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MAGNETIC GEODESICS ON THE SPACE OF KÄHLER POTENTIALS

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Abstract. In this work, magnetic geodesics over the space of Kähler potentials are studied through a variational method for a generalized Landau-Hall functional. The magnetic geodesic equation is calculated in this setting and its relation to a perturbed complex Monge-Ampère equation is given. Lastly, the magnetic geodesic equation is considered over the special case of toric Kähler potentials over toric Kähler manifolds.

1. Introduction

Let (X,ω) be a compact Kähler manifold. X. X. Chen et al. examined the metric and geometric aspects of the space of all Kähler metrics $\mathcal{H}_\alpha$ and showed a remarkable result that $\mathcal{H}_\alpha$ is a path metric space [1–5]. Weak geodesics (a special type of path between two points in $\mathcal{H}_\alpha$) on this space play an important role in the variational approach for solving complex Monge-Ampère equations (CMAE) and for understanding the applications of CMAE to find the Kähler-Einstein metrics on various varieties. As we will give in detail in the following parts of this study, Semmes [10] showed that the geodesic equation can be reformulated as a homogenous CMAE of one degree higher.

Magnetic curves (or magnetic geodesics) are generalizations of geodesics. Such a curve actually describes the trajectory of a particle moving under the effect of a magnetic field. As it is known geodesics are extremals of the energy functional and the geodesic equation can be calculated via Euler-Lagrange equations, in the case of magnetic geodesics one considers the extremals of Landau-Hall functional with the magnetic force (known as Lorentz force) and the magnetic trajectories are the curves γ that satisfy the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \phi(\gamma')$$

with the magnetic field ϕ.

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In this study we will take this generalization of magnetic geodesics to the setting of Kähler potentials over compact Kähler manifolds and examine its relation to a perturbed CMAE. The organization of the paper is as follows:

In Section 2, we give the necessary background about Kähler potentials, geodesics over the space of Kähler potentials $\mathcal{H}_\alpha$ and basics about magnetic geodesics on Riemannian manifolds. In Section 3, we introduce magnetic geodesics over $\mathcal{H}_\alpha$ and we will give the main result of this study about the magnetic geodesic equation. As it is known, toric Kähler manifolds and toric potentials are very special cases. However they are quite useful to test your hypothesis about the non toric environment. Hence in the last part of this paper we examine the toric magnetic geodesics over toric compact Kähler manifolds $(X, \omega, \mathbb{T})$.

2. Preliminaries

Throughout this work we will work on the magnetic geodesics on the space of Kähler potentials. So let us first introduce the setting and the classical geodesics of these potentials.

Definition. Let (X, ω) be a compact Kähler manifold of dimension n. Any other Kähler metric on X that is in the same cohomology class as ω is given by

$$\omega_\varphi = \omega + dd^c\varphi,$$

where $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. Then we define the space of Kähler potentials as

$$\mathcal{H} = \{\varphi \in C^\infty(X) : \omega_\varphi = \omega + dd^c\varphi > 0\}.$$

Notation 1. (1) We know that two Kähler potentials generate the same metric if and only if they differ by a constant hence

$$\mathcal{H}_\alpha = \mathcal{H}/\mathbb{R}$$

is the space of Kähler metrics on X in the cohomology class $\alpha = \{\omega\} \in H^{1,1}(X, \mathbb{R})$.

(2) Let $V_\alpha = \alpha^n = \int_X \omega^n$ be the volume of the space X. Then for any $\varphi \in \mathcal{H}$ we denote the Monge-Ampère measure associated to φ as

$$MA(\varphi) = \frac{\omega_\varphi^n}{V_\alpha}.$$

Part 1: Classical geodesics on the space of Kähler potentials

Definition. We define the geodesics between two points φ_0, φ_1 in $\mathcal{H}$ as the extremals of the energy functional

$$\varphi \longrightarrow H(\varphi) = \frac{1}{2}\int_0^1 \int_X (\dot{\varphi}_t)^2 MA(\varphi_t)dt,$$

where $\varphi = \varphi_t$ is a path in $\mathcal{H}$ joining φ_0 and φ_1.

