Bull. Korean Math. Soc. **59** (2022), No. 4, pp. 801–810
https://doi.org/10.4134/BKMS.b200606
pISSN: 1015-8634 / eISSN: 2234-3016

INTEGRABILITY OF AN ALMOST COMPLEX STRUCTURE ON $S^4 \times_f V^2$

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Abstract. In this paper, we prove that any orthogonal almost complex structure on a warped product manifold of any oriented closed surface and a round 4-sphere for a concircular warping function on the sphere is never integrable. This gives a partial answer to Calabi's problem.

1. Introduction

In [2], Calabi raised the problem concerning the integrability condition of an almost complex structure on a 6-dimensional almost complex manifold. One of the questions that he left is whether the product manifold $V^2 \times S^4$ ($V^2 =$ any oriented closed surface) admits an integrable almost complex structure or not. In previous works [4, 5], Euh and Sekigawa gave some partial answers to this problem.

Theorem 1.1 ([4]). *Any orthogonal almost complex structure on a Riemannian product of a round* 2*-sphere and a round* 4*-sphere is never integrable.*

Theorem 1.2 ([5]). *Let $V^2 \times_f S^4$ be a warped product Riemannian manifold of an oriented closed surface V^2 with nonnegative Gaussian curvature and a round* 4*-sphere S^4, where f is a positive-valued smooth function on V^2. Then, any orthogonal almost complex structure on $V^2 \times_f S^4$ is never integrable.*

Let M be an n-dimensional Riemannian manifold with metric tensor g. Tashiro [8] introduced the concept of a *concircular function*: A function $f : M \to \mathbb{R}$ is called concircular if the Hessian of f, Hess f, satisfies

$$\text{Hess } f(X, Y) = \phi g(X, Y)$$

Received July 15, 2020; Revised January 3, 2022; Accepted May 9, 2022.
2020 *Mathematics Subject Classification.* 53C15, 53C21, 53C55.
Key words and phrases. Orthogonal complex structure, warped product, concircular function.
This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2019R1F1A1040829).
S. H. Chun was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018R1D1A1B07045729).

for any smooth vector fields X and Y on M, where ϕ is a function on M which is called the *characteristic function* of f. A concircular transformation is by definition a conformal transformation preserving geodesic circles. Due to Tashiro's result ([8], Theorem 1), if a complete Riemannian manifold M of dimension $n \geq 2$ admits a concircular function f, then the number N of isolated stationary points of f is less than or equal to 2, and M is conformal to a spherical space form when $N = 2$. In fact, Chen found a concircular function on an n-dimensional unit sphere $S^n(1)$ in terms of isothermal coordinates ([3], Proposition 1.10). So it is natural to consider a concircular function on a sphere.

In this paper, we deal with the warped product manifold $S^4 \times_f V^2$, where the function f is a positive-valued concircular function on S^4. The main purpose is to prove the following theorem.

Theorem 1.3. *Let $S^4 \times_f V^2$ be a warped product Riemannian manifold of a round 4-sphere S^4 and an oriented closed surface V^2 with nonnegative Gaussian curvature, where f is a positive-valued concircular function on S^4. Then, any orthogonal almost complex structure on $S^4 \times_f V^2$ is never integrable.*

Theorem 1.3 makes a progress for solving Calabi's problem completely.

2. Preliminaries

Let $M = (M, g)$ be an n-dimensional Riemannian manifold. Denote by $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on M. For any smooth function h on M, the gradient of h, grad h, is the vector field defined by $g(\text{grad } h, X) = Xh$ for any $X \in \mathfrak{X}(M)$. The hessian of h, Hess h, is defined by Hess $h(X,Y) = g(\nabla_X \text{grad } h, Y)$ for any X, $Y \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of g. Then we see that the trace of Hess $h = \triangle h$, where $\triangle h$ is the Laplacian of h.

