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INTEGRAL OPERATORS ON CESÀRO FUNCTION SPACES

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Abstract. This paper studies the boundedness of integral operators on the Cesàro function spaces. As applications of the main result, we obtain the Hilbert inequalities, the boundedness of the Erdélyi-Kober fractional integrals and the Mellin fractional integrals on the Cesàro function spaces.

1. Introduction

This paper studies the boundedness of integral operators on the Cesàro function spaces on $[0, \infty)$.

The Cesàro function space is one of the most important examples of function spaces, and it is the extension of the Cesàro sequence spaces to $[0, \infty)$ while the Cesàro sequence spaces appeared explicitly in 1968 when the Dutch Mathematical Society posted the problem to find a representation of their duals [3, p. 14]. For the studies of duality, the geometry and the other issues of the Cesàro function spaces, the reader is referred to [1–8, 14, 16, 17, 27, 30–32, 39, 41].

In view of the definition of the Cesàro function spaces, Hardy's inequality on the Cesàro function spaces is a consequence of Hardy's inequality on Lebesgue spaces. Thus, it motivates us to investigate the validity of the other famous inequality on the Cesàro function spaces, such as the Hilbert inequality.

We establish a general result for the boundedness of integral operators on the Cesàro function spaces by the mapping properties of dilation operators and the Minkowski inequalities on the Cesàro function spaces.

As consequences of the main result on the boundedness of integral operators, we extend the Hilbert inequalities, the boundedness of the Erdélyi-Kober fractional integrals and the Mellin fractional integrals to the Cesàro function spaces on $[0, \infty)$.

The Cesàro function spaces have been generalized to the Cesàro second order function spaces in [9, 24]. We also extend the Hilbert inequalities, the boundedness of the Erdélyi-Kober fractional integrals and the Mellin fractional integrals to the Cesàro second order function spaces on $[0, \infty)$.

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This paper is organized as follows. The main result on the boundedness of integral operators on the Cesàro function spaces is established in Section 2. the applications of the main result on the Hilbert inequalities, the Erdélyi-Kober fractional integrals and the Mellin fractional integrals are presented in Section 3.

2. Integral operators

Let $\mathcal{M}$ be the set of Lebesgue measurable functions on $[0,\infty)$.

Definition 1. Let $p \in [1,\infty)$. The Cesàro function space $Ces_p[0,\infty)$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{Ces_p[0,\infty)} = \left(\int_0^\infty \left(\frac{1}{x}\int_0^x |f(t)|dt\right)^p dx\right)^{1/p} < \infty.$$

The Cesàro second order function space $Ces_p^2[0,\infty)$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{Ces_p^2[0,\infty)} = \left(\int_0^\infty \left(\frac{1}{x^2}\int_0^x |(2x-t)f(t)|dt\right)^p dx\right)^{1/p} < \infty.$$

The classical Hardy inequality yields the embedding $L^p[0,\infty) \hookrightarrow Ces_p[0,\infty)$. In addition, according to [3, Theorem 1], $Ces_p[0,\infty)$ is not rearrangement-invariant nor reflexive. Thus, $L^p(0,\infty) \neq Ces_p[0,\infty)$.

The Cesàro second order function space subset of $Ces_p[0,\infty)$ [9, Proposition 1] and for any $f \in Ces_p^2[0,\infty)$, f is locally integrable [9, Lemma 1]. In addition, $Ces_p^2[0,\infty)$ is also not rearrangement-invariant nor reflexive [9, Theorem 7].

We study the dilation operators on the Cesàro function spaces and the Cesàro second order function spaces. For any $s > 0$ and Lebesgue measurable function f, define $E_s f(t) = f(st)$.

