Bull. Korean Math. Soc. **59** (2022), No. 4, pp. 1019–1044
https://doi.org/10.4134/BKMS.b210607
pISSN: 1015-8634 / eISSN: 2234-3016

APPROXIMATE PROJECTION ALGORITHMS FOR SOLVING EQUILIBRIUM AND MULTIVALUED VARIATIONAL INEQUALITY PROBLEMS IN HILBERT SPACE

Nguyen Minh Khoa and Tran Van Thang

Abstract. In this paper, we propose new algorithms for solving equilibrium and multivalued variational inequality problems in a real Hilbert space. The first algorithm for equilibrium problems uses only one approximate projection at each iteration to generate an iteration sequence converging strongly to a solution of the problem underlining the bifunction is pseudomonotone. On the basis of the proposed algorithm for the equilibrium problems, we introduce a new algorithm for solving multivalued variational inequality problems. Some fundamental experiments are given to illustrate our algorithms as well as to compare them with other algorithms.

1. Introduction

Let H denote a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. First, the article is interested in a method for finding a solution of the following equilibrium problems:

$$(EPs) \qquad \text{Find } \ p \in C \ \text{ such that } f(p, y) \geq 0 \quad \forall y \in C,$$

where $f : C \times C \to \mathbb{R}$ is a bifunction such that $f(x, x) = 0$ for all $x \in C$ and the nonempty subset C in H is defined by

$$C := \{x \in \mathrm{H} : g_i(x) \leq 0\},$$

$g_i : \mathrm{H} \to \mathbb{R}$ is a subdifferentiable, lower semicontinuous, convex function on H for every $i = 1, 2, \ldots, m$. In the framework of this paper, we denote the solution set of Problem (EPs) by $Sol(EPs)$. A bifunction $f : C \times C \to \mathrm{H}$ is said to be

- (C$_1$) *γ-strongly monotone on C*, if $f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2 \ \ \forall x, y \in C$;
- (C$_2$) *monotone on C*, if $f(x, y) + f(y, x) \leq 0 \ \ \forall x, y \in C$;
- (C$_3$) *pseudomonotone on C*, if $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \ \forall x, y \in C$;

Received August 15, 2021; Revised January 27, 2022; Accepted February 23, 2022.
2020 *Mathematics Subject Classification.* 65K10, 90C25, 47J25, 47J20, 91B50.
Key words and phrases. Equilibrium problem, multivalued variational inequality problem, subgradient, approximate projection, pseudomonotone, Tseng's extragradient method.

(C_4) *Lipschitz-type continuous* with constants $c_1 > 0$ and $c_2 > 0$ (introduced first by Mastroeni in [22]), if

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2 \quad \forall x,y,z \in C.$$

Problem (EPs) is a general model of some important mathematical models such as optimization, variational inequality, Kakutani fixed point, and so on (see, for example, [6,12]). Therefore, the problem has received a lot of research attention from mathematicians. In order to solve (EPs), many iterative methods have been proposed, among them, the projection and the extragradient (or double projection) algorithms are widely used (see [8,13,18,24,26–28] and the references therein). Some other methods for solving (EPs) can be found, for example, in [1,15,16,23,30].

In the special case, when $f(x,y) = \langle F(x), y-x\rangle$, where $F : C \to \mathrm{H}$ is a mapping, Problem (EPs) is equivalent to the following variational inequality problem:

$$(VIPs) \qquad \text{Find } \; p \in C \; \text{ such that } \langle F(p), x-p\rangle \geq 0 \quad \forall y \in C.$$

There are many ways to solve this problem but projection method and extragradient method are still the most popular methods (see [4,5,9,18,20,25,29]). The metric projection from H onto C is defined by P_C and

$$P_C(x) = \operatorname{argmin}\{\|x-y\| : y \in C\}, \; \forall x \in \mathrm{H}.$$

It is easy to check that a point p is a solution of Problem $(VIPs)$ if and only if it is a fixed point of the following mapping:

$$S(x) = P_C(x - \lambda F(x))$$

for any $\lambda > 0$. From the above idea, several algorithms that only use one projection at each iteration are proposed (see [5,11]). However, these methods require too harsh assumptions to obtain convergence theorems, such as the strong monotonicity or inverse strong monotonicity of the mapping $F(x)$. In general, the projection algorithm is not convergent even if F is a monotone mapping [10]. To obtain the convergence results of the projection algorithms, the extragradient algorithms have been proposed. In [18], Korpelevich introduced an extragradient algorithm that is determined by the following iterative formula:

$$(1) \qquad \begin{cases} x^0 \in C, \\ y^k = P_C(x^k - \lambda_k F(x^k)), \\ x^{k+1} = P_C(x^k - \lambda_k F(y^k)), \end{cases}$$

where $\lambda_k \in (0, \frac{1}{L})$ and L is the Lipschitz constant of F. The author showed that the algorithm is convergent when F is monotone and L-Lipschitz continuous. Afterward, Korpelevich's extragradient method (1) has been extended and improved by many mathematicians in different ways [7,9]. Notice that this algorithm requires computing two projections onto the feasible set C at each iteration. This can be computationally expensive when the set C is not so simple. To overcome this drawback, in [29], Tseng has proposed the following extragradient algorithm for solving Problem ($VIPs$):

$$\begin{cases} x^0 \in C, \\ y^k = P_C(x^k - \lambda F(x^k)), \\ x^{k+1} = y^k + \lambda(F(x^k) - F(y^k)), \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$. However, this algorithm only obtains weak convergence in real Hilbert spaces. Very recently, Tseng's extragradient method has received great attention from many researchers (see [20] and the references therein).

The first aim of this article is to introduce a new algorithm for solving equilibrium Problem (EPs) by modifying Tseng's extragradient method. The proposed algorithm only uses one approximate projection on the feasible C at each iteration to generate the iteration sequence converging strongly to a solution of the problem when the bifunction f is pseudomonotone and satisfies the following assumption:

$$(\mathrm{C}_5) \qquad \rho\left(\partial_2 f(x,\cdot)(x), \partial_2 f(y,\cdot)(y)\right) \le L\|x-y\|, \ \forall x, y \in C,$$

where $\rho(\mathcal{M},\mathcal{N})$ is the Hausdorff distance between two subsets $\mathcal{M}$ and $\mathcal{N}$ of H, i.e.,

$$\rho(\mathcal{M},\mathcal{N}) := \max\{d(\mathcal{M},\mathcal{N}), d(\mathcal{N},\mathcal{M})\},$$

$d(\mathcal{M},\mathcal{N}) := \sup_{x\in\mathcal{M}} \inf_{y\in\mathcal{N}} \|x-y\|$. Observe that if F is a pseudomonotone and L-Lipschitz continuous mapping on C, then $g(x,y) = \langle F(x), y-x\rangle$ is pseudomonotone and satisfies the assumption (C_5). In the later part of this paper, we can show that there doesn't exist an inclusive relationship between the set of all bifunctions satisfying (C_5) and the set of all Lipschitz-type continuous bifunctions. In many research results, in order to obtain the strong convergence theorem, the bifunction f must satisfy the assumption (C_4) and one of the assumptions (C_1)-(C_3) (see [13, 14, 24]). Moreover, the Lipschitz constants need to be known to be able to implement these algorithms. In practice, these constants are often unknown or difficult to calculate accurately. Our algorithm only needs to choose an arbitrarily approximate parameter $\overline{L}$ satisfying $\overline{L} > L$ and does not use any line search procedure to update the step size at each iteration. If $f(x,\cdot)$ is convex and differentiable on C, then the proposed algorithm can be implemented without prior knowledge of constants L, $\overline{L}$. On the basis of the proposed algorithm for problem (EPs), we develop a new algorithm for the following multivalued variational inequality problem in real Hilbert space:

$$(MVIPs) \qquad \text{Find } \ (p, u_p) \in C \times F(p) \text{ such that } \langle u_p, x-p\rangle \ge 0, \quad \forall x \in C,$$

where $F : \mathrm{H} \to 2^{\mathrm{H}}$ is a multivalued mapping with nonempty values. We can prove that the algorithm is strongly convergent when F is L-Lipschitz Hausdorff continuous and pseudomonotone on H. We also provide some numerical examples to illustrate the performance of the proposed method where the bifunction f is not Lipschitz-type continuous.

The paper is organized as follows. We first recall some necessary concepts and lemmas in Section 2. In Section 3, we introduce an approximate projection algorithm for solving Problem (EPs) and prove the strong convergence theorem of the algorithm. Section 4 gives the algorithm for the multivalued variational inequality problem $(MVIPs)$. In the last section, several fundamental experiments are provided to illustrate the convergence of our algorithms and compare them with others.

2. Preliminaries

In this section, we review some concepts and results that are used to prove the main results of this paper.

We first recall some well-known definitions of monotonicity for nonlinear operators.

Definition. Let $F : C \to 2^{\mathrm{H}}$ be a multivalued mapping. Then F is said to be

(i) *monotone on* C, if

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in C,\ u \in F(x),\ v \in F(y);$$

(ii) *pseudomonotone on* C, if

$$\langle v, x - y\rangle \geq 0 \Rightarrow \langle u, x - y\rangle \geq 0, \quad \forall x, y \in C,\ u \in F(x),\ v \in F(y).$$

It is easy to check that if F is monotone, then F is pseudomonotone. But the converse is not true. Indeed, let F be a multivalued mapping defined by:

$$F(x) = \{tMx : t \in [a, b]\} \ \ \forall x \in \mathbb{R}^n,$$

where $0 < a < b$ and M is an $n \times n$ positive semidefinite matrix. Let $t_1, t_2 \in [a, b]$ and $\langle t_2 My, x - y\rangle \geq 0$. Then, we have $\langle My, x - y\rangle \geq 0$. From the positive semidefinite assumption of M, it follows that $\langle Mx, x - y\rangle \geq 0$, equivalently, $\langle t_1 Mx, x - y\rangle \geq 0$. Therefore, F is pseudomonotone on $\mathbb{R}^n$. Clearly, F is not monotone.