Let (B, g_B) and (F, g_F) be Riemannian manifolds and f be a positive-valued smooth function on B. By definition, a warped product Riemannian manifold $(M, g) = (B, g_B) \times_f (F, g_F)$, briefly, $B \times_f F$, is the product manifold $M = B \times F$ equipped with the Riemannian metric g given by $g = g_B + f^2 g_F$. We denote by ∇^B and ∇^F the Levi-Civita connections of g_B and g_F, respectively. Then, we see that the following relations hold ([1], Lemma 7.3):

$$\nabla_X Y = \nabla^B_X Y, \tag{2.1}$$

$$\nabla_U X = \frac{1}{f} X f U = \frac{1}{f} g_B(\text{grad}^B f, X) U, \tag{2.2}$$

$$\nabla_X U = \frac{1}{f} X f U = \frac{1}{f} g_B(\text{grad}^B f, X) U, \tag{2.3}$$

$$\nabla_U V = \nabla^F_U V - f g_F(U, V) \text{grad}^B f \tag{2.4}$$

for X, $Y \in \mathfrak{X}(B)$ and U, $V \in \mathfrak{X}(F)$. We denote the curvature tensors of (M, g), (B, g_B) and (F, g_F) by R, R^B and R^F defined by

$$R(\bar{X}, \bar{Y})\bar{Z} = [\nabla_{\bar{X}}, \nabla_{\bar{Y}}]\bar{Z} - \nabla_{[\bar{X}, \bar{Y}]}\bar{Z}, \tag{2.5}$$

$$R^B(X, Y)Z = [\nabla^B_X, \nabla^B_Y]Z - \nabla^B_{[X,Y]}Z, \tag{2.6}$$

$$R^F(U, V)W = [\nabla^F_U, \nabla^F_V]W - \nabla^F_{[U,V]}W \tag{2.7}$$

for $X, Y, Z \in \mathfrak{X}(B)$, $U, V, W \in \mathfrak{X}(F)$ and $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(M)$. Then, from (2.1)∼(2.7), we have

$$R(X, Y)Z = R^B(X, Y)Z, \tag{2.8}$$

$$R(X, Y)U = 0, \tag{2.9}$$

$$R(X, U)Y = \frac{1}{f}\,\mathrm{Hess}^B\, f(X, Y)U, \tag{2.10}$$

$$R(U, V)X = 0, \tag{2.11}$$

$$R(U, V)W = R^F(U, V)W - |\,\mathrm{grad}^B f|^2_B\big(g_F(V, W)U - g_F(U, W)V\big) \tag{2.12}$$

for $X, Y, Z \in \mathfrak{X}(B)$ and $U, V, W \in \mathfrak{X}(F)$ ([1], Lemma 7.4). From (2.8)∼(2.12), we have further

$$R(X, Y, Z, Z') = R^B(X, Y, Z, Z'), \tag{2.13}$$

$$R(X, Y, Z, U) = 0, \tag{2.14}$$

$$R(X, Y, U, V) = 0, \tag{2.15}$$

$$R(X, U, Y, V) = f\mathrm{Hess}^B\, f(X, Y)g_F(U, V), \tag{2.16}$$

$$R(U, V, W, X) = 0, \tag{2.17}$$

$$\begin{aligned} R(U, V, W, W') = f^2\Big\{ & R^F(U, V, W, W') \\ & - |\mathrm{grad}^B f|^2_B\big(g_F(V, W)g_F(U, W') \\ & - g_F(U, W)g_F(V, W')\big)\Big\} \end{aligned} \tag{2.18}$$

for $X, Y, Z, Z' \in \mathfrak{X}(B)$ and $U, V, W, W' \in \mathfrak{X}(F)$.

3. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3 by making use of the fundamental formulas prepared in Section 2. In the sequel, we assume that $(B, g_B) = (S^4(\alpha), g_{S^4(\alpha)})$ and $(F, g_F) = (V^2, g_{V^2})$, where $(S^4(\alpha), g_{S^4(\alpha)})$ is a round 4-sphere of constant sectional curvature α and (V^2, g_{V^2}) is an oriented closed surface with nonnegative Gaussian curvature β and further $(M, g) = (S^4(\alpha), g_{S^4(\alpha)}) \times_f (V^2, g_{V^2})$, where f is a positive-valued concircular function on S^4. First, we recall Gray's result [6] which plays an essential role in the proof of Theorem 1.3.