As it is the standard procedure in most of the variational problems, the geodesic equation (the equation whose solution gives a geodesic path for certain boundary conditions) for the classical geodesics on the space of Kähler potentials is calculated via Euler-Lagrange equation:

Lemma 2.1 ([7], Lemma 1.2)**.** *The geodesic equation is*

$$\ddot{\varphi} = \|\nabla \dot{\varphi}\|_{\varphi}^2, \tag{1}$$

where the gradient is relative to the metric ω_φ. This identity is also written as

$$\ddot{\varphi} MA(\varphi) = \frac{n}{V_\alpha} d\dot{\varphi} \wedge d^c \dot{\varphi} \wedge \omega_\varphi^{n-1}.$$

Boundary problem for geodesic equation:

Given φ_0, φ_1 two distinct potentials in $\mathcal{H}$ can one find a path $\varphi = (\varphi_t)_{0\le t\le 1} \in \mathcal{H}$ which is a solution of the geodesic equation (1) with endpoints $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_1$?

The answer to this problem was given by Semmes [10] and the solution is totally determined by the solvability of the complex Monge-Ampère equation over a certain region. Before giving Semmes' solution, let us introduce the specific setting of the problem:

For each path $(\varphi_t)_{0\le t\le 1} \in \mathcal{H}$, we set $\varphi(t,x,s) = \varphi_t(x)$, $x \in X$, $e^{t+is} \in A = [0,1] \times \partial\mathbb{D}$. Set $z = e^{t+is}$ and $\omega(x,z) \coloneqq \omega(x)$.

Proposition 2.2 ([10], Theorem 8.11)**.** *The path φ_t is a geodesic in $\mathcal{H}$ if and only if the associated radial function φ on $A \times X$ is a solution of the homogeneous complex Monge-Ampère equation*

$$(\omega + dd^c_{x,z}\varphi)^{n+1} = 0,$$

where the derivatives are taken in all variables x, z.

Remark 2.3. In ([10], Theorem 8.11) if we take $\mathcal{O}$ as $A = [0,1]$ and $f(z,w)$ on $\mathcal{O} \times X$ as $\varphi_t(x)$ on $A \times X$, then we obtain the proposition above.

Part 2: Magnetic geodesics on Riemannian manifolds

As we have seen before, geodesics are obtained as the critical points of the energy functional. Finding the critical points of a specific perturbation of the energy functional however results in another type of curves namely magnetic geodesics.

Let us follow the definition of [8].

Definition. Let (M,g) be a Riemannian manifold and ω be a 1-form (potential). For a smooth curve $\gamma : [a,b] \to M$ consider the functional

$$LH(\gamma) \coloneqq \int_a^b \frac{1}{2}\left(\langle \gamma'(t), \gamma'(t)\rangle + \omega(\gamma'(t))\right) dt$$

which is called the Landau-Hall functional for the curve γ.

The critical points of the LH-functional satisfy the following Lorentz equation

$$\nabla_{\gamma'}\gamma' - \phi(\gamma') = 0,$$

where ϕ is a $(1,1)$-tensor field on M and determined by $g(\phi(X,Y)) = d\omega(X,Y)$ for all X, Y tangent to M.

Remark 2.4. For a map $f : (M,g) \to (N,h)$, the Landau-Hall functional is given as

$$LH(f) \coloneqq E(f) + \int_N \omega(df(\xi))d\upsilon_h,$$

where $E(f)$ is the energy functional.

A map is called magnetic if it is a critical point of the Landau-Hall integral above.

3. Magnetic geodesics on the space of Kähler potentials

The notion of magnetic geodesics can also be generalized to Kähler potentials on a compact Kähler manifold and for the rest of the study we will focus on how this can be done.

Definition. Let (X,ω) be a compact Kähler manifold and α_σ be a 1-form on $\mathcal{H}$ for some $\sigma \in \mathcal{H}$. We define magnetic geodesics between two points $\varphi_0, \varphi_1 \in \mathcal{H}$ as the extremals of the generalized Landau-Hall functional

$$\varphi \longrightarrow LH(\varphi) \coloneqq \frac{1}{2}\int_0^1 \int_X (\dot{\varphi}_t)^2 MA(\varphi_t)dt + \int_0^1 \alpha_\sigma(\dot{\varphi}_t)dt, \tag{2}$$

where $\varphi = \varphi_t$ is a path connecting φ_0 and φ_1 in $\mathcal{H}$ and α_σ is the closed 1-form (potential) on $\mathcal{H}$ defined as $\alpha_\sigma(\psi) = \int_X \psi MA(\sigma)$.

Remark 3.1. For the details of the closedness of the 1-form α_σ, see [9, Lemma 6.19, pp. 245–246].

