 + 2|\nabla_\xi A|^2 + 2g(\phi(\nabla_\xi A)\phi A\xi, \phi A\xi) - |(\nabla_\xi A)\xi|^2 - 2|\phi A\xi|^4,$$

and the equality occurs if and only if the hypersurface is ruled or of type (A).

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $\mathbb{C}M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $\mathbb{C}M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX \tag{2.1}$$

for any vector fields X and Y tangent to M, where A denotes the shape operator of M in $\mathbb{C}M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi. \tag{2.2}$$

On M there is an almost contact metric structure (ϕ, ξ, η, g) defined as follows:

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi) \tag{2.4}$$

for any vector fields X and Y on M. Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla} J = 0$) of $\mathbb{C}M^n(c)$ and using (2.1), (2.2) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any vector fields X and Y. We denote by R the Riemannian curvature tensor of M. Since $\mathbb{C}M^n(c)$ is assumed to be of constant holomorphic sectional curvature c, then the Gauss and Codazzi equations of M in $\mathbb{C}M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.8}$$

for any vector fields X, Y on M.

In this paper, all manifolds are assumed to be connected and of class C^∞.

3. Proofs of main results

In order to obtain a characterization result of ruled hypersurfaces, Y. J. Suh in [10] considered the following condition

$$(\nabla_X A)Y = \Big\{\beta^2 g(X, U)g(Y\phi U) + \beta^2 g(X, \phi U)g(Y, U) - \frac{c}{4}g(\phi X, Y)\Big\}\xi \tag{3.1}$$

for any vector fields X, Y orthogonal to ξ, where U is a unit vector field and is defined by

$$U := \frac{1}{\beta}(A\xi - \eta(A\xi)\xi)$$

on a non-Hopf hypersurface on which there is $\beta \neq 0$ and β denotes the length of $A\xi - \eta(A\xi)\xi$. In order to avoid involving U, (3.1) can be rewritten as

$$(\nabla_X A)Y = \Big\{\eta(AX)g(Y, \phi A\xi) + \eta(AY)g(X, \phi A\xi) - \frac{c}{4}g(\phi X, Y)\Big\}\xi \tag{3.2}$$

for any vector fields X and Y orthogonal to ξ (cf. [5]). It has been proved in [10] that if a real hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$, $n > 2$, satisfies (3.2), then the hypersurface is locally congruent to a real hypersurface of type (A) or a ruled one. Conversely, any ruled hypersurface satisfies (3.2) (cf. [10]). In fact, (3.2) means that the shape operator is η-parallel (namely $g((\nabla_X A)Y, Z) = 0$ for any vector fields X, Y, Z orthogonal to the structure vector field ξ) and therefore any real hypersurface of dimension > 3 satisfying such an equation is of type (A) or ruled (cf. [4, 8, 9]). Very recently, the present author in [12] extended this result from real hypersurfaces of dimension > 3 to dimension three. Based on these analyses, we have:

Lemma 3.1. *On a real hypersurface in nonflat complex space forms $\mathbb{C}M^n(c)$, (3.2) holds if and only if the hypersurface is of type (A) or is ruled.*

Because in general on a ruled hypersurface M there exists a certain singular point p and on this point $|A\xi - \eta(A\xi)\xi|_p = \infty$. M is not smooth on such a subset $\{p \in M : |A\xi - \eta(A\xi)\xi|_p = \infty\}$ (cf. [6, 7]). So in this paper by a ruled hypersurface we consider the open subset of it, i.e., $\{p \in M : |A\xi - \eta(A\xi)\xi|_p \neq 0, \neq \infty\}$.