If $\{x^k\}$ is a sequence in H, then we denote by $x^k \to p$ the strong convergence of $\{x^k\}$ to p and $x^k \rightharpoonup p$ the weak convergence. We now recall some weak continuity concepts of a function.

Definition. A mapping $g : \mathrm{H} \to (-\infty, +\infty]$ is said to be

(i) *sequentially weakly continuous* at $\bar{x}$, if $\lim_{x \rightharpoonup \bar{x}} g(x) = g(\bar{x})$, and *sequentially weakly continuous* on H if this holds for every $\bar{x}$ in H.

(ii) *sequentially weakly lower semicontinuous at* $\bar{x}$, if $\liminf_{x \rightharpoonup \bar{x}} g(x) \geq g(\bar{x})$, and sequentially weakly lower semicontinuous on H if this holds for every $\bar{x}$ in H.

(iii) *sequentially weakly upper semicontinuous at* $\bar{x}$, if $\limsup_{x \rightharpoonup \bar{x}} g(x) \leq g(\bar{x})$, and sequentially weakly upper semicontinuous on H if this holds for every $\bar{x}$ in H.

We have already known that the norm mapping $g(x) = \|x\|$ is sequentially weakly lower semicontinuous on H.

Next, we recall that the subdifferential of a convex function $g : C \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial g(p) = \{x^* \in \mathrm{H} : g(x) - g(p) \geq \langle x^*, x - p\rangle \ \ \forall x \in C\}.$$

If $\partial g(x) \neq \emptyset$, then g is called subdifferentiable at x. The function g is said to be differentiable on C if f is differentiable at every x in C. The outer normal cone N_C of C at $p \in C$ is defined by

$$N_C(x) = \{x^* \in \mathrm{H} : \langle x^*, x - p\rangle \leq 0 \ \ \forall x \in C\}.$$

To obtain the main results that are presented in Sections 3 and 4, we will use the following lemmas in the sequel.

Lemma 2.1. *For every u, $v \in \mathrm{H}$, we have the following assertions.*

(i) $\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$;
(ii) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$.

From the definition of projection, it is easy to see that P_C has the following characteristic properties.

Lemma 2.2. *For any $x \in \mathrm{H}$, we have*

(i) $p = P_C(x)$ *if and only if* $\langle p - x, y - p\rangle \leq 0, \quad \forall y \in C$;
(ii) $\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathrm{H}$.

Lemma 2.3. *Let C be a convex subset of a real Hilbert space* H*, $g : C \to (-\infty, +\infty]$ be convex and subdifferentiable. Then, p is an optimal solution of the following convex minimization problem*

$$\min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(p) + N_C(p)$.

Lemma 2.4 ([21])**.** *Let $\{\xi_k\}$ be a sequence of nonnegative real numbers satisfying the following condition*

$$\xi_{k+1} \leq (1 - t_k)\xi_k + t_k\alpha_k + \beta_k, \ \forall k \geq 1,$$

where $\{t_k\} \subset [0, 1]$, $\sum_{k=0}^{\infty} t_k = +\infty$, $\limsup_{k\to\infty} \alpha_k \leq 0$ and $\beta_k \geq 0$, $\sum_{n=1}^{\infty} \beta_k < \infty$. Then, $\lim_{k\to\infty} \xi_k = 0$.

Lemma 2.5 ([21, Remark 4.4])**.** *Let $\{\xi_k\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer M such that $M \geq m$ and $\xi_M \leq \xi_{M+1}$. Let $\bar{k}$ be an integer such that $\xi_{\bar{k}} \leq \xi_{\bar{k}+1}$ and define, for all integer $k \geq \bar{k}$,*

$$\tau(k) = \max\{i \in \mathrm{N} : \bar{k} \leq i \leq k, \ \xi_i \leq \xi_{i+1}\}.$$

Then, $0 \leq \xi_k \leq \xi_{\tau(k)+1}$ for all $k \geq \bar{k}$ the and sequence $\{\tau(k)\}_{k\geq\bar{k}}$ is nondecreasing and tends to $+\infty$ as $k \to \infty$.

3. Algorithm for equilibrium problem

In this section, we develop a new iterative algorithm for solving the equilibrium problem (EPs). In order to prove convergence of the sequences generated by the proposed algorithm, we need to use the following assumptions imposed on the bifunction f.

$\mathcal{T}_1$. $f(x,x)=0$ for all $x\in C$, $f(x,y)$ is pseudomonotone on $C\times C$ and $f(\cdot,y)$ is sequentially weakly upper semicontinuous on C;

$\mathcal{T}_2$. there exists a real positive number L such that

$$\rho\Big(\partial_2 f(x,\cdot)(x),\partial_2 f(y,\cdot)(y)\Big)\le L\|x-y\|,\ \forall x,y\in C,$$

where $\partial_2 f(x,\cdot)(x)$ is subdifferential of $f(x,\cdot)$ at x, i.e.,

$$\partial_2 f(x,\cdot)(x)=\{\xi\in \mathrm{H}:\langle\xi,z-y\rangle\le f(x,z),\ \forall z\in C\};$$

$\mathcal{T}_3$. $Sol(EPs)$ is nonempty;

$\mathcal{T}_4$. $f(x,\cdot)$ is convex and lower semicontinuous, subdifferentiable on C.

Remark 3.1. There isn't an inclusive relationship between the set of all functions satisfying the assumption $(\mathcal{T}_2)$ and the set of all Lipschitz-type continuous functions. Indeed, let bifunction $f:\mathrm{R}\times\mathrm{R}\to\mathrm{R}$ be defined by

$$f(x,y)=y^3-x^3-y^2+xy.$$

It is easy to check that the bifunction f is Lipschitz-type continuous with constants $c_1=c_2=\frac{1}{2}$, i.e.,

$$f(x,y)+f(y,z)\ge f(x,z)-\frac{1}{2}\|x-y\|^2-\frac{1}{2}\|y-z\|^2\ \ \forall\ x,y,z\in\mathbb{R}.$$

On the other hand, we have from the definition of f that

$$\rho\Big(\partial_2 f(x,\cdot)(x),\partial_2 f(y,\cdot)(y)\Big)=|x-y||3y+3x+1|,\ \forall\ x,y\in\mathbb{R}.$$

In order that f satisfies the assumption $(\mathcal{T}_2)$, it must have $|3y+3x+1|\le L,\ \forall\ x,y\in\mathbb{R}$ for some $L>0$. This is a contradiction. We next show that there exists a bifunction that satisfies the assumptions $(\mathcal{T}_1)$, $(\mathcal{T}_2)$ and $(\mathcal{T}_4)$ but isn't Lipschitz-type continuous. For every $x,y\in\mathbb{R}^n$, setting

$$f(x,y)=\langle Px+Qy+q,y-x\rangle+\alpha\|B(y-x)\|^2\|x\|^2,$$

where $\alpha\in\mathbb{R}$, $q\in\mathbb{R}^n$, B, P and Q are $n\times n$ matrices such that Q is symmetric positive semidefinite and $P-Q$ is negative semidefinite. As known in [28], the bifunction $g(x,y)=\langle Px+Qy+q,y-x\rangle$ is often found in Nash-Cournot equilibrium models and satisfies the assumptions $(\mathcal{T}_1)$ and $(\mathcal{T}_4)$. Hence, it is not difficult to prove that $f(x,y)$ satisfies the assumptions $(\mathcal{T}_1)$, $(\mathcal{T}_4)$ and $\partial_2 f(x,\cdot)(x)=\{Px+Qx+q\}$, $\partial_2 f(y,\cdot)(y)=\{Py+Qy+q\}$. It follows that

$$\rho\left(\partial_2 f(x,\cdot)(x),\partial_2 f(y,\cdot)(y)\right)=\|(Q-P)(x-y)\|\le\|Q-P\|\|x-y\|,\ \forall x,y\in\mathbb{R}^n.$$

Finally, we show that the bifunction f is not Lipschitz-type continuous on $\mathbb{R}\times\mathbb{R}$. For simplicity, we consider $n=1$ and $\alpha B\ne 0$. Assume that f is Lipschitz-type continuous with constants $c_1 > 0$ and $c_2 > 0$ on H, i.e., for every $x, y, z \in \mathbb{R}$ we have

$$\begin{aligned}&(Px+Qy+q)(y-x)+\alpha B^2(y-x)^2x^2+(Py+Qz+q)(z-y)\\&\quad+\alpha B^2(z-y)^2y^2\\&\geq (Px+Qz+q)(z-x)+\alpha B^2(z-x)^2x^2-c_1\|x-y\|^2-c_2\|y-z\|^2.\end{aligned}$$

Replacing $x = k+1,\ y = k,\ z = k-1$ on the last inequality, we obtain the following relation

$$\alpha B^2(-2k^2+8k-7)-\alpha B^2 Q \geq -c_1-c_2\ \forall k.$$

Taking the limit as $k \to +\infty$ on both sides of the above inequality, we get a contradiction. Hence, f is not Lipschitz-type continuous.

As is known, in some previous studies [13,24], to obtain a strong convergence algorithm for non-monotone problems, it is necessary to add the assumption that the function f is Lipschitz-type continuous. Moreover, the algorithms in [13, 24] use more than one projection at each iteration. Now, we propose a new algorithm for (EPs) which only uses one projection at each iteration. It is described as follows.

In order to find a point of the set C, we can use the following procedure:

Procedure A: Data: A Point $x \in$ H. Output: A point $R(x) \in C$.