Proposition 2.1. *Let $s > 0$ and $p \in [1,\infty)$. We have*

$$\|E_s f\|_{Ces_p[0,\infty)} = s^{-1/p}\|f\|_{Ces_p[0,\infty)},$$
$$\|E_s f\|_{Ces_p^2[0,\infty)} = s^{-1/p}\|f\|_{Ces_p^2[0,\infty)}.$$

The above results follow from the definitions of $Ces_p^2[0,\infty)$. For brevity, we skip the details.

Next, we establish the Minkowski inequalities on $Ces_p[0,\infty)$ and $Ces_p^2[0,\infty)$.

Theorem 2.2. *Let $p \in [1,\infty)$. For any Lebesgue measurable function $F : [0,\infty)\times[0,\infty) \to \mathbb{R}$, we have*

$$\left\|\int_0^\infty F(y,\cdot)dy\right\|_{Ces_p[0,\infty)} \le \int_0^\infty \|F(y,\cdot)\|_{Ces_p[0,\infty)} dy,$$
$$\left\|\int_0^\infty F(y,\cdot)dy\right\|_{Ces_p^2[0,\infty)} \le \int_0^\infty \|F(y,\cdot)\|_{Ces_p^2[0,\infty)} dy.$$

Proof. As the proofs for the Minkowski inequalities on $Ces_p[0,\infty)$ and $Ces_p^2[0,\infty)$ are similar, for brevity, we just present the proof for Minkowski inequalities on $Ces_p[0,\infty)$.

We have

$$\begin{aligned}\left\| \int_0^\infty F(y,\cdot)dy \right\|_{Ces_p[0,\infty)} &= \left(\int_0^\infty \left(\frac{1}{x} \int_0^x \left| \int_0^\infty F(y,t)dy \right| dt \right)^p dx \right)^{1/p} \\ &\leq \left(\int_0^\infty \left(\frac{1}{x} \int_0^x \int_0^\infty |F(y,t)|dydt \right)^p dx \right)^{1/p} \\ &= \left(\int_0^\infty \left(\int_0^\infty \frac{1}{x} \int_0^x |F(y,t)|dtdy \right)^p dx \right)^{1/p}.\end{aligned}$$

By using the Minkowski's inequality for integrals [15, (6.19)], we get

$$\begin{aligned}\left\| \int_0^\infty F(y,\cdot)dy \right\|_{Ces_p[0,\infty)} &\leq \int_0^\infty \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |F(y,t)|dt \right)^p dx \right)^{1/p} dy \\ &= \int_0^\infty \|F((y,\cdot)\|_{Ces_p[0,\infty)} dy.\end{aligned}$$

□

For the rest of the paper, we present the results for the Cesàro function spaces and the Cesàro second order function spaces while the proofs for them are similar, therefore, for simplicity, we just present the proof for $Ces_p[0,\infty)$.

We are now ready to study the integral operators on $Ces_p[0,\infty)$. Let $K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function. We consider the integral operator

$$Tf(t) = \int_0^\infty K(s,t)f(s)ds, \quad t \geq 0.$$

The following is the main result on the boundedness of integral operators on $Ces_p[0,\infty)$.

Theorem 2.3. *Let $p \in [1,\infty)$ and $K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function. Suppose that for any $\lambda > 0$*

$$K(\lambda s, \lambda t) = \lambda^{-1} K(s,t) \tag{1}$$

and

$$\int_0^\infty |K(u,1)|u^{-\frac{1}{p}} du < \infty. \tag{2}$$

For any $f \in Ces_p[0,\infty)$ and $g \in Ces_p^2[0,\infty)$, we have

$$\begin{aligned}\|Tf\|_{Ces_p[0,\infty)} &\leq \left(\int_0^\infty |K(u,1)|u^{-\frac{1}{p}} du \right) \|f\|_{Ces_p[0,\infty)}, \\ \|Tg\|_{Ces_p^2[0,\infty)} &\leq \left(\int_0^\infty |K(u,1)|u^{-\frac{1}{p}} du \right) \|g\|_{Ces_p^2[0,\infty)}.\end{aligned}$$