Lemma 3.1. *Let $M = (M, J, g)$ be a Hermitian manifold. Then, we have*

$$R(\bar{X},\bar{Y},\bar{Z},\bar{W}) + R(J\bar{X},J\bar{Y},J\bar{Z},J\bar{W}) - R(J\bar{X},J\bar{Y},\bar{Z},\bar{W}) - R(J\bar{X},\bar{Y},J\bar{Z},\bar{W}) - R(J\bar{X},\bar{Y},\bar{Z},J\bar{W}) - R(\bar{X},J\bar{Y},J\bar{Z},\bar{W}) - R(\bar{X},J\bar{Y},\bar{Z},J\bar{W}) - R(\bar{X},\bar{Y},J\bar{Z},J\bar{W}) = 0$$

for $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W} \in \mathfrak{X}(M)$.

It is known that $M = S^4(\alpha)\times V^2$ admits an almost complex structure [2,7]. Let J be an orthogonal almost complex structure on (M, g). We may identify $T_{(p_1,p_2)}(S^4(\alpha) \times V^2)$ with $T_{p_1}S^4(\alpha) \oplus T_{p_2}V^2$ for each point $p = (p_1, p_2) \in S^4(\alpha) \times V^2$ in the natural way. Let $\{e_i\}_{1\le i\le 6}$ be a local orthonormal frame field on (M, g) such that $\{e_1, e_2, e_3, e_4\}$ and $\{e_5, e_6\}$ are tangent to $S^4(\alpha)$ and V^2, respectively. We here set

$$Je_a = \sum_b J_{ab}e_b + \sum_v J_{av}e_v, \quad Je_u = \sum_b J_{ub}e_b + \sum_v J_{uv}e_v \tag{3.1}$$

for $1 \le a, b \le 4$ and $5 \le u, v \le 6$. Then, we may easily check that the following equalities hold:

$$J_{ij} = -J_{ji}, \quad \sum_{k=1}^{6} J_{ik}J_{jk} = \delta_{ij} \tag{3.2}$$

for $1 \le i, j \le 6$. Then, from (2.13)∼(2.18), taking account of (3.1) and (3.2), we have

$$R(e_5, e_6, e_5, e_6) = -\frac{1}{f^2}\left(\beta - |\mathrm{grad}^{S^4} f|^2_{S^4}\right), \tag{3.3}$$