Step a. If $x \in C$, set $R(x) = x$. Otherwise, set $y^0 = x, k = 0$.

Step b. Choose $w^k \in \partial g(y^k)$, compute $y^{k+1} = y^k - 2g(y^k)\frac{w^k}{\|w^k\|^2}$, where

$$g(x) = \max\{g_i(x) : i = 1, 2, \ldots, m\}, \forall x \in \text{H}.$$

Step c. If $y^{k+1} \in C$, then set $R(x) := y^{k+1}$ and stop. Otherwise, set $k = k+1$, go to **Step b**.

Note that if $w^k = 0$, then we have from $w^k \in \partial g(y^k)$ that

$$g(x) - g(y^k) \geq \langle w^k, x - \bar{x}^k\rangle = 0,\ \forall x \in C.$$

It follows that $g(y^k) \leq g(x) \leq 0$ for all $x \in C$, and so $y^k \in C$. We know in [17] that the number of iterations in Procedure A is finite and

$$\|R(x) - y\| \leq \|x - y\|,\quad \forall y \in C. \tag{2}$$

Algorithm 3.2. Take arbitrary starting point $x^0 \in C$, $\lambda_0 > 0,\ 0 < \nu < 1,\ \overline{L} > L$ and control parameter sequences $\{t_k\},\ \{\epsilon_k\},\ \{\eta_k\},\ \{\lambda_k\},\ \{\rho_k\}$ satisfying conditions:

$$\begin{cases} 0<\rho_k,\ \sum\limits_{k=0}^{+\infty}\rho_k<+\infty,\ t_k\in(0,1),\ \lim\limits_{k\to\infty}t_k=0,\ \sum\limits_{k=0}^{+\infty}t_k=+\infty\\ \epsilon_k\in(0,1),\ \lim\limits_{k\to\infty}\frac{\epsilon_k}{t_k}=0,\ \sum\limits_{k=0}^{+\infty}\epsilon_k<+\infty,\ \eta_k\in[0,1),\ \lim\limits_{k\to\infty}\frac{\eta_k}{t_k}=0\\ \sum\limits_{k=0}^{+\infty}\eta_k<+\infty.\end{cases} \tag{3}$$

Step 1. (Apply **Procedure A**) Set $\bar{x}^k := R(x^k)$.

Step 2. Choose $u^k \in \partial_2 f(\bar{x}^k, \bar{x}^k)$. If $u^k = 0$, then Stop. Otherwise, find $y^k \in C$ such that

$$\left\langle y^k - \bar{x}^k + \lambda_k u^k, x - y^k \right\rangle \geq -\epsilon_k \quad \forall x \in C.$$

Step 3. Take

$$v^k \in B\left(u^k, \overline{L}\|\bar{x}^k - y^k\|\right) \cap \partial_2 f(y^k, y^k),$$

where $B\left(u^k, \overline{L}\|\bar{x}^k - y^k\|\right) := \{u \in \mathrm{H} : \|u - u^k\| \leq \overline{L}\|\bar{x}^k - y^k\|\}$. If $v^k = 0$, then Stop. Otherwise, compute $z^k = (1+\theta_k)y^k - \theta_k\bar{x}^k + \lambda_k(u^k - v^k)$, where

$$\theta_k = \begin{cases} \min\{\frac{\eta_k}{\|u^k\|\|\bar{x}^k - y^k\|}, \eta_k\}, & \text{if } \ \bar{x}^k - y^k \neq 0, \\ \eta_k & \text{otherwise.} \end{cases} \tag{4}$$

Compute $x^{k+1} = t_k x^0 + (1 - t_k)z^k$ and

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\nu\|\bar{x}^k - y^k\|}{\|u^k - v^k\|}, \lambda_k + \rho_k\right\}, & \text{if } u^k - v^k \neq 0, \\ \lambda_k + \rho_k & \text{otherwise,} \end{cases} \tag{5}$$

Step 4. Let $k := k + 1$ and return to **Step 1**.

Remark 3.3. (i) If $u^k = 0$, from $u^k \in \partial_2 f(\bar{x}^k, \bar{x}^k)$ in **Step 2**, we have

$$f(\bar{x}^k, x) = f(\bar{x}^k, x) - f(\bar{x}^k, \bar{x}^k) \geq \langle u^k, x - \bar{x}^k\rangle = 0, \ \forall x \in C.$$

Hence, the algorithm terminates at iteration k and $\bar{x}^k$ is a solution of Problem (EPs).

(ii) Let $y^k = P_C(\bar{x}^k - \lambda_k u^k)$. Applying Lemma 2.2(i), we deduce that

$$\left\langle y^k - \bar{x}^k + \lambda_k u^k, x - y^k \right\rangle \geq 0 \quad \forall x \in C.$$

Thus, $y^k \in C$ satisfying **Step 2** is always determined.

(iii) We have from the assumption $(\mathcal{T}_4)$ that $\partial_2 f(y^k, y^k)$ is a nonempty, closed and convex set in H. Which together the assumption $(\mathcal{T}_2)$ implies that

$$\|u^k - P_{\partial_2 f(y^k,y^k)}(u^k)\| \leq \rho\Big(\partial_2 f(\bar{x}^k, \cdot)(\bar{x}^k), \partial_2 f(y^\cdot \cdot)(y^k)\Big) \leq L\|\bar{x}^k - y^k\|.$$

Therefore, we can always choose v^k satisfying **Step 3**. If the $f(x, \cdot)$ is differentiable on H, then v^k is uniquely determined and $v^k = \nabla f(y^k, \cdot)(y^k)$.

(iv) From (4) and Condition (3), it is easy to check that

$$\theta_k \in (0, 1), \ \theta_k\|u^k\|\|\bar{x}^k - y^k\| \leq \eta_k, \ \forall k \geq 0 \text{ and } \lim_{k\to\infty} \theta_k = 0.$$

We first obtain the following important lemma.

Lemma 3.4. *Assume that* $(\mathcal{T}_1)$-$(\mathcal{T}_4)$ *hold. Let* $p \in Sol(EPs)$ *and* $\{x^k\}$, $\{\bar{x}^k\}$, $\{y^k\}$, $\{z^k\}$, $\{\lambda_k\}$ *be the sequences generated by Algorithm 3.2. Then,*

(i) $\lambda_k \in [\min\{\frac{\nu}{L}, \lambda_0\}, \lambda_0 + M]$, $\forall k \geq 0$ *and* $\lim_{k\to\infty} \lambda_k = \lambda$, *where* $\sum_{k=0}^{+\infty} \rho_k = M$;

(ii) $\|z^k - p\|^2 \le \|x^k - p\|^2 - \left[1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2\right] \|y^k - \bar{x}^k\|^2 + 2(\lambda_0 + M)\eta_k + 4\epsilon_k$.

Proof. If $u^k - v^k \neq 0$, then we have from L-Lipschitz continuity of F and $L \le \overline{L}$ that

$$\frac{\nu\|\bar{x}^k - y^k\|}{\|u^k - v^k\|} \ge \frac{\nu\|\bar{x}^k - y^k\|}{\overline{L}\|\bar{x}^k - y^k\|} = \frac{\nu}{\overline{L}}. \tag{6}$$

Using mathematical induction proof method, we can prove that $\{\lambda_k\}$ belongs to $[\min\{\frac{\nu}{L}, \lambda_0\},\ \lambda_0 + M]$, $\forall k \ge 0$. Indeed, assume that $\lambda_k \in [\min\{\frac{\nu}{L}, \lambda_0\},\ \lambda_0 + M]$. Then, we have from $\min\{\frac{\nu}{L}, \lambda_0\} < \lambda_k + \rho_k$, (5) and (6) that $\min\{\frac{\nu}{L}, \lambda_0\} < \lambda_{k+1}$. Using (5) and Condition (3), we get

$$\lambda_{k+1} \le \lambda_k + \rho_k \le \cdots \le \lambda_0 + \sum_{i=0}^{k} \rho_i \le \lambda_0 + \sum_{i=0}^{+\infty} \rho_i = \lambda_0 + M.$$

Hence, $\lambda_{k+1} \in [\min\{\frac{\nu}{L}, \lambda_0\},\ \lambda_0 + M]$. Set $(\lambda_{k+1} - \lambda_k)^+ = \max\{0, \lambda_{k+1} - \lambda_k\}$ and $(\lambda_{k+1} - \lambda_k)^- = \max\{0, -(\lambda_{k+1} - \lambda_k)\}$. It follows from (5) that

$$\sum_{k=0}^{+\infty} (\lambda_{k+1} - \lambda_k)^+ \le \sum_{k=0}^{+\infty} \rho_k < +\infty. \tag{7}$$

Assume that $\sum_{k=0}^{+\infty} (\lambda_{k+1} - \lambda_k)^- = +\infty$. From the following quality

$$\lambda_{k+1} - \lambda_k = (\lambda_{k+1} - \lambda_k)^+ - (\lambda_{k+1} - \lambda_k)^-,$$

it follows that

$$\lambda_{k+1} - \lambda_0 = \sum_{i=0}^{k} (\lambda_{i+1} - \lambda_i) = \sum_{i=0}^{k} (\lambda_{i+1} - \lambda_i)^+ - \sum_{i=0}^{k} (\lambda_{i+1} - \lambda_i)^-.$$

Taking the limit as $k \to \infty$ on both sides of the last inequality and using (7), we have $\lambda_k \to -\infty$. That is a contradiction. Hence, $\sum_{k=0}^{+\infty} (\lambda_{k+1} - \lambda_k)^- < +\infty$. This together with (7) implies that $\lim_{k\to\infty} \lambda_k = \lambda \in [\min\{\frac{\nu}{L}, \lambda_0\},\ \lambda_0 + M]$.