Now we will give the main result of this study and calculate the magnetic geodesic equation for Kähler potentials:

Theorem 3.2. *Magnetic geodesic equation on the space of Kähler potentials is given as*

$$\ddot{\varphi}MA(\varphi) = \frac{n}{V_\alpha}\left[d\dot{\varphi} \wedge d^c\dot{\varphi} \wedge \omega_\varphi^{n-1} - dd^c\dot{\sigma} \wedge \omega_\sigma^{n-1}\right].$$

Proof. We need to calculate the critical points of the generalized Landau-Hall functional therefore we will compute the Euler-Lagrange equation of this functional. Suppose that $\phi_{s,t}$ is a variation of φ_t with fixed end points such that

$$\phi_{0,t} = \varphi_t,\ \phi_{s,0} = \varphi_0,\ \phi_{s,1} = \varphi_1.$$

Let $\psi_t = \left.\frac{\partial \phi}{\partial s}\right|_{s=0}$. Then $\psi_0 \equiv \psi_1 \equiv 0$ (†) and we have

$$\phi_{s,t} = \varphi_t + s\psi_t + o(s) \text{ and } \frac{\partial \phi_{s,t}}{\partial t} = \dot{\varphi}_t + s\dot{\psi}_t + o(s).$$

Then,

$$\begin{aligned} LH(\phi_{s,t}) &= \frac{1}{2}\int_0^1\int_X (\dot{\phi_{s,t}})^2 MA(\phi_{s,t})dt + \int_0^1\int_X (\dot{\phi_{s,t}})MA(\dot{\phi_{s,t}})dt \\ &= \frac{1}{2}\int_0^1\int_X (\dot{\varphi}_t)^2 MA(\varphi_t) + \frac{ns}{2V_\alpha}\int_0^1\int_X (\dot{\varphi}_t)^2 dd^c\psi_t\wedge\omega_{\varphi_t}^{n-1}dt \\ &+ s\int_0^1\int_X \dot{\varphi}_t\dot{\psi}_t MA(\varphi_t)dt + \int_0^1\int_X (\dot{\varphi}_t)MA(\sigma)dt \\ &+ s\int_0^1\int_X \dot{\psi}_t MA(\sigma)dt + o(s). \end{aligned}$$

Now using the boundary values (†) let us calculate the s-dependent terms,

- $$\int_0^1\int_X (\dot{\varphi}_t)^2 dd^c\psi_t\wedge\omega_{\varphi_t}^{n-1}dt$$
$$= 2\int_0^1\int_X \psi_t\{d\dot{\varphi}_t\wedge d^c\dot{\varphi}_t + \dot{\varphi}_t\wedge dd^c\dot{\varphi}_t\}\wedge\omega_{\varphi_t}^{n-1}dt.$$
- $$\int_0^1\int_X \dot{\varphi}_t\dot{\psi}_t MA(\varphi_t)dt$$
$$= -\int_0^1\int_X \psi_t\{\ddot{\varphi}_t MA(\varphi_t) + \frac{n}{V_\alpha}\dot{\varphi}_t dd^c\dot{\varphi}_t\wedge\omega_{\varphi_t}^{n-1}\}dt.$$
- $$\int_0^1\int_X \dot{\psi}_t MA(\sigma)dt = -\int_0^1\int_X \psi_t\left(\frac{n}{V_\alpha}dd^c\dot{\sigma}\wedge\omega_\sigma^{n-1}\right)dt.$$

If we combine all these equations, we obtain

$$\begin{aligned} &LH(\phi_{s,t}) \\ = &LH(\varphi_t) \\ &+ s\int_0^1\int_X \psi_t\left\{-\ddot{\varphi}_t MA(\varphi_t) + \frac{n}{V_\alpha}d\dot{\varphi}_t\wedge d^c\dot{\varphi}_t\wedge\omega_{\varphi_t}^{n-1} - \frac{n}{V_\alpha}dd^c\dot{\sigma}\wedge\omega_\sigma^{n-1}\right\}dt \\ &+ o(s). \end{aligned}$$

Hence, if φ is a critical point of the generalized Landau-Hall functional, then it satisfies the following equation

$$\ddot{\varphi}MA(\varphi) = \frac{n}{V_\alpha}\left[d\dot{\varphi}\wedge d^c\dot{\varphi}\wedge\omega_\varphi^{n-1} - dd^c\dot{\sigma}\wedge\omega_\sigma^{n-1}\right].$$

□

4. A special case: toric magnetic geodesics

In this section we will consider the magnetic geodesic equation on a special setting where $(X, \omega, \mathbb{T})$ is a toric, compact, Kähler manifold and our potentials are toric Kähler potentials. Before passing to magnetic geodesic equation let us first give the details about toric manifolds and the structure of toric potentials:

Definition. A toric, compact, Kähler manifold is obtained by an equivariant compactification of the torus $\mathbb{T} = (\mathbb{C}^*)^n$ with an $(S^1)^n$-invariant Kähler metric ω and given by the triple $(X, \omega, \mathbb{T})$.