$$\begin{aligned}
&R(Je_5, Je_6, Je_5, Je_6) \\
&= R\big(\sum_a J_{5a}e_a + \sum_u J_{5u}e_u, \sum_b J_{6b}e_b + \sum_v J_{6v}e_v, \\
&\qquad \sum_c J_{5c}e_c + \sum_w J_{5w}e_w, \sum_d J_{6d}e_d + \sum_z J_{6z}e_z\big) \\
&= \sum_{a,b,c,d} J_{5a}J_{6b}J_{5c}J_{6d}R(e_a, e_b, e_c, e_d) + \sum_{a,v,c,z} J_{5a}J_{6v}J_{5c}J_{6z}R(e_a, e_v, e_c, e_z) \\
&\quad + \sum_{a,v,w,d} J_{5a}J_{6v}J_{5w}J_{6d}R(e_a, e_v, e_w, e_d) + \sum_{u,b,c,z} J_{5u}J_{6b}J_{5c}J_{6z}R(e_u, e_b, e_c, e_z) \\
&\quad + \sum_{u,b,w,d} J_{5u}J_{6b}J_{5w}J_{6d}R(e_u, e_b, e_w, e_d) + \sum_{u,v,w,z} J_{5u}J_{6v}J_{5w}J_{6z}R(e_u, e_v, e_w, e_z) \\
&= \sum_{a,b,c,d} J_{5a}J_{6b}J_{5c}J_{6d}R^{S^4}(e_a, e_b, e_c, e_d) + \frac{1}{f}\sum_{a,v,c,z} J_{5a}J_{6v}J_{5c}J_{6z}\mathrm{Hess}^{S^4} f(e_a, e_c)\delta_{vz} \\
&\quad + \frac{1}{f}\sum_{a,v,w,d} J_{5a}J_{6v}J_{5w}J_{6d}\mathrm{Hess}^{S^4} f(e_a, e_d)\delta_{vw}
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&- \frac{1}{f} \sum_{u,b,c,z} J_{5u} J_{6b} J_{5c} J_{6z} \mathrm{Hess}^{S^4} f(e_b, e_c) \delta_{uz} \\
&+ \frac{1}{f} \sum_{u,b,w,d} J_{5u} J_{6b} J_{5w} J_{6d} \mathrm{Hess}^{S^4} f(e_d, e_b) \delta_{uw} \\
&+ \sum_{u,v,w,z} J_{5u} J_{6v} J_{5w} J_{6z} \big(g(R^{V^2}(e_u, e_v) e_w, e_z) \\
&\qquad - \frac{1}{f^2} |\operatorname{grad}^{S^4} f|^2_{S^4} g(\delta_{vw} e_u - \delta_{uw} e_v, e_z) \big) \\
= &- \alpha (1 - J_{56}^2)^2 \\
&+ \frac{1}{f} J_{56}^2 \Big\{ J_{15}^2 \mathrm{Hess}^{S^4} f(e_1, e_1) + J_{25}^2 \mathrm{Hess}^{S^4} f(e_2, e_2) \\
&\qquad + J_{35}^2 \mathrm{Hess}^{S^4} f(e_3, e_3) + J_{45}^2 \mathrm{Hess}^{S^4} f(e_4, e_4) \\
&\qquad + 2 \big(J_{15} J_{25} \mathrm{Hess}^{S^4} f(e_1, e_2) + J_{15} J_{35} \mathrm{Hess}^{S^4} f(e_1, e_3) \\
&\qquad\quad + J_{15} J_{45} \mathrm{Hess}^{S^4} f(e_1, e_4) + J_{25} J_{35} \mathrm{Hess}^{S^4} f(e_2, e_3) \\
&\qquad\quad + J_{25} J_{45} \mathrm{Hess}^{S^4} f(e_2, e_4) + J_{35} J_{45} \mathrm{Hess}^{S^4} f(e_3, e_4) \big) \Big\} \\
&+ \frac{1}{f} J_{56}^2 \Big\{ J_{16}^2 \mathrm{Hess}^{S^4} f(e_1, e_1) + J_{26}^2 \mathrm{Hess}^{S^4} f(e_2, e_2) \\
&\qquad + J_{36}^2 \mathrm{Hess}^{S^4} f(e_3, e_3) + J_{46}^2 \mathrm{Hess}^{S^4} f(e_4, e_4) \\
&\qquad + 2 \big(J_{16} J_{26} \mathrm{Hess}^{S^4} f(e_1, e_2) + J_{16} J_{36} \mathrm{Hess}^{S^4} f(e_1, e_3) \\
&\qquad\quad + J_{16} J_{46} \mathrm{Hess}^{S^4} f(e_1, e_4) + J_{26} J_{36} \mathrm{Hess}^{S^4} f(e_2, e_3) \\
&\qquad\quad + J_{26} J_{46} \mathrm{Hess}^{S^4} f(e_2, e_4) + J_{36} J_{46} \mathrm{Hess}^{S^4} f(e_3, e_4) \big) \Big\} \\
&- \frac{1}{f^2} J_{56}^4 (\beta - |\operatorname{grad}^{S^4} f|^2_{S^4}),
\end{aligned}
$$

$$
\begin{aligned}
(3.5) \qquad & R(Je_5, Je_6, e_5, e_6) \\
&= R\big(\sum_a J_{5a} e_a + \sum_u J_{5u} e_u, \sum_b J_{6b} e_b + \sum_v J_{6v} e_v, e_5, e_6 \big) \\
&= \sum_{a,b} J_{5a} J_{6b} R(e_a, e_b, e_5, e_6) + \sum_{a,v} J_{5a} J_{6v} R(e_a, e_v, e_5, e_6) \\
&\quad + \sum_{b,u} J_{5u} J_{6b} R(e_u, e_b, e_5, e_6) + \sum_{u,v} J_{5u} J_{6v} R(e_u, e_v, e_5, e_6) \\
&= \sum_{u,v} J_{5u} J_{6v} R(e_u, e_v, e_5, e_6) \\
&= - \frac{1}{f^2} J_{56}^2 (\beta - |\operatorname{grad}^{S^4} f|^2_{S^4}),
\end{aligned}
$$