Now we prove (ii). By $z^k = (1 + \theta_k)y^k - \theta_k \bar{x}^k + \lambda_k(u^k - v^k)$ and Lemma 2.1 (i), we have

$$\begin{aligned}
&\|z^k - p\|^2 \\
&= \|(1 + \theta_k)(y^k - \bar{x}^k) + (\bar{x}^k - p) + \lambda_k(u^k - v^k)\|^2 \\
&= \|(1 + \theta_k)(y^k - \bar{x}^k) + (\bar{x}^k - p)\|^2 + 2\lambda_k(1 + \theta_k)\langle u^k - v^k, y^k - \bar{x}^k\rangle \\
&\quad + 2\lambda_k\langle u^k - v^k, \bar{x}^k - p\rangle + \lambda_k^2\|v^k - u^k\|^2 \\
&= (1 + \theta_k)^2\|y^k - \bar{x}^k\|^2 + 2(1 + \theta_k)\langle y^k - \bar{x}^k, \bar{x}^k - p\rangle \\
&\quad + \|\bar{x}^k - p\|^2 + \lambda_k^2\|v^k - u^k\|^2 + 2\lambda_k(1 + \theta_k)\langle u^k - v^k, y^k - \bar{x}^k\rangle \\
&\quad + 2\lambda_k\langle u^k - v^k, \bar{x}^k - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|\bar{x}^k - p\|^2 + (1+\theta_k)^2\|y^k - \bar{x}^k\|^2 + 2\lambda_k(1+\theta_k)\langle u^k - v^k, y^k - \bar{x}^k\rangle \\
&\quad + \lambda_k^2\|v^k - u^k\|^2 + 2(1+\theta_k)\langle y^k - \bar{x}^k, \bar{x}^k - p\rangle - 2\lambda_k\langle v^k - u^k, \bar{x}^k - y^k\rangle \\
&\quad - 2\lambda_k\langle v^k - u^k, y^k - p\rangle \\
&= \|\bar{x}^k - p\|^2 + (\theta_k^2 - 1)\|y^k - \bar{x}^k\|^2 + 2\lambda_k\theta_k\langle u^k - v^k, y^k - \bar{x}^k\rangle \\
&\quad + \lambda_k^2\|v^k - u^k\|^2 + 2(1+\theta_k)\langle y^k - \bar{x}^k, y^k - p\rangle - 2\lambda_k\langle v^k, y^k - p\rangle \\
&\quad + 2\lambda_k\langle u^k, y^k - p\rangle. \qquad (8)
\end{aligned}
$$

It follows from (5) that

$$
\|v^k - u^k\| \leq \frac{\nu}{\lambda_{k+1}}\|y^k - \bar{x}^k\|. \qquad (9)
$$

Using Cauchy-Schwarz Theorem and (9), we obtain

$$
\langle u^k - v^k, y^k - \bar{x}^k\rangle \leq \|u^k - v^k\|\|y^k - \bar{x}^k\| \leq \frac{\nu}{\lambda_{k+1}}\|y^k - \bar{x}^k\|^2.
$$

This together with (8) and (9) implies that

$$
\begin{aligned}
&\|z^k - p\|^2 \\
&\leq \|\bar{x}^k - p\|^2 + (\theta_k^2 - 1)\|y^k - \bar{x}^k\|^2 \\
&\quad + 2\lambda_k\theta_k\frac{\nu}{\lambda_{k+1}}\|y^k - \bar{x}^k\|^2 + \lambda_k^2\frac{\nu^2}{\lambda_{k+1}^2}\|y^k - \bar{x}^k\|^2 \\
&\quad + 2(1+\theta_k)\langle y^k - \bar{x}^k, y^k - p\rangle - 2\lambda_k\langle v^k, y^k - p\rangle + 2\lambda_k\langle u^k, y^k - p\rangle. \\
&\leq \|\bar{x}^k - p\|^2 - \left[1 - \left(\theta_k + \lambda_k\frac{\nu}{\lambda_{k+1}}\right)^2\right]\|y^k - \bar{x}^k\|^2 \\
&\quad + 2(1+\theta_k)\langle y^k - \bar{x}^k, y^k - p\rangle - 2\lambda_k\langle v^k, y^k - p\rangle + 2\lambda_k\langle u^k, y^k - p\rangle. \qquad (10)
\end{aligned}
$$

We have from the definition of y^k in **Step 2** that $\langle y^k - \bar{x}^k + \lambda_k u^k, x - y^k\rangle \geq -\epsilon_k \ \ \forall x \in C$. Substituting x by $p \in Sol(EPs) \subset C$ on the last inequality, we obtain

$$
\langle y^k - p, \bar{x}^k - y^k - \lambda_k u^k\rangle \geq -\epsilon_k.
$$

It is equivalent to

$$
\langle y^k - \bar{x}^k, y^k - p\rangle \leq -\lambda_k\langle u^k, y^k - p\rangle + \epsilon_k. \qquad (11)
$$

On the other hand, by using $v^k \in \partial_2 f(y^k, y^k)$, $p \in Sol(EPs)$ and the pseudomonotone assumption of $f(x,y)$, we get

$$
\langle v^k, y^k - p\rangle \geq -f(y^k, p) \geq 0, \ \forall k \geq 0,
$$

which together with (10) and (11) implies that

$$
\begin{aligned}
&\|z^k - p\|^2 \\
&\leq \|\bar{x}^k - p\|^2 - \left[1 - \left(\theta_k + \lambda_k\frac{\nu}{\lambda_{k+1}}\right)^2\right]\|y^k - \bar{x}^k\|^2
\end{aligned}
$$

$$
\begin{aligned}
&-2(1+\theta_k)\lambda_k\langle u^k, y^k - p\rangle + 2\lambda_k\langle u^k, y^k - p\rangle + 2(1+\theta_k)\epsilon_k \\
&\leq \|\bar{x}^k - p\|^2 - \left[1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2\right]\|y^k - \bar{x}^k\|^2 - 2\theta_k\lambda_k\langle u^k, y^k - p\rangle \\
&\quad + 2(1+\theta_k)\epsilon_k \\
&\leq \|\bar{x}^k - p\|^2 - \left[1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2\right]\|y^k - \bar{x}^k\|^2 + 2\theta_k\lambda_k\langle u^k, \bar{x}^k - y^k\rangle \\
&\quad - 2\theta_k\lambda_k\langle u^k, \bar{x}^k - p\rangle + 2(1+\theta_k)\epsilon_k. \qquad (12)
\end{aligned}
$$

It follows from $u^k \in \partial_2 f(\bar{x}^k, \bar{x}^k)$, $p \in Sol(EPs)$ and the pseudomonotone assumption of $f(x,y)$ that

$$\langle u^k, \bar{x}^k - p\rangle \geq -f(\bar{x}^k, p) \geq 0, \ \forall k \geq 0.$$

Combining the last inequality, Remark 3.3(iv), $\lambda_k \in [\min\{\frac{\nu}{L}, \lambda_0\}, \ \lambda_0 + M]$, (2) and (12), we obtain the inequality in (ii). □

Lemma 3.5. *Assume that* $(\mathcal{T}_1)$-$(\mathcal{T}_4)$ *hold. Then, the sequences* $\{x^k\}$, $\{y^k\}$, $\{z^k\}$ *and* $\{u^k - v^k\}$ *are bounded.*

Proof. Let $p \in Sol(EPs)$. We have from Lemma 3.4(i), Remark 3.3(iv) and Condition (3) that

$$\lim_{k\to\infty}\left[1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2\right] = 1 - \nu^2 > 0,$$

which implies that there exists a nonnegative integer K_0 such that

$$1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2 > 0, \ \forall k \geq K_0.$$

From the above inequality and Lemma 3.4(ii), it follows that

$$\|z^k - p\|^2 \leq \|x^k - p\|^2 + 2(\lambda_0 + M)\eta_k + 4\epsilon_k, \ \forall k \geq K_0.$$

Therefore, from the definition of x^{k+1}, for every $k \geq K_0$, we have

$$
\begin{aligned}
\|x^{k+1} - p\|^2 &= \|t_k x^0 + (1-t_k)z^k - p\|^2 \\
&\leq t_k\|x^0 - p\|^2 + (1-t_k)\|z^k - p\|^2 \\
&\leq t_k\|x^0 - p\|^2 + (1-t_k)(\|x^k - p\|^2 + 2(\lambda_0 + M)\eta_k + 4\epsilon_k) \\
&\leq \max\left\{\|x^0 - p\|^2, \|x^k - p\|^2 + A_k\right\}, \qquad (13)
\end{aligned}
$$

where $A_k = 2(\lambda_0 + M)\eta_k + 4\epsilon_k$ for every $k \geq K_0$. Similarly, we have

$$\|x^k - p\|^2 \leq \max\{\|x^0 - p\|^2, \|x^{k-1} - p\|^2 + A_{k-1}\}.$$

This together with (13) implies that

$$
\begin{aligned}
\|x^{k+1} - p\|^2 &\leq \max\left\{\|x^0 - p\|^2, \|x^0 - p\|^2 + A_k, \|x^{k-1} - p\|^2 + A_{k-1} + A_k\right\} \\
&= \max\left\{\|x^0 - p\|^2 + A_k, \|x^{k-1} - p\|^2 + A_{k-1} + A_k\right\}
\end{aligned}
$$

$$\begin{aligned}
&\dots\\
&\le \max\left\{\|x^0-p\|^2+\sum_{i=K_0+1}^{k}A_i, \|x^{K_0}-p\|^2+\sum_{i=K_0}^{k}A_i\right\}\\
&\le \max\left\{\|x^0-p\|^2,\ \|x^{K_0}-p\|^2\right\}+\sum_{k=K_0}^{\infty}A_k\\
&< +\infty,
\end{aligned}$$

where the latest equality holds because $\sum_{k=K_0}^{\infty} A_k < +\infty$. This implies that $\{x^k\}$ is bounded. The boundness of $\{\bar{x}^k\}$ follows from (2). Again, by Lemma 3.4(ii), for all $k \ge K_0$, we have

$$\|z^k-p\|^2 \le \|x^k-p\|^2 + 2(\lambda_0+M)\eta_k + 4\epsilon_k \tag{14}$$

and

$$\left[\left(\theta_k+\lambda_k\frac{\nu}{\lambda_{k+1}}\right)^2-1\right]\|y^k-\bar{x}^k\|^2 \le \|x^k-p\|^2-\|z^k-p\|^2+2(\lambda_0+M)\eta_k+4\epsilon_k.$$