Proof. As the proof of (1) is similar to the proof of (2), we just give the proof of (1). Let $f \in Ces_p[0,\infty)$. Proposition 2.1 and Theorem 2.2 give

$$\begin{aligned}\|Tf\|_{Ces_p[0,\infty)} &= \left\|\int_0^\infty K(u,1)E_u f(\cdot)du\right\|_{Ces_p[0,\infty)}\\ &\le \int_0^\infty |K(u,1)|\|E_u f\|_{Ces_p[0,\infty)}du\\ &= \left(\int_0^\infty |K(u,1)|u^{-\frac{1}{p}}du\right)\|f\|_{Ces_p[0,\infty)}. \qquad \square\end{aligned}$$

3. Applications

The study of the Cesàro function spaces is related to Hardy's inequality. Hardy's inequalities on the Cesàro function spaces are well known. They are consequences of Definition 1 and Hardy's inequality on Lebesgue spaces, see [30, Remark 7]. For the history and recent developments of Hardy's inequalities, the reader is referred to [28, 29, 35]. For any locally integrable function f on $[0,\infty)$, the Hardy operator H is defined as

$$Hf(t) = \frac{1}{t}\int_0^t f(s)ds, \quad t \ge 0.$$

Let f be an integrable function on the intervals $[t,\infty)$ for $t > 0$. The Copson operator H^* is defined as

$$H^*f(t) = \int_t^\infty \frac{f(s)}{s}ds, \quad t > 0.$$

Hardy's inequalities on the Cesàro function spaces are well known. They are consequences of Definition 1 and Hardy's inequality on Lebesgue spaces, see [30, Remark 7]. Theorem 2.3 gives the best constant for the Hardy's inequality on $Ces_p[0,\infty)$,

$$\|Hf\|_{Ces_p[0,\infty)} \le \frac{p}{p-1}\|f\|_{Ces_p[0,\infty)},$$
$$\|H^*f\|_{Ces_p[0,\infty)} \le p\|f\|_{Ces_p[0,\infty)}.$$

We also have Hardy's inequalities on some other function spaces such as function space of bounded mean oscillation, Herz-Morrey spaces function spaces of q-integral p-variation [19, 23, 42].

Theorem 2.3 can be applied to obtain the Hilbert inequality on $Ces_p[0,\infty)$. Recall that the Hilbert operator is defined as

$$\mathcal{H}f(t) = \int_0^\infty \frac{f(s)}{t+s}ds$$

when $f(\cdot)/(t+\cdot)$ is integrable for almost all $t \in (0,\infty)$.

Thus

$$\mathcal{H}f(t) = \int_0^\infty K(s,t)f(s)ds,$$

where $K(s,t) = \frac{1}{s+t}$.

Theorem 3.1. *Let* $p \in (1,\infty)$. *For any* $f \in Ces_p[0,\infty)$, *we have*

$$\|\mathcal{H}f\|_{Ces_p[0,\infty)} \le \left(\int_0^\infty \frac{u^{-\frac{1}{p}}}{1+u}du\right)\|f\|_{Ces_p[0,\infty)},$$
$$\|\mathcal{H}f\|_{Ces_p^2[0,\infty)} \le \left(\int_0^\infty \frac{u^{-\frac{1}{p}}}{1+u}du\right)\|f\|_{Ces_p^2[0,\infty)}.$$

Proof. We find that $K(s,t) = \frac{1}{s+t}$ satisfies (1). Additionally,

$$\int_0^\infty |K(u,1)|u^{-\frac{1}{p}}du = \int_0^\infty \frac{u^{-\frac{1}{p}}}{1+u}du$$
$$\le \int_0^1 u^{-\frac{1}{p}}du + \int_1^\infty u^{-1-\frac{1}{p}}du < \infty$$

because $p \in (1,\infty)$. Theorem 2.3 yields the Hilbert inequality on $Ces_p[0,\infty)$. □

For the Hilbert inequality on Herz-Morrey spaces and function spaces of q-integral p-variation, see [19, 23, 42].