$$
\begin{aligned}
(3.6)\qquad & R(Je_5, e_6, Je_5, e_6) \\
&= R\big(\sum_a J_{5a}e_a + \sum_u J_{5u}e_u, e_6, \sum_b J_{5b}e_b + \sum_v J_{5v}e_v, e_6\big) \\
&= \sum_{a,b} J_{5a}J_{5b}R(e_a, e_6, e_b, e_6) + \sum_{a,v} J_{5a}J_{5v}R(e_a, e_6, e_v, e_6) \\
&\quad + \sum_{b,u} J_{5u}J_{5b}R(e_u, e_6, e_b, e_6) + \sum_{u,v} J_{5u}J_{5v}R(e_u, e_6, e_v, e_6) \\
&= \sum_{a,b} J_{5a}J_{5b}R(e_a, e_6, e_b, e_6) \\
&= \frac{1}{f}\Big\{J_{15}^2 \mathrm{Hess}^{S^4} f(e_1, e_1) + J_{25}^2 \mathrm{Hess}^{S^4} f(e_2, e_2) \\
&\qquad + J_{35}^2 \mathrm{Hess}^{S^4} f(e_3, e_3) + J_{45}^2 \mathrm{Hess}^{S^4} f(e_4, e_4) \\
&\qquad + 2\big(J_{15}J_{25}\mathrm{Hess}^{S^4} f(e_1, e_2) + J_{15}J_{35}\mathrm{Hess}^{S^4} f(e_1, e_3) \\
&\qquad\quad + J_{15}J_{45}\mathrm{Hess}^{S^4} f(e_1, e_4) + J_{25}J_{35}\mathrm{Hess}^{S^4} f(e_2, e_3) \\
&\qquad\quad + J_{25}J_{45}\mathrm{Hess}^{S^4} f(e_2, e_4) + J_{35}J_{45}\mathrm{Hess}^{S^4} f(e_3, e_4)\big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(3.7)\qquad & R(Je_5, e_6, e_5, Je_6) \\
&= R\big(\sum_a J_{5a}e_a + \sum_u J_{5u}e_u, e_6, e_5, \sum_b J_{6b}e_b + \sum_v J_{6v}e_v\big) \\
&= \sum_{a,b} J_{5a}J_{6b}R(e_a, e_6, e_5, e_b) + \sum_{a,v} J_{5a}J_{6v}R(e_a, e_6, e_5, e_v) \\
&\quad + \sum_{b,u} J_{5u}J_{6b}R(e_u, e_6, e_5, e_b) + \sum_{u,v} J_{5u}J_{6v}R(e_u, e_6, e_5, e_v) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
(3.8)\qquad & R(e_5, Je_6, e_5, Je_6) \\
&= R\big(e_5, \sum_a J_{6a}e_a + \sum_u J_{6u}e_u, e_5, \sum_b J_{6b}e_b + \sum_v J_{6v}e_v\big) \\
&= \sum_{a,b} J_{6a}J_{6b}R(e_5, e_a, e_5, e_b) + \sum_{a,v} J_{6a}J_{6v}R(e_5, e_a, e_5, e_v) \\
&\quad + \sum_{b,u} J_{6b}J_{6u}R(e_5, e_u, e_5, e_b) + \sum_{u,v} J_{6u}J_{6v}R(e_5, e_u, e_5, e_v) \\
&= \sum_{a,b} J_{6a}J_{6b}R(e_5, e_a, e_5, e_b) \\
&= \frac{1}{f}\Big\{J_{16}^2 \mathrm{Hess}^{S^4} f(e_1, e_1) + J_{26}^2 \mathrm{Hess}^{S^4} f(e_2, e_2)
\end{aligned}
$$