What follows is to present proofs of all main results. First, applying the tensorial property it is easy to see

$$\begin{aligned}&(\nabla_{X-\eta(X)\xi}A)(Y-\eta(Y)\xi)\\ &= (\nabla_X A)Y - \eta(Y)(\nabla_X A)\xi - \eta(X)(\nabla_\xi A)Y + \eta(X)\eta(Y)(\nabla_\xi A)\xi\end{aligned}$$

for any vector fields X, Y. Suppose (3.2) is true, from the above equation we have

$$\begin{aligned}&(\nabla_X A)Y\\ &= \eta(Y)(\nabla_X A)\xi + \eta(X)(\nabla_\xi A)Y - \eta(X)\eta(Y)(\nabla_\xi A)\xi - \frac{c}{4}g(\phi X, Y)\xi\\ &\quad + (\eta(AX) - \eta(X)\eta(A\xi))g(Y, \phi A\xi)\xi + (\eta(AY) - \eta(Y)\eta(A\xi))g(X, \phi A\xi)\xi\end{aligned}$$

for any vector fields X, Y. By applying the Codazzi equation (2.8) we get

$$(\nabla_X A)\xi = (\nabla_\xi A)X - \frac{c}{4}\phi X \tag{3.3}$$

for any vector field X, which is substituted into the previous relation implying

$$\begin{aligned}&(\nabla_X A)Y\\ &= \eta(X)(\nabla_\xi A)Y + \eta(Y)(\nabla_\xi A)X - \eta(X)\eta(Y)(\nabla_\xi A)\xi - \frac{c}{4}\eta(Y)\phi X - \frac{c}{4}g(\phi X, Y)\xi\\ &\quad + (\eta(AX) - \eta(X)\eta(A\xi))g(Y, \phi A\xi)\xi + (\eta(AY) - \eta(Y)\eta(A\xi))g(X, \phi A\xi)\xi\end{aligned}$$

for any vector fields X, Y. According to Lemma 3.1, the above equation is a necessary and sufficient condition for a real hypersurface to be of type (A) or ruled.

Now on a real hypersurface let us define a tensor field T of $(1, 2)$-type by

$$\begin{aligned}T(X, Y) := {}& (\nabla_X A)Y - \eta(X)(\nabla_\xi A)Y - \eta(Y)(\nabla_\xi A)X + \eta(X)\eta(Y)(\nabla_\xi A)\xi\\ &+ \frac{c}{4}\eta(Y)\phi X + \frac{c}{4}g(\phi X, Y)\xi + (\eta(X)\eta(A\xi) - \eta(AX))g(Y, \phi A\xi)\xi\\ &+ (\eta(Y)\eta(A\xi) - \eta(AY))g(X, \phi A\xi)\xi\end{aligned}$$

for any vector fields X, Y. Note that $\nabla_\xi A$ is symmetric and ϕ is anti-symmetric, then

$$\sum_{i=1}^{2n-1} g((\nabla_\xi A)e_i, \phi e_i) = \sum_{i=1}^{2n-1} g((\nabla_\xi A)\phi e_i, e_i) = 0,$$

where $\{e_1, e_2, \dots, e_{2n-1}\}$ denotes a local orthonormal frame of the tangent space of the hypersurface at each point. With the help of the above two equations, by applying (2.3)-(2.6) together with (3.3) and direct calculations we obtain the following several equations.

$$\sum_{i=1}^{2n-1} \eta(e_i) g((\nabla_{e_i} A)e_j, (\nabla_\xi A)e_j) = \sum_{i=1}^{2n-1} \eta(e_j) g((\nabla_{e_i} A)e_j, (\nabla_\xi A)e_i) = |\nabla_\xi A|^2.$$

$$\sum_{i=1}^{2n-1} \eta(e_j) g((\nabla_{e_i} A)e_j, \phi e_i) = \sum_{i=1}^{2n-1} g(\phi e_i, e_j) \eta((\nabla_{e_i} A)e_j) = -\frac{1}{2}(n-1)c.$$