It follows from (14) that $\{z^k\}$ is bounded. This together with the last inequality and the boundedness of $\{x^k\}$ implies that $\{y^k\}$ is bounded. Finally, we can deduce from $v^k \in B\left(u^k, \overline{L}\|\bar{x}^k-y^k\|\right)$ that

$$\|u^k-v^k\| \le \overline{L}\|\bar{x}^k-y^k\|,$$

and so the sequence $\{u^k-v^k\}$ is bounded. □

Lemma 3.6. *Assume that* $(\mathcal{T}_1)$-$(\mathcal{T}_4)$ *hold. Let* $\|\bar{x}^k-y^k\| \to 0$ *and a subsequence* $\{\bar{x}^{k_i}\}$ *of* $\{\bar{x}^k\}$ *converge weakly to* p. *Then,* $p \in Sol(EPs)$.

Proof. Since $\|\bar{x}^k-y^k\| \to 0$ and the subsequence $\{\bar{x}^{k_i}\}$ converges weakly to p, the sequence $\{y^{k_i}\}$ also converges weakly to p. From $v^k \in B\left(u^k, \overline{L}\|\bar{x}^k-y^k\|\right)$, it follows that

$$\|u^k-v^k\| \le \overline{L}\|\bar{x}^k-y^k\|,$$

and so $\lim_{k\to\infty}\|u^k-v^k\|=0$. We get from **Step 2** that

$$\langle y^{k_i}-\bar{x}^{k_i}+\lambda_{k_i}u^{k_i}, x-y^{k_i}\rangle \ge -\epsilon_{k_i}\ \forall x \in C,$$

which together with $u^{k_i} \in \partial_2 f(\bar{x}^{k_i}, \bar{x}^{k_i})$ implies that

$$\begin{aligned}
\langle \bar{x}^{k_i}-y^{k_i}, x-y^{k_i}\rangle &\le \lambda_{k_i}\langle u^{k_i}, x-y^{k_i}\rangle+\epsilon_{k_i}\\
&\le \lambda_{k_i}(\langle v^{k_i}, x-y^{k_i}\rangle+\langle u^{k_i}-v^{k_i}, x-y^{k_i}\rangle)+\epsilon_{k_i}\\
&\le \lambda_{k_i}f(y^{k_i},x)+\lambda_{k_i}\langle u^{k_i}-v^{k_i}, x-y^{k_i}\rangle+\epsilon_{k_i}.
\end{aligned}$$

It follows that

$$\frac{1}{\lambda_{k_i}}\langle \bar{x}^{k_i}-y^{k_i}, x-y^{k_i}\rangle \le f(y^{k_i},x)+\langle u^{k_i}-v^{k_i}, x-y^{k_i}\rangle+\frac{1}{\lambda_{k_i}}\epsilon_{k_i}.$$

For each fixed point $x \in C$, taking the limit as $i \to \infty$ on both sides of the last inequality, using $\lim_{i\to\infty} \|\bar{x}^{k_i} - y^{k_i}\| = 0$, $\lim_{i\to\infty} \|u^{k_i} - v^{k_i}\| = 0$, the weak upper semicontinuity of the function $f(\cdot, y)$ and the boundedness of the sequence $\{y^k\}$, we get

$$f(p, x) \geq 0 \quad \forall x \in C.$$

It means that $p \in Sol(EPs)$. □

Now we state and prove the main convergence result of the algorithm in the following theorem.

Theorem 3.7. *Let bifunction $f : C \times C \to \mathbb{R}$ satisfy the assumptions $(\mathcal{T}_1)$-$(\mathcal{T}_4)$. Then, the sequence $\{x^k\}$ generated by Algorithm 3.2 converges strongly to a solution $p \in Sol(EPs)$, where $p = P_{Sol(EPs)}(x^0)$.*

Proof. Set $\xi_k = \|x^k - p\|^2$, $\alpha_k = 2\langle x^0 - p, x^{k+1} - p\rangle$ and $\beta_k = 2(\lambda_0 + M)\eta_k + 4\epsilon_k$. To prove this theorem, we consider two following cases.

Case 1. Suppose that there exists $\bar{k} \in \mathrm{N}$ such that $\xi_{k+1} \leq \xi_k$ for all $k \geq \bar{k}$. Then, there exists the limit $\lim_{k\to\infty} \xi_k \in [0, \infty)$. By using the definition of x^{k+1} in **Step 3**, Condition 3 and the relation $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$ for all $u, v \in \mathrm{H}$, we obtain

$$\begin{aligned}\|x^{k+1} - p\|^2 &= \|(1 - t_k)(z^k - p) + t_k(x^0 - p)\|^2\\ &\leq (1 - t_k)^2\|z^k - p\|^2 + 2t_k\langle x^0 - p, x^{k+1} - p\rangle\\ &\leq \|z^k - p\|^2 + 2t_k\langle x^0 - p, x^{k+1} - p\rangle.\end{aligned}$$

Therefore, from Lemma 3.4(ii), it follows that

$$\begin{aligned}\|x^{k+1} - p\|^2 \leq \|x^k - p\|^2 &- \left[1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2\right] \|y^k - \bar{x}^k\|^2\\ &+ 2t_k\langle x^0 - p, x^{k+1} - p\rangle + 2(\lambda_0 + M)\eta_k + 4\epsilon_k,\end{aligned}$$

which implies that

$$\begin{aligned}&\left[1 - \left(\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)^2\right] \|y^k - \bar{x}^k\|^2\\ \leq\ &- \xi_{k+1} + \xi_k + 2(\lambda_0 + M)\eta_k + 4\epsilon_k + t_k Q_0\end{aligned} \tag{15}$$

for every $k \geq 0$, where $Q_0 := \sup\{2\langle x^0 - p, x^{k+1} - p\rangle : k = 0, 1, \ldots\} < \infty$ (since the sequence $\{x^k\}$ is bound). Taking the limit as $k \to \infty$ on both sides of the last inequality and using Lemma 3.4(i), Remark 3.3(iv) and Condition (3), we obtain

$$\lim_{k\to\infty} \|\bar{x}^k - y^k\| = 0. \tag{16}$$

Observe that

$$\|z^k - \bar{x}^k\| = \|(1 + \theta_k)y^k - \theta_k\bar{x}^k + \lambda_k(u^k - v^k) - \bar{x}^k\|$$

$$\le (1+\theta_k)\|y^k - \bar{x}^k\| + \lambda_k \frac{\nu}{\lambda_{k+1}}\|y^k - \bar{x}^k\|$$
$$= \left(1+\theta_k + \lambda_k \frac{\nu}{\lambda_{k+1}}\right)\|y^k - \bar{x}^k\|,$$

which together with (16) implies that $\lim_{k\to\infty}\|z^k - \bar{x}^k\| = 0$. By the definition of x^{k+1} and boundedness of the sequence $\{z^k\}$, we have

$$\|x^{k+1} - z^k\| = t_k\|x^0 - z^k\| \le t_k Q_1 \to 0 \quad \text{as} \quad k \to \infty,$$

where $Q_1 = \sup\{\|x^0 - z^k\| : k = 0, 1, \ldots\} < +\infty$. This together with $\lim_{k\to\infty}\|z^k - \bar{x}^k\| = 0$ implies that

$$\|x^{k+1} - \bar{x}^k\| \le \|x^{k+1} - z^k\| + \|z^k - \bar{x}^k\| \to 0 \text{ as } k \to \infty. \tag{17}$$

By the definition of x^{k+1} in **Step 3** and the inequality $\|u+v\|^2 \le \|u\|^2 + 2\langle v, u+v\rangle \quad \forall u, v \in \mathrm{H}$, we get

$$\begin{aligned}\|x^{k+1} - p\|^2 &= \|t_k(x^0 - p) + (1-t_k)(z^k - p)\|^2\\ &\le (1-t_k)^2\|z^k - p\|^2 + 2t_k(1-t_k)\langle x^0 - p, x^{k+1} - p\rangle\\ &\le (1-t_k)\|z^k - p\|^2 + 2t_k\langle x^0 - p, x^{k+1} - p\rangle.\end{aligned}$$

From the last inequality and Lemma 3.4(ii), it follows that

$$\begin{aligned}&\|x^{k+1} - p\|^2\\ &\le (1-t_k)\|x^k - p\|^2 + 2t_k\langle x^0 - p, x^{k+1} - p\rangle + (1-t_k)[2(\lambda_0 + M)\eta_k + 4\epsilon_k]\\ &\le (1-t_k)\|x^k - p\|^2 + 2t_k\langle x^0 - p, x^{k+1} - p\rangle + 2(\lambda_0 + M)\eta_k + 4\epsilon_k,\end{aligned}$$

which implies that

$$\xi_{k+1} \le (1-t_k)\xi_k + t_k\alpha_k + \beta_k. \tag{18}$$

On the other hand, since the sequence $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i+1}\}$ such that $x^{k_i+1} \rightharpoonup z$ as $i \to \infty$ and

$$\limsup_{k\to\infty}\langle x^0 - p, x^{k+1} - p\rangle = \lim_{i\to\infty}\langle x^0 - p, x^{k_i+1} - p\rangle. \tag{19}$$

We deduce from (17) that $\bar{x}^{k_i} \rightharpoonup z$ as $i \to \infty$. Applying $\lim_{k\to\infty}\|\bar{x}^k - y^k\| = 0$ and Lemma 3.6, we get $z \in Sol(EPs)$. From this, (19) and Lemma 2.2(i), it follows that

$$\limsup_{k\to\infty}\alpha_k = 2\lim_{k\to\infty}\langle x^0 - p, x^{k_i+1} - p\rangle = 2\langle x^0 - p, z - p\rangle \le 0.$$

By using Lemma 2.4, the last inequality, $\limsup_{k\to\infty}\alpha_k \le 0$ and Condition (3), we deduce

$$\lim_{k\to\infty}\xi_k = \lim_{k\to\infty}\|x^k - p\|^2 = 0.$$

Thus, $\{x^k\}$ converges strongly to the solution $p = Pr_{Sol(EPs)}(x^0)$.