$$+ J_{36}^2 \mathrm{Hess}^{S^4} f(e_3, e_3) + J_{46}^2 \mathrm{Hess}^{S^4} f(e_4, e_4)$$
$$+ 2\big(J_{16}J_{26}\mathrm{Hess}^{S^4} f(e_1, e_2) + J_{16}J_{36}\mathrm{Hess}^{S^4} f(e_1, e_3)$$
$$+ J_{16}J_{46}\mathrm{Hess}^{S^4} f(e_1, e_4) + J_{26}J_{36}\mathrm{Hess}^{S^4} f(e_2, e_3)$$
$$+ J_{26}J_{46}\mathrm{Hess}^{S^4} f(e_2, e_4) + J_{36}J_{46}\mathrm{Hess}^{S^4} f(e_3, e_4)\big)\Big\}$$

for $1 \leq a, b, c, d \leq 4$ and $5 \leq u, v, w, z \leq 6$. Here, since f is a concircular function on $S^4(\alpha)$, we have Hess $f(e_a, e_b) = \phi(p_1)\delta_{ab}$ for a point $p_1 \in S^4(\alpha)$, where δ_{ab} denotes the Kronecker delta and ϕ is the characteristic function of f on $S^4(\alpha)$. Thus, from (3.3)$\sim$(3.8) and Lemma 3.1, we have

$$
\begin{aligned}
0 &= R(e_5, e_6, e_5, e_6) + R(Je_5, Je_6, Je_5, Je_6) - 2R(Je_5, Je_6, e_5, e_6) \\
&\quad - R(Je_5, e_6, Je_5, e_6) - 2R(Je_5, e_6, e_5, Je_6) - R(e_5, Je_6, e_5, Je_6) \\
&= -\frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2) - \frac{1}{f^2}J_{56}^4(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2) \\
&\quad + \frac{2}{f^2}J_{56}^2(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2) - \frac{1}{f}(1 - J_{56}^2)\Big[(J_{15}^2 + J_{16}^2)\mathrm{Hess}^{S^4} f(e_1, e_1) \\
&\quad + (J_{25}^2 + J_{26}^2)\mathrm{Hess}^{S^4} f(e_2, e_2) + (J_{35}^2 + J_{36}^2)\mathrm{Hess}^{S^4} f(e_3, e_3) \\
&\quad + (J_{45}^2 + J_{46}^2)\mathrm{Hess}^{S^4} f(e_4, e_4) + 2\big\{(J_{15}J_{25} + J_{16}J_{26})\mathrm{Hess}^{S^4} f(e_1, e_2) \\
&\quad + (J_{15}J_{35} + J_{16}J_{36})\mathrm{Hess}^{S^4} f(e_1, e_3) + (J_{15}J_{45} + J_{16}J_{46})\mathrm{Hess}^{S^4} f(e_1, e_4) \\
&\quad + (J_{25}J_{35} + J_{26}J_{36})\mathrm{Hess}^{S^4} f(e_2, e_3) + (J_{25}J_{45} + J_{26}J_{46})\mathrm{Hess}^{S^4} f(e_2, e_4) \\
&\quad + (J_{35}J_{45} + J_{36}J_{46})\mathrm{Hess}^{S^4} f(e_3, e_4)\big\}\Big] \\
&\quad - \alpha(1 - J_{56}^2)^2 \\
&= -(1 - J_{56}^2)^2\{\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2)\} \\
&\quad - \frac{1}{f}(1 - J_{56}^2)\Big[(J_{15}^2 + J_{16}^2)\mathrm{Hess}^{S^4} f(e_1, e_1) + (J_{25}^2 + J_{26}^2)\mathrm{Hess}^{S^4} f(e_2, e_2) \\
&\quad + (J_{35}^2 + J_{36}^2)\mathrm{Hess}^{S^4} f(e_3, e_3) + (J_{45}^2 + J_{46}^2)\mathrm{Hess}^{S^4} f_(e_4, e_4) \\
&\quad + 2\big\{(J_{15}J_{25} + J_{16}J_{26})\mathrm{Hess}^{S^4} f(e_1, e_2) + (J_{15}J_{35} + J_{16}J_{36})\mathrm{Hess}^{S^4} f(e_1, e_3) \\
&\quad + (J_{15}J_{45} + J_{16}J_{46})\mathrm{Hess}^{S^4} f(e_1, e_4) + (J_{25}J_{35} + J_{26}J_{36})\mathrm{Hess}^{S^4} f(e_2, e_3) \\
&\quad + (J_{25}J_{45} + J_{26}J_{46})\mathrm{Hess}^{S^4} f(e_2, e_4) + (J_{35}J_{45} + J_{36}J_{46})\mathrm{Hess}^{S^4} f(e_3, e_4)\big\}\Big] \\
&= -(1 - J_{56}^2)^2\{\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2)\} - \frac{1}{f}(1 - J_{56}^2)\{(J_{15}^2 + J_{16}^2)\phi(p_1) \\
&\quad + (J_{25}^2 + J_{26}^2)\phi(p_1) + (J_{35}^2 + J_{36}^2)\phi(p_1) + (J_{45}^2 + J_{46}^2)\phi(p_1)\} \\
&= -(1 - J_{56}^2)^2\{\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2)\}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{f}(1 - J_{56}^2)\{(J_{15}^2 + J_{25}^2 + J_{35}^2 + J_{45}^2)\phi(p_1) \\
&\quad + (J_{16}^2 + J_{26}^2 + J_{36}^2 + J_{46}^2)\phi(p_1)\} \\
&= -(1 - J_{56}^2)^2\{\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2)\} \\
&\quad - \frac{1}{f}(1 - J_{56}^2)\{(1 - J_{56}^2)\phi(p_1) + (1 - J_{56}^2)\phi(p_1)\} \\
&= -(1 - J_{56}^2)^2\big\{\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2) + \frac{2}{f}\phi(p_1)\big\}.
\end{aligned}
$$