Theorem 2.3 can also be used to study the mapping properties of the Erdélyi-Kober fractional integrals and the Mellin fractional integrals. We recall the definition of the Erdélyi-Kober fractional integrals from [25, (0.7)] and [40]. Let $\delta,\eta \in (0,\infty)$ and $\gamma \in (-\infty,\infty)$. For any locally integrable function f, the Erdélyi-Kober fractional integrals are defined as

$$I_\eta^{\gamma,\delta}f(t) = \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t s^{\eta(\gamma+1)-1}(t^\eta - s^\eta)^{\delta-1}f(s)ds, \quad t \ge 0,$$
$$J_\eta^{\gamma,\delta}f(t) = \frac{t^{\eta\gamma}}{\Gamma(\delta)}\int_t^\infty (s^\eta - t^\eta)^{\delta-1}s^{-\eta(\gamma+\delta)+\eta-1}f(s)ds, \quad t \ge 0.$$

When $\gamma = 0$ and $\eta = 1$, we have

$$I_1^{0,\delta}f(t) = \frac{t^{-\delta}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}f(s)ds = t^{-\delta}R_\delta f(t), \quad t \ge 0,$$

where $R_\delta f$ is the Riemann-Liouville integral of order δ.

For any $\nu > -\frac{1}{2}$, the Poisson transform is defined as

$$P_\nu f(x) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}f(xt)dt.$$

The Poisson transformation was introduced by Delsarte in [38, Vol 1, p. 439].

By using the substitution $y = xt$, we have

$$\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}f(xt)dt = \frac{1}{x^{2\nu}}\int_0^x (x^2-y^2)^{\nu-\frac{1}{2}}f(y)dy.$$

Thus,

$$P_\nu f(x) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}} I_2^{-\frac{1}{2},\nu+\frac{1}{2}} f(x).$$

Therefore, the Poisson transform is a member of the Erdélyi-Kober fractional integrals. The reader is referred to [18, 25, 33, 34, 36, 37, 40] for the uses of the Erdélyi-Kober fractional integrals on geometry, applied analysis and statistic. The Poisson transform is the transmutation operator between the second order derivative operator and the Bessel-Clifford operator, see [26, p. 303].

We establish the boundedness of the Erdélyi-Kober fractional integrals on the Cesàro function spaces.

Theorem 3.2. *Let $p \in [1,\infty)$, $\delta, \eta \in (0,\infty)$ and $\gamma \in (-\infty,\infty)$.*

(1) *If $\eta(\gamma+1) > \frac{1}{p}$, then there exists a constant $C > 0$ such that for any $f \in Ces_p[0,\infty)$*

$$\|I_\eta^{\gamma,\delta} f\|_{Ces_p[0,\infty)} \le C\|f\|_{Ces_p[0,\infty)}.$$

(2) *If $\eta\gamma > -\frac{1}{p}$, then there exists a constant $C > 0$ such that for any $f \in Ces_p[0,\infty)$*

$$\|J_\eta^{\gamma,\delta} f\|_{Ces_p[0,\infty)} \le C\|f\|_{Ces_p[0,\infty)}.$$

Proof. We have

$$I_\eta^{\gamma,\delta} f(t) = \int_0^\infty I(s,t) f(s)\,ds,$$

where

$$I(s,t) = \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \chi_{(0,t)}(s) s^{\eta(\gamma+1)-1}(t^\eta - s^\eta)^{\delta-1}.$$

For any $\lambda > 0$, we have

$$\begin{aligned} I(\lambda s, \lambda t) &= \frac{(\lambda t)^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \chi_{(0,\lambda t)}(\lambda s)(\lambda s)^{\eta(\gamma+1)-1}((\lambda t)^\eta - (\lambda s)^\eta)^{\delta-1} \\ &= \lambda^{-1} \frac{t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \chi_{(0,t)}(s) s^{\eta(\gamma+1)-1}(t^\eta - s^\eta)^{\delta-1} = \lambda^{-1} I(s,t). \end{aligned}$$