Case 2. We now assume that there is not $\bar{k} \in \mathcal{N}$ such that $\{\xi_k\}_{k=\bar{k}}^{\infty}$ is monotonically decreasing. Then, there exists an integer $k_0 \ge \bar{k}$ such that

$\xi_{k_0} \le \xi_{k_0+1}$. We have from Lemma 2.5 that there exists a subsequence $\{\xi_{\tau(k)}\}$ of $\{\xi_k\}$ such that

$$0 \le \xi_k \le \xi_{\tau(k)+1},\ \xi_{\tau(k)} \le \xi_{\tau(k)+1} \quad \forall k \ge k_0,$$

where $\tau(k) = \max\{i \in \mathrm{N} : k_0 \le i \le k,\ \xi_i \le \xi_{i+1}\}$. Using $\xi_{\tau(k)} \le \xi_{\tau(k)+1},\ \forall k \ge k_0$ and (15), one has

$$\begin{aligned} 0 &\le \left[1 - \left(\theta_{\tau(k)} + \lambda_{\tau(k)}\frac{\nu}{\lambda_{\tau(k)+1}}\right)^2\right] \|y^{\tau(k)} - \bar{x}^{\tau(k)}\| \\ &\le -\xi_{\tau(k)+1} + \xi_{\tau(k)} + t_{\tau(k)}Q_0 + 2(\lambda_0 + M)\eta_{\tau(k)} + 4\epsilon_{\tau(k)} \\ &\le t_{\tau(k)}Q_0 + 2(\lambda_0 + M)\eta_{\tau(k)} + 4\epsilon_{\tau(k)}. \end{aligned}$$

Passing to the limit in the above unequal and taking into account Condition (3), we obtain $\lim_{k\to\infty} \|y^{\tau(k)} - \bar{x}^{\tau(k)}\| = 0$. By the same arguments as in the **Case 1**, we can show that

$$\lim_{n\to\infty} \|x^{\tau(k)+1} - \bar{x}^{\tau(k)}\| = \lim_{n\to\infty} \|\bar{x}^{\tau(k)} - z^{\tau(k)}\| = \lim_{n\to\infty} \|z^{\tau(k)} - y^{\tau(k)}\| = 0. \tag{20}$$

Since $\{x^{\tau(k)}\}$ is bounded, there exists a subsequence of $\{x^{\tau(k)}\}$, still denoted by $\{x^{\tau(k)+1}\}$, which converges weakly to z. Following similar arguments as in **Case 1**, we conclude that $z \in Sol(EPs)$ and

$$\limsup_{k\to\infty} \alpha_{\tau(k)} \le 0. \tag{21}$$

We deduce from (18) and $\xi_{\tau(k)} \le \xi_{\tau(k)+1},\ \forall k \ge k_0$ that

$$t_{\tau(k)}\xi_{\tau(k)} \le \xi_{\tau(k)} - \xi_{\tau(k)+1} + t_{\tau(k)}\alpha_{\tau(k)} + \beta_{\tau(k)} \le t_{\tau(k)}\alpha_{\tau(k)} + \beta_{\tau(k)}.$$

It is equivalent to $\xi_{\tau(k)} \le \alpha_{\tau(k)} + \frac{\beta_{\tau(k)}}{t_{\tau(k)}}$. From (21), Condition (3) and the last inequality, it follows that

$$\limsup_{k\to\infty} \xi_{\tau(k)} \le \limsup_{k\to\infty} \alpha_{\tau(k)} \le 0,$$

which implies that $\lim_{k\to\infty} \xi_{\tau(k)} = 0$. Using (2), we have

$$\begin{aligned} \sqrt{\xi_{\tau(k)+1}} &= \|x^{\tau(k)+1} - p\| \\ &\le \|x^{\tau(k)+1} - \bar{x}^{\tau(k)}\| + \|\bar{x}^{\tau(k)} - p\| \\ &\le \|x^{\tau(k)+1} - \bar{x}^{\tau(k)}\| + \xi_{\tau(k)}. \end{aligned}$$

Taking the limit as $k \to \infty$ on both sides of the last inequality and using (20), we obtain $\lim_{k\to\infty} \xi_{\tau(k)+1} = 0$. This together with $0 \le \xi_k \le \xi_{\tau(k)+1}$ for all $k \ge k_0$ implies that $\lim\limits_{k\to\infty} \xi_k = 0$. Is means that the sequence $\{x^k\}$ converges strongly to $p \in Sol(EPs)$. The proof is complete. □

4. Algorithm for multivalued variational inequality

In this section, by modifying the Algorithm 3.2, we develop a new algorithm for solving the multivalued variational inequality problem ($MVIPs$) in real Hilbert space.

Algorithm 4.1. Take arbitrary starting point $x^0 \in C$, $\lambda_0 > 0,\ 0 < \nu < 1,\ \overline{L} > L$ and control parameter sequences $\{t_k\}$, $\{\epsilon_k\}$, $\{\eta_k\}$, $\{\lambda_k\}$, $\{\rho_k\}$ satisfying

$$\begin{cases} 0 < \rho_k,\ \sum\limits_{k=0}^{+\infty} \rho_k < +\infty,\ t_k \in (0,1),\ \lim\limits_{k\to\infty} t_k = 0,\ \sum\limits_{k=0}^{+\infty} t_k = +\infty \\ \epsilon_k \in (0,1),\ \lim\limits_{k\to\infty} \frac{\epsilon_k}{t_k} = 0,\ \sum\limits_{k=0}^{+\infty} \epsilon_k < +\infty,\ \eta_k \in [0,1),\ \lim\limits_{k\to\infty} \frac{\eta_k}{t_k} = 0 \\ \sum\limits_{k=0}^{+\infty} \eta_k < +\infty. \end{cases}$$

Step 1. (Apply **Procedure A**) Set $\bar{x}^k := R(x^k)$.
Step 2. Choose $u^k \in F(\bar{x}^k)$. If $u^k = 0$, then Stop. Otherwise, find $y^k \in C$ such that

$$\left\langle y^k - \bar{x}^k + \lambda_k u^k, x - y^k \right\rangle \geq -\epsilon_k \quad \forall x \in C. \tag{22}$$

Step 3. Take $v^k \in B\left(u^k, \overline{L}\|\bar{x}^k - y^k\|\right) \cap F(y^k)$, where $B\left(u^k, \overline{L}\|\bar{x}^k - y^k\|\right) := \{u \in \mathrm{H} : \|u - u^k\| \leq \overline{L}\|\bar{x}^k - y^k\|\}$. If $v^k = 0$, then Stop. Otherwise, compute $z^k = (1+\theta_k)y^k - \theta_k \bar{x}^k + \lambda_k(u^k - v^k)$, where

$$\theta_k = \begin{cases} \min\{\frac{\eta_k}{\|u^k\|\|\bar{x}^k - y^k\|}, \eta_k\}, & \text{if } \ \bar{x}^k - y^k \neq 0, \\ \eta_k & \text{otherwise.} \end{cases}$$

Compute $x^{k+1} = t_k x^0 + (1-t_k)z^k$ and

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\nu\|\bar{x}^k - y^k\|}{\|u^k - v^k\|}, \lambda_k + \rho_k\right\}, & \text{if } u^k - v^k \neq 0, \\ \lambda_k + \rho_k & \text{otherwise,} \end{cases}$$

Step 4. Let $k := k+1$ and return to **Step 1**.

Obviously, if $u^k = 0$, then $\bar{x}^k$ is a solution of ($MVIPs$), and so Algorithm 4.1 stops at the k-th iteration. Hence, in the rest of this paper, we assume that $u^k \neq 0$ for all $k \geq 0$. To get the strong convergence theorem of Algorithm 4.1, we need the following assumptions of mapping F.

$\mathcal{A}_1$. The F is pseudomonotone and L-Lipschitz Hausdorff continuous on H, i.e.,

$$\rho(F(x), F(y)) \leq L\|x - y\|,\ \forall x,\ y \in C;$$

$\mathcal{A}_2$. The solution set $Sol(MVIPs)$ of Problem ($MVIPs$) is nonempty;
$\mathcal{A}_3$. Iif $x^k \rightharpoonup \bar{x}$ and $u^k \in F(x^k)$, then there exists a subsequence $\{u^{k_j}\}$ of $\{u^k\}$ such that $u^{k_j} \rightharpoonup \bar{u} \in F(\bar{x})$;

Regarding the assumption $(\mathcal{A}_3)$, we recall the concept and characterization of upper semicontinuity of a set-valued operator in topological spaces ([19]). Let S_1 and S_2 be topological spaces. A set-valued operator with nonempty values $A : S_1 \to 2^{S_2}$ is said to be upper semicontinuous if for all $x \in S_1$ and for every open set V containing $A(x)$, there exists a neighborhood U of x such that $A(U) \subset V$. It is well-known that if A is compact-valued, then A is upper semicontinuous if and only if, for every net $\{x^k\}$ such that $x^k \to \bar{x}$ and for all $u^k \in A(x^k)$, then exists a subsequence $\{u^{k_j}\}$ of $\{u^k\}$ such that $u^{k_j} \to \bar{u} \in A(\bar{x})$ (see [19, Lemma 2.1]). Hence, if F is weakly compact-valued and weakly upper semicontinuous on H, then F satisfies the assumption $(\mathcal{A}_3)$.