In this setting the Kähler metric ω can be written in the form

$$\omega = dd^c F_0 \circ L$$

over $\mathbb{T}$ where $F_0 : \mathbb{R}^n \to \mathbb{R}$ is a smooth, strictly convex function and $L : \mathbb{T} \to \mathbb{R}^n$ is the logarithmic transformation function

$$L(z_1, \ldots, z_n) = (\log |z_1|, \ldots, \log |z_n|).$$

As it is very well known, in the compact setting as a result of the maximum modulus principle there is no non-constant plurisubharmonic function. However we have a general class of functions analogous to plurisubharmonic functions namely quasiplurisubharmonic functions in this setting:

Definition. A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called ω-plurisubharmonic if

(i) it is locally the sum of a plurisubharmonic function and a smooth function,
(ii) the current $\omega + dd^c\varphi$ is positive on X.

An ω-plurisubharmonic function is called *toric* if it is invariant under the $(S^1)^n$-action induced by the $(\mathbb{C}^*)^n$ action on X. Toric ω-plurisubharmonic functions on X are denoted as $PSH_{tor}(X, \omega)$.

From the definition we obtain a representation of $PSH_{tor}(X, \omega)$ functions over $\mathbb{T}$ such that there exists a convex function $F_\varphi : \mathbb{R}^n \to \mathbb{R}$ such that

$$F_\varphi \circ L = F_0 \circ L + \varphi \text{ on } \mathbb{T} \subset X. \tag{3}$$

The representation (3) gives a relation between toric ω-plurisubharmonic functions and real convex functions, now we will continue with a result which takes this relation one step further, i.e., the connection between complex and real Monge-Ampère measures [6, Lemma 2.3]:

Proposition 4.1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If χ is a continuous function with compact support on $\mathbb{R}^n$, then*

$$\int_{\mathbb{T}} (\chi \circ L)(dd^c F \circ L)^n = \int_{\mathbb{R}^n} \chi MA_{\mathbb{R}}(F),$$

where $MA_{\mathbb{R}}(F)$ is the real Monge-Ampère measure of F defined as

$$MA_{\mathbb{R}}(F) = n! \det \left[\frac{\partial^2 F}{\partial x_i \partial x_j} \right] dV.$$

Remark 4.2. For a detailed study of the concepts related to toric pluripotential theory, see [6].

Now let us introduce the space of toric-Kähler potentials:

Definition. We define the space of toric-Kähler potentials as

$$\mathcal{H}_{tor} = \mathcal{H} \cap PSH_{tor}(X, \omega),$$

where a toric-Kähler potential is represented by a strictly convex function.

Lastly, let us give the magnetic geodesic equation for toric-Kähler potentials in totally real terms:

Corollary 4.3. *In the space of toric-Kähler potentials $\mathcal{H}_{tor}(X, \omega)$ the magnetic geodesic equation can be written in the following form*:

$$\begin{aligned}\ddot{F_\varphi} MA_{\mathbb{R}}(F_\varphi) = \frac{n}{V} \sum_{i,j} \left(\frac{\partial \dot{F_\varphi}}{\partial x_i} \right) \left(\frac{\partial \dot{F_\varphi}}{\partial x_j} \right) \left(\sum_{i,j} \frac{\partial^2 F_\varphi}{\partial x_i \partial x_j} \right)^{n-1} \\ - \frac{n}{V} \left(\sum_{i,j} \frac{\partial^2 \dot{F_\sigma}}{\partial x_i \partial x_j} \right) \left(\sum_{i,j} \frac{\partial^2 F_\sigma}{\partial x_i \partial x_j} \right)^{n-1},\end{aligned} \tag{4}$$

where F_σ is the representing strictly convex function for the potential α_σ given in (2), *F_φ is the corresponding path of strictly convex functions for the path φ connecting $\varphi_0, \varphi_1 \in \mathcal{H}_{tor}$ and $V = \int_{\mathbb{R}^n} MA_{\mathbb{R}}(F_0)$.*

Proof. Representation of toric functions together with the previous proposition and the main theorem, (Theorem 3.2) give the result. □

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