It satisfies (1). Furthermore,

$$\int_0^\infty |I(u,1)| u^{-\frac{1}{p}}\,du = \frac{1}{\Gamma(\delta)} \int_0^1 u^{\eta(\gamma+1)-1-\frac{1}{p}}(1-u^\eta)^{\delta-1}\,du.$$

We find that

$$\int_0^{\frac{1}{2}} u^{\eta(\gamma+1)-1-\frac{1}{p}}(1-u^\eta)^{\delta-1}\,du \le \int_0^{\frac{1}{2}} u^{\eta(\gamma+1)-1-\frac{1}{p}}\,du < \infty$$

because $\eta(\gamma+1) > \frac{1}{p}$. Since $\lim_{u\to 1} \frac{1-u^\eta}{1-u} = \eta$, we find that

$$\int_{\frac{1}{2}}^1 u^{\eta(\gamma+1)-1-\frac{1}{p}}(1-u^\eta)^{\delta-1}\,du \le C \int_{\frac{1}{2}}^1 (1-u)^{\delta-1}\,du < \infty$$

for some $C > 0$ because $\delta > 0$. Thus, (2) is fulfilled. Theorem 2.3 yields the boundedness of $I_\eta^{\gamma,\delta} : Ces_p[0,\infty) \to Ces_p[0,\infty)$.

Similarly, we have

$$J_\eta^{\gamma,\delta} f(t) = \int_0^\infty J(s,t) f(s) ds,$$

where

$$J(s,t) = \frac{t^{\eta\gamma}}{\Gamma(\delta)} \chi_{(t,\infty)}(s)(s^\eta - t^\eta)^{\delta-1} s^{-\eta(\gamma+\delta)+\eta-1}.$$

It is easy to see that it satisfies (1). Moreover,

$$\int_0^\infty |J(u,1)| u^{-\frac{1}{p}} du = \frac{1}{\Gamma(\delta)} \int_1^\infty (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1-\frac{1}{p}} du.$$

Since $\delta > 0$ and $\lim_{u\to 1} \frac{u^\eta - 1}{u-1} = \eta$, we have

$$\int_1^2 (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1-\frac{1}{p}} du \leq C \int_1^2 (u-1)^{\delta-1} du < \infty$$

for some $C > 0$. Furthermore,

$$\begin{aligned}\int_2^\infty (u^\eta - 1)^{\delta-1} u^{-\eta(\gamma+\delta)+\eta-1-\frac{1}{p}} du &\leq C \int_2^\infty u^{\eta(\delta-1)-\eta(\gamma+\delta)+\eta-1-\frac{1}{p}} du \\ &= C \int_2^\infty u^{-\eta\gamma-1-\frac{1}{p}} du < \infty\end{aligned}$$

for some $C > 0$ because $\eta\gamma > -\frac{1}{p}$. Theorem 2.3 assures the boundedness of $J_\eta^{\gamma,\delta} : Ces_p[0,\infty) \to Ces_p[0,\infty)$. ☐

We also have the boundedness of the Erdélyi-Kober fractional integrals on $Ces_p^2[0,\infty)$.

Theorem 3.3. *Let $p \in [1,\infty)$, $\delta, \eta \in (0,\infty)$ and $\gamma \in (-\infty,\infty)$.*

(1) *If $\eta(\gamma+1) > \frac{1}{p}$, then there exists a constant $C > 0$ such that for any $f \in Ces_p^2[0,\infty)$*

$$\|I_\eta^{\gamma,\delta} f\|_{Ces_p^2[0,\infty)} \leq C\|f\|_{Ces_p^2[0,\infty)}.$$

(2) *If $\eta\gamma > -\frac{1}{p}$, then there exists a constant $C > 0$ such that for any $f \in Ces_p^2[0,\infty)$*

$$\|J_\eta^{\gamma,\delta} f\|_{Ces_p^2[0,\infty)} \leq C\|f\|_{Ces_p^2[0,\infty)}.$$

For the boundedness of the Erdélyi-Kober fractional integrals on Hardy space and Morrey spaces, the reader may consult [20–22].