Next, we show that Algorithm 4.1 is strongly convergent in the following theorem.

Theorem 4.2. *Let $F : \mathrm{H} \to 2^{\mathrm{H}}$ satisfy the assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_3)$. Then, the sequence $\{x^k\}$ generated by Algorithm 4.1 converges strongly to a solution $p \in Sol(MVIPs)$, where $p = P_{Sol(MVIPs)}(x^0)$.*

Proof. Assume that $\|\bar{x}^k - y^k\| \to 0$ and $\{\bar{x}^{k_i}\}$ is a subsequence of $\{\bar{x}^k\}$ converging weakly to p. Then, the sequence $\{y^{k_i}\}$ also converges weakly to p. We have from (22) that

$$\langle y^{k_i} - \bar{x}^{k_i} + \lambda_{k_i} u^{k_i}, x - y^{k_i}\rangle \geq -\epsilon_{k_i} \ \forall x \in C,$$

which implies that

$$\langle \bar{x}^{k_i} - y^{k_i}, x - y^{k_i}\rangle + \lambda_{k_i}\langle u^{k_i}, y^{k_i} - \bar{x}^{k_i}\rangle \leq \lambda_{k_i}\langle u^{k_i}, x - \bar{x}^{k_i}\rangle + \epsilon_{k_i} \ \ \forall x \in C. \tag{23}$$

By the assumption $(\mathcal{A}_3)$, we can assume that $u^{k_i} \rightharpoonup u_p \in F(p)$ as $i \to \infty$. For each fixed point $x \in C$, passing to the limit for i tending to $+\infty$ in (23) and taking into account that $\lim_{i\to\infty} \|\bar{x}^{k_i} - y^{k_i}\| = 0 = \lim_{i\to\infty} \epsilon_{k_i} = 0$, we obtain

$$\liminf_{i\to\infty}\langle u^{k_i}, x - \bar{x}^{k_i}\rangle \geq 0 \ \ \forall x \in C.$$

Let $\{\gamma_j\}$ be a positive decreasing sequence and $\gamma_j \to 0$ as $j \to \infty$. Then, for each $j \in \mathrm{N}$, there exists a smallest positive integer h_j such that

$$\left\langle u^{h_j}, x - \bar{x}^{h_j}\right\rangle + \gamma_j \geq 0 \ \ \forall x \in C.$$

Observe that $\{h_j\}$ is increasing. Setting $\varrho^{h_j} := \frac{1}{\|u^{h_j}\|^2} u^{h_j}$, we have $\langle u^{h_j}, \varrho^{h_j}\rangle = 1$ for every $j \in \mathrm{N}$ and

$$\left\langle u^{h_j}, x + \gamma_j \varrho^{h_j} - \bar{x}^{h_j}\right\rangle \geq 0 \ \ \forall x \in C.$$

It follows from the above inequality and the pseudomonotonicity of F that

$$\left\langle u_{x+\gamma_j\varrho^{h_j}}, x + \gamma_j\varrho^{h_j} - \bar{x}^{h_j}\right\rangle \geq 0 \ \ \forall x \in C, u_{x+\gamma_j\varrho^{h_j}} \in F(x + \gamma_j\varrho^{h_j}). \tag{24}$$

If $u_p = 0$, then p is a solution of Problem $(MVIPs)$. So, we can assume that $u_p \neq 0$. It follows from lower weak semicontinuity of the norm mapping that

$0 < \|u_p\| \le \liminf_{j\to\infty} \|u^{h_j}\|$, which implies

$$0 \le \limsup_{j\to\infty} \gamma_j \|\varrho^{h_j}\| = \limsup_{j\to\infty} \frac{\gamma_j}{\|u^{h_j}\|} \le \frac{\limsup_{j\to\infty} \gamma_j}{\liminf_{j\to\infty} \|u^{h_j}\|} = 0.$$

Consequently

$$\lim_{j\to\infty} \gamma_j \|\varrho^{h_j}\| = 0. \tag{25}$$

For each $u_x \in F(x)$, using L-Lipschitz continuity of the mapping F, we can always take $u_{\bar{x}^{h_j}} \in F\left(x + \gamma_j \varrho^{h_j}\right)$ such that

$$\|u_x - u_{\bar{x}^{h_j}}\| \le \rho\left(F(x), F\left(x + \gamma_j \varrho^{h_j}\right)\right) \le \overline{L}\|\gamma_j \varrho^{h_j}\|.$$

From (25) and the last inequality, it follows that

$$\lim_{j\to\infty} \|u_x - u_{\bar{x}^{h_j}}\| = 0. \tag{26}$$

Replacing $u_{x+\gamma_j\varrho^{h_j}} := u_{\bar{x}^{h_j}} \in F\left(x + \gamma_j \varrho^{h_j}\right)$ into (24), we get

$$\left\langle u_{\bar{x}^{h_j}}, x + \gamma_j \varrho^{h_j} - \bar{x}^{h_j} \right\rangle \ge 0 \quad \forall x \in C.$$

Passing the limit as $j \to +\infty$ into the last inequality, using (25), (26) and $\lim_{j\to\infty} \gamma_j = 0$, we have

$$\langle u_x, x - p\rangle \ge 0 \quad \forall x \in C.$$

Set $\bar{x}^i := \frac{1}{i}x + (1 - \frac{1}{i})p \in C$ for all $i > 0$. Then, there exists $\bar{u}^i \in F(\bar{x}^i)$ such that

$$0 \le \langle \bar{u}^i, \bar{x}^i - p\rangle = \langle \bar{u}^i, \frac{1}{i}x + (1 - \frac{1}{i})p - p\rangle = \frac{1}{i}\langle \bar{u}^i, x - p\rangle \quad \forall x \in C.$$

By the assumption $(\mathcal{A}_3)$, we can assume that $\{\bar{u}^i\}$ converges weakly to $\bar{u}_p \in F(p)$. Taking the limit as $i \to +\infty$ on the last inequality, we have

$$\langle \bar{u}_p, x - p\rangle \ge 0 \quad \forall x \in C.$$

It implies $p \in Sol(MVIPs)$. Similar to the proof of Lemmas 3.4, 3.5 and Theorem 3.7, we can prove that the sequence $\{x^k\}$ converges strongly to p. □

Remark 4.3. If F is bounded for some $x \in \mathrm{H}$, then the above theorem's result is still true if $(\mathcal{A}_3)$ is replaced by the following more general assumption: for every $x^k \rightharpoonup \bar{x}$ and $u^k \in F(x^k)$ such that $u^{k_j} \rightharpoonup \bar{u}$, then $\bar{u} \in F(\bar{x})$. Indeed, since $F(x)$ is bounded for some x, there exist a constant $\Gamma > 0$ and a point $x^0 \in \mathrm{H}$ such that $\|x\| \le \Gamma$ for every $y \in F(x^0)$. From L-Lipschitz continuity of the mapping F, we can always choose $\bar{u}^k \in F(x^0)$ such that $\|u^k - \bar{u}^k\| \le \bar{L}\|x^k - x^0\|$. Then

$$\|u^k\| \le \|u^k - \bar{u}^k\| + \|\bar{u}^k\| \le \bar{L}\|x^k - x^0\| + \Gamma < +\infty.$$

It follows that $\{u^k\}$ is bounded. Which together with $\bar{x}^{k_i} \rightharpoonup p$ implies that there exists a subsequence $\{u^{K_i}\}$ of $\{u^{k_i}\}$ such that $u^{K_i} \rightharpoonup u_p \in F(p)$. From here, the proof is completely similar in the proof of the above theorem.

5. Computational experiments

In this final, we present some numerical examples to illustrate proposed algorithms. All programming is coded in Matlab R2016a and the program was run on a PC Intel(R) Core(TM) i5-2430M CPU @ 2.40 GHz 4GB Ram. We used the Optimization Toolbox (fmincon) to solve strongly convex subproblems that are generated by proposed algorithms.

Example 5.1. Let $\mathrm{H} = \mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n : \langle a, x\rangle = q\}$ $(0 \neq a \in \mathbb{R}^n,\ q \in \mathbb{R})$. Consider Problem (EPs) with the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\begin{aligned} f(x, y) =\ & \max\{\frac{1}{2}\|y\|^2 + q, \frac{1}{2}\|y\|^2 + \langle a, y\rangle\} - \max\{\frac{1}{2}\|x\|^2 + q, \frac{1}{2}\|x\|^2 + \langle a, x\rangle\} \\ & + \alpha\|B(y - x)\|^2\|x\|^2. \end{aligned}$$

By the same argument as in Remark 3.1, we can prove that $f(x, y)$ is not Lipschitz-type continuous on C. It is well-known that $h(x) = \max\{\frac{1}{2}\|x\|^2 + q, \frac{1}{2}\|x\|^2 + \langle a, x\rangle\}$ is convex, subdifferentiable on $\mathbb{R}^n$ and

$$\partial h(x) = \begin{cases} \{x + a\} & \text{if } \langle a, x\rangle > 0, \\ \{x\} & \text{if } \langle a, x\rangle > 0, \\ [x, x + a] & \text{if } \langle a, x\rangle = q, \end{cases}$$

where $[x, x + a] = \{tx + (1 - t)(x + a) : t \in [0, 1]\}$. It follows that

$$\rho\Big(\partial_2 f(x, \cdot)(x), \partial_2 f(y, \cdot)(y)\Big) = \|x - y\|,\ \forall x, y \in C.$$

Therefore, $f(x, y)$ satisfies assumptions $(\mathcal{T}_1)$, $(\mathcal{T}_2)$ and $(\mathcal{T}_4)$ on C.