$$\begin{aligned}&\sum_{i=1}^{2n-1} (\eta(e_i)\eta(A\xi) - \eta(Ae_i)) g(e_j, \phi A\xi) \eta((\nabla_{e_i} A)e_j)\\ &= \eta(A\xi) g(\phi A\xi, (\nabla_\xi A)\xi) - g(A\xi, (\nabla_\xi A)\phi A\xi) + \frac{1}{4}|\phi A\xi|^2 c.\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{2n-1} (\eta(e_j)\eta(A\xi) - \eta(Ae_j)) g(e_i, \phi A\xi) \eta((\nabla_{e_i} A)e_j)\\ &= \eta(A\xi) g(\phi A\xi, (\nabla_\xi A)\xi) - g(A\xi, (\nabla_\xi A)\phi A\xi) - \frac{1}{4}|\phi A\xi|^2 c.\end{aligned}$$

$$\sum_{i=1}^{2n-1} g(\phi e_i, e_j)(\eta(e_i)\eta(A\xi) - \eta(Ae_i)) g(e_j, \phi A\xi) = -|\phi A\xi|^2.$$

$$\sum_{i=1}^{2n-1} g(\phi e_i, e_j)(\eta(e_j)\eta(A\xi) - \eta(Ae_j)) g(e_i, \phi A\xi) = |\phi A\xi|^2.$$

Finally, with the aid of the above equations, by the definition of T and the symmetry of $\nabla_\xi A$ we have

$$\begin{aligned}|T|^2 = |\nabla A|^2 &- \frac{1}{4}(n-1)c^2 + |(\nabla_\xi A)\xi|^2 + 2|\phi A\xi|^4 - 2|\nabla_\xi A|^2\\ &+ 2\eta(A\xi) g(\phi A\xi, (\nabla_\xi A)\xi) - 2g(\phi A\xi, (\nabla_\xi A)A\xi).\end{aligned}$$

Moreover, by simplification it is proved that on any real hypersurface we always have

$$|\nabla A|^2 \geq \frac{1}{4}(n-1)c^2 + 2|\nabla_\xi A|^2 - |(\nabla_\xi A)\xi|^2 - 2|\phi A\xi|^4 + 2g(\phi(\nabla_\xi A)\phi A\xi, \phi A\xi).$$

If the equality sign in the above inequality holds, then we have $T \equiv 0$. By Lemma 3.1 we conclude that it is a necessary and sufficient condition for a hypersurface to be of type (A) or ruled. This completes the proof of Corollary 1.6 and Theorem 1.4.

As claimed before, the Hopf principal curvature of a Hopf hypersurface is always a constant (cf. [8, Corollary 2.6]). Therefore, on any Hopf hypersurface we always have $(\nabla_\xi A)\xi = 0$ and hence the above inequality reduces to

$$|\nabla A|^2 \geq \frac{1}{4}(n-1)c^2 + 2|\nabla_\xi A|^2.$$

If the equality sign in the inequality relation occurs, by the definition of T we obtain

$$(\nabla_X A)Y = \eta(X)(\nabla_\xi A)Y + \eta(Y)(\nabla_\xi A)X - \frac{c}{4}\eta(Y)\phi X - \frac{c}{4}g(\phi X, Y)\xi \tag{3.4}$$

for any vector fields X, Y. Obviously, this means that the shape operator is η-parallel and hence the Hopf hypersurface satisfying (3.4) must be of type (A) (cf. [4, 8, 9]). The converse is trivial. This completes the proof of Theorem 1.4.

Remark 3.2. Our Theorem 1.5 can be regarded as an extension of Theorem 1.3. In fact, according to Theorem 8.120 in [2], the shape operator A of a type (A) hypersurface is always Reeb parallel, i.e., $\nabla_\xi A = 0$. Then the two inequalities in Theorems 1.3 and 1.5 are the same for type (A) hypersurfaces. In this case, by (3.4) the shape operator satisfies

$$(\nabla_X A)Y = -\frac{c}{4}\eta(Y)\phi X - \frac{c}{4}g(\phi X, Y)\xi$$

for any vector fields X, Y.

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