Since f is the concircular function and the Laplacian of f is the trace of Hessian of f, we have $\triangle^{S^4} f = \sum_{i=1}^{4} \mathrm{Hess}^{S^4} f(e_i, e_i) = 4\phi(p_1)$. Therefore we have the following equation:

$$
0 = -(1 - J_{56}^2)^2\{\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2) + \frac{1}{2f}\triangle^{S^4} f\}. \tag{3.9}
$$

Furthermore, since $S^4(\alpha)$ is compact, f takes its minimum at p_1 and hence, $\mathrm{grad}^{S^4} f = 0$ and $\triangle^{S^4} f \geq 0$ at the point p_1. Thus, from (3.9), we have

$$
\alpha + \frac{1}{f^2}(\beta - |\mathrm{grad}^{S^4} f|_{S^4}^2) + \frac{1}{2f}\triangle^{S^4} f = \alpha + \frac{\beta}{f^2} + \frac{1}{2f}\triangle^{S^4} f > 0 \tag{3.10}
$$

along $\{p_1\} \times V^2$. Thus, from (3.9) and (3.10), we see that $J_{56}^2 = 1$ holds along $\{p_1\} \times V^2$ with respect to any local orthonormal frame field $\{e_i\}$ such that $\{e_1, e_2, e_3, e_4\}$ and $\{e_5, e_6\}$ are tangent to $S^4(\alpha)$ and V^2, respectively. This means that the subspace $T_{p_2}V^2$ of $T_{(p_1,p_2)}M$ for any $p_2 \in V^2$ is J-invariant, and hence the subspace $T_{p_1}S^4(\alpha)$ of $T_{(p_1,p_2)}M$ is also J-invariant for any $p_1 \in S^4(\alpha)$. But, since among the round spheres S^{2n}, only S^2 and S^6 admit orthogonal almost complex structures [7], this is impossible. This completes the proof of Theorem 1.3. □

4. Warped product manifolds of surfaces

In this section, we study the integrability of orthogonal almost complex structures on warped product manifolds of surfaces.