As a consequence of Theorem 3.2, we have the boundedness of the Poisson transform on the Cesàro function spaces.

Corollary 3.4. *Let $\nu > -\frac{1}{2}$ and $p \in (1, \infty)$. There exists a constant $C > 0$ such that for any $f \in Ces_p[0, \infty)$*

$$\|P_\nu f\|_{Ces_p[0,\infty)} \leq C\|f\|_{Ces_p[0,\infty)}.$$

We turn to the Mellin fractional integral. Let $\mu \in (-\infty, \infty)$ and $\alpha \in (0, \infty)$. The Mellin fractional integral is defined as

$$\mathcal{J}_\mu^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^\mu \left(\log \frac{x}{u}\right)^{\alpha-1} f(u) \frac{du}{u},$$

see [10, (2)].

Theorem 3.5. *Let $p \in [1, \infty)$, $\mu \in (-\infty, \infty)$ and $\alpha \in (0, \infty)$. If $\mu > \frac{1}{p}$, then $\mathcal{J}_\mu^\alpha$ is bounded on $Ces_p[0, \infty)$ and $Ces_p^2[0, \infty)$.*

Proof. We see that

$$\mathcal{J}_\mu^\alpha f(t) = \int_0^\infty K(s, t) dt,$$

where

$$K(s, t) = \frac{1}{\Gamma(\alpha)} \chi_{(0,t)}(s) \left(\frac{s}{t}\right)^\mu \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s}.$$

For any $\lambda > 0$, we find that

$$\begin{aligned} K(\lambda s, \lambda t) &= \frac{1}{\Gamma(\alpha)} \chi_{(0,\lambda t)}(\lambda s) \left(\frac{\lambda s}{\lambda t}\right)^\mu \left(\log \frac{\lambda t}{\lambda s}\right)^{\alpha-1} \frac{1}{\lambda s} \\ &= \frac{1}{\lambda} \frac{1}{\Gamma(\alpha)} \chi_{(0,t)}(s) \left(\frac{s}{t}\right)^\mu \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} = \lambda^{-1} K(s, t). \end{aligned}$$

Thus, it satisfies (1).

Take a $\epsilon \in (0, \mu - \frac{1}{p})$. We have a constant $C > 0$ such that

$$(-\log u)^{\alpha-1} < Cu^{-\epsilon}, \quad u \in (0, 1).$$

Consequently,

$$\begin{aligned} \int_0^\infty |K(u, 1)| u^{-\frac{1}{p}} du &= \frac{1}{\Gamma(\alpha)} \int_0^1 u^\mu \left(\log \frac{1}{u}\right)^{\alpha-1} u^{-\frac{1}{p}} \frac{du}{u} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\mu-1-\frac{1}{p}} (-\log u)^{\alpha-1} du \\ &\leq C \int_0^1 u^{\mu-1-\frac{1}{p}-\epsilon} du < \infty \end{aligned}$$

because $\mu > \frac{1}{p} + \epsilon$. Thus, (2) is fulfilled and Theorem 2.3 asserts the boundedness of $\mathcal{J}_\mu^\alpha : Ces_p[0, \infty) \to Ces_p[0, \infty)$. □

For the mapping properties of the Mellin fractional integrals on space of bounded mean oscillation and Campanato spaces, see [19].

The Mellin fractional integrals have been generalized in [11–13]. Theorem 2.3 also applies to these generalizations, for brevity, we omit the details.

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