Test 1. In this test, we perform an experiment to show the numerical behaviors of Algorithm 3.2 for solving Example 5.1 in space $\mathbb{R}^5$. The initial point is $x^0 = (-34, 0, 0, 0, 0)^\top$ and the data is chosen as follows:

$$a = (1, 1, 2, 3, -1)^\top,\ q = -34,\ B = \begin{pmatrix} 1 & 2 & 3 & 8 & 0 \\ -2 & 3 & 0 & -1 & -9 \\ 0 & 1 & 9 & 8 & -3 \\ 6 & -1 & 2 & 3 & -5 \\ -2 & 9 & 8 & -6 & 8 \end{pmatrix},$$

$\lambda_0 = 0.5,\ \nu = 0.5,\ \overline{L} = 2,\ t_k = \frac{1}{25k+1},\ \rho_k = \frac{1}{(k+1)^{1.5}},\ \epsilon_k = 0,\ \eta_k = \frac{1}{(25k+1)^{2.2}}.$

Note that if $\epsilon_k = 0$, then y^k of Step 2 is defined by

$$y^k = P_C(x^k - \lambda_k u^k).$$

The stopping criteria is $Err = \|x^k - x^{k-1}\| \leq \epsilon$ with $\epsilon = 10^{-3}$. Figure 1 shows convergent results of $x^k(i) - x^{k-1}(i)$, $i = 1, 2, \ldots, 5$, where $x^k(i)$ is the i-th coordinate of x^k.

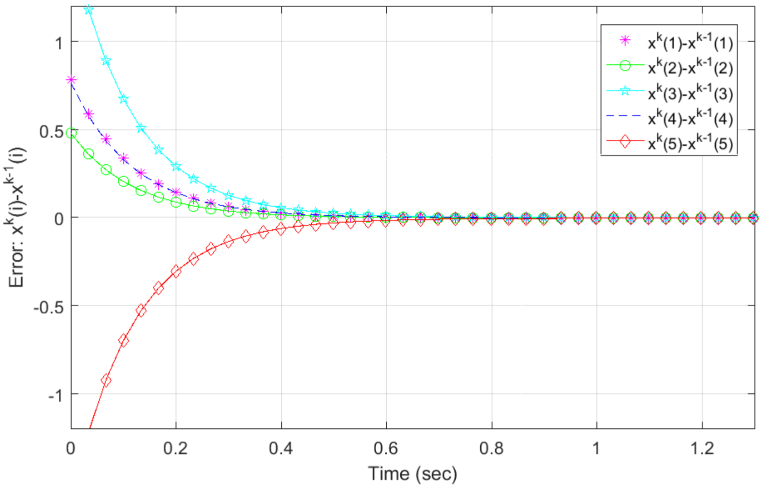

FIGURE 1. Convergence of Algorithm 3.2.

Example 5.2. Let $\mathrm{H} = \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A be an $m \times n$ matrix. Consider Problem (EPs) with the feasible C is a polyhedral convex set given by

$$C = \{x \in \mathbb{R}^n : Ax \leq b\},$$

and the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as in Remark 3.1:

$$f(x, y) = \langle Px + Qy + q, y - x\rangle + \|B(y - x)\|^2\|x\|^2.$$

As shown in Remark 3.1, the $f(x, y)$ satisfies the assumptions $(\mathcal{T}_1)$, $(\mathcal{T}_2)$ and $(\mathcal{T}_4)$ but is not Lipschitz-type continuous.

Test 2. Algorithm 3.2 combines the approximate projection method with the Halpern iteration technique and uses self-adaptive step sizes. Hence, the efficiency of the algorithm depends very much on the choice of the initial point x^0 and the parameters λ_k, t_k, where λ_k is the step size of the approximate projection on C of $\bar{x}^k - \lambda_k u^k$ and updated by formula (5) via the previous iteration points and ρ_k; t_k is the parameter in the Halpern iterative formula

$$x^{k+1} = t_k x^0 + (1 - t_k)z^k.$$

In this test, we apply Algorithm 3.2 to solve Example 5.2 with different given initial points and parameters ρ_k, t_k. We will use $\lambda_0 = 0.5$, $\nu = 0.5$, $\overline{L} = \|P - Q\| + 1$, $\rho_k = \frac{1}{k^2+1}$, $\eta_k = \epsilon_k = 0$ for all k. The matrix B is chosen as in

Test 1, the matrices $P,\ Q,\ q$ are chosen as in [13,28]:

$$Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},\ P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},\ q = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix},$$

and the matrices $A,\ b$ are chosen as follows:

$$A^T = \begin{pmatrix} -1 & 1 & -2 & 0 & 1 & -1 & 1 & -3 & -2 & -2 \\ -1 & -2 & -1 & -2 & -1 & -2 & -3 & 3 & 4 & 2 \\ -1 & -1 & -0.5 & 1 & -1 & -2 & -4 & -3 & -5 & 2 \\ 0 & 2 & 1 & -2 & 1 & 1 & 2 & 2 & -3 & 1 \\ -1 & -0.5 & 2 & 1.5 & -2 & -1 & 3 & 2 & 5 & 0 \end{pmatrix},$$

$$b^T = (0, 1, 0, 1, -1, 2, 2, -1, -1, -2).$$

The stopping criteria is $Err = \|x^k - x^{k-1}\| \le \epsilon$ with $\epsilon = 10^{-3}$ and the approximate solution computed by Algorithm 3.2 is

$$x^* = (2.3129, 0.5307, 0.7121, 0.2040, 1.1518)^T.$$

The computation results are shown in Table 1. From this table, we can make the following comments about the algorithm.

(a) The efficiency of the algorithm depends very much on the choice of the parameters t_k. For example, in this test, we have chosen $t_k = \frac{1}{5k+1}$ and $t_k = \frac{1}{k+1}$. In the first case, the program that encodes the proposed algorithm runs much quickly than in the second case.

(b) The speed of our algorithm is less affected by the parameters ρ_k. This shows that the parameter λ_k is mostly updated based on previous iteration points.

(c) The program that encodes the proposed algorithm runs quickly if the initial point x^0 is close to a solution of the problem. Conversely, if the initial point x^0 is far from a solution, then the program takes much more time.

Test 3. In this test, we solve Example 5.2 with the bifunction:

$$f(x, y) = \langle Px + Qy + q, y - x\rangle$$

and perform some experiments to show the numerical behaviors of Algorithm 3.2, Halpern subgradient method (HSM) in [27], Halpern subgradient extragradient method ($HSEM$) in [13] and the extragradient-viscosity method (EVM) in [30]. Calculations are done with the following data:

- The matrices $P,\ Q,\ q,\ A,\ b$ and x^0 are chosen as in [13,27], in detail, A is a matrix of the size $m \times n$ with its entries generated randomly in $[-2, 2]$, elements of b generated randomly in $[1, 3]$; the matrix $P = Q-T$ where the symmetric positive semidefinite matrix Q is made by using Q_1 and a random orthogonal matrix, the negative semidefinite T is made by Q_2 and another random orthogonal matrix; Q_1, Q_2 are random

TABLE 1. The comparative results for different starting points and parameters.

	Parameters		Algorithm 3.2	
Init.point x^0	t_k	ρ_k	Iterations.	CPU times
$(1,3,1,1,-2)^\top$	$\frac{1}{k+1}$	$\frac{1}{k^2+1}$	55	2.0904
$(1,3,1,1,-2)^\top$	$\frac{1}{2k+1}$	$\frac{1}{k^2+1}$	40	1.6124
$(1,3,1,1,-2)^\top$	$\frac{1}{3k+1}$	$\frac{1}{k^2+1}$	34	1.5508
$(1,3,1,1,-2)^\top$	$\frac{1}{4k+1}$	$\frac{1}{k^2+1}$	30	1.4344
$(1,3,1,1,-2)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^2+1}$	27	1.1856
$(1,3,1,1,-2)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^4+1}$	27	1.1544
$(1,3,1,1,-2)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^6+1}$	27	1.3572
$(1,3,1,1,-2)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^8+1}$	27	1.3260
$(1,3,1,1,-2)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^{10}+1}$	29	1.3480
$(2.4,0.6,1,0.25,1.3)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^2+1}$	18	0.9828
$(4,6,5,3,7)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^2+1}$	38	1.8408
$(7,8,6,6,13)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^2+1}$	50	2.2308
$(11,13,12,21,24)^\top$	$\frac{1}{5k+1}$	$\frac{1}{k^2+1}$	76	3.2448

diagonal matrices with their diagonal elements in $[1,m]$ and $[-m,0]$, respectively; the initial point x^0 generated randomly in $[0,1]$.

- Alg. 3.2: $\lambda_0 = 0.5$, $\nu = 0.5$, $\overline{L} = \|P-Q\|+1$, $t_k = \frac{1}{5k+1}$, $\rho_k = \frac{1}{k^2+1}$, $\eta_k = \epsilon_k = 0$ for all k.
- HSM: $\overline{L} = \|P-Q\|$, $\lambda_k = \frac{1}{2\overline{L}}$, $\epsilon_k = 0$ and $\alpha_k = \frac{1}{55k+1}$ for all k.
- $HSEM$: $\overline{L} = \|P-Q\|$, $\lambda_k = \frac{1}{2\overline{L}}$ and $\alpha_k = \frac{1}