Lemma 4.1. *Let $(M, g) = N_1 \times_f N_2$ be a warped product manifold of an oriented closed 2-dimensional Riemannian manifold (N_1, g_1) with nonnegative Gaussian curvature and an oriented 2-dimensional Riemannian manifold (N_2, g_2) with strictly positive Gaussian curvature, and let J be a Hermitian structure on M. Then the subspaces T_pN_1 and T_qN_2 of $T_{(p,q)}M$ are J-invariant, respectively.*

We omit the proof of Lemma 4.1, instead of it, we refer to the proof of Theorem 1.2 in [5]. Now we prove the following theorem.

Theorem 4.2. *Let $(M, g) = N_1 \times_f N_2$ be a warped product manifold of an oriented closed 2-dimensional Riemannian manifold (N_1, g_1) with nonnegative Gaussian curvature and an oriented 2-dimensional Riemannian manifold (N_2, g_2) with strictly positive Gaussian curvature, and let J be an orthogonal almost complex structure on M. Then (M, J, g) is a Kähler manifold if and only if f is a constant function on N_1.*

Proof. First, recalling that every almost complex structure on an oriented 2-dimensional manifold is integrable, let J_1 and J_2 be the naturally induced complex structure on (N_1, g_1) and (N_2, g_2), respectively. Then by Lemma 4.1 we find that $J|_{N_1} = J_1$ and $J|_{N_2} = J_2$. Moreover, since every Riemannian metric on an oriented 2-dimensional Riemannian manifold is a Kähler metric with respect to the induced complex structure, using (2.1) $\sim$ (2.4) we compute

$$\begin{aligned} (\nabla_X J)Y &= \nabla_X JY - J(\nabla_X Y) \\ &= \nabla^{N_1}_X J_1 Y - J\nabla^{N_1}_X Y \\ &= J_1 \nabla_X Y - J_1 \nabla_X Y = 0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} (\nabla_X J)V &= \nabla_X JV - J(\nabla_X V) \\ &= \nabla_X J_2 V - J_2 \nabla_X V \\ &= \frac{1}{f} g_1(\mathrm{grad}^{N_1} f, X) J_2 V - \frac{1}{f} g_1(\mathrm{grad}^{N_1} f, X) J_2 V = 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} (\nabla_V J)X &= \nabla_V JX - J(\nabla_V X) \\ &= \nabla_V J_1 X - J_2 \nabla_V X \\ &= \frac{1}{f} g_1(\mathrm{grad}^{N_1} f, J_1 X)V - \frac{1}{f} g_1(\mathrm{grad}^{N_1} f, X) J_2 V, \end{aligned} \tag{4.3}$$

$$\begin{aligned} (\nabla_V J)W &= \nabla_V JW - J(\nabla_V W) \\ &= \nabla_V J_2 W - J(\nabla_V W) \\ &= \nabla^{N_2}_V J_2 W - f g_1(V, J_2 W)\mathrm{grad}^{N_1} f - J_2 \nabla^{N_2}_V W \\ &\quad + f g_2(V, W) J_1 \mathrm{grad}^{N_1} f \\ &= -f g_2(V, J_2 W)\mathrm{grad}^{N_1} f + f g_2(V, W) J_1 \mathrm{grad}^{N_1} f \end{aligned} \tag{4.4}$$

for $X, Y \in X(N_1)$ and $V, W \in X(N_2)$. Suppose that (J, g) is a Kähler structure. Then, from (4.4), we have $\mathrm{grad}^{N_1} f = 0$, which yields f is a constant function. Conversely, we see that $N_1 \times_c N_2$ is a Kähler manifold. □

Corollary 4.3. *Let $(M, g) = N_1 \times_f S^2$ be a warped product of an oriented closed 2-dimensional Riemannian manifold (N_1, g_1) with nonnegative Gaussian curvature and a round sphere, and let J be a Hermitian structure with respect to g. Then (M, J, g) is a Kähler manifold if and only if f is a constant function on N_1.*

Acknowledgments. The authors would like to thank the referee for carefully reading the manuscript and for valuable comments to improve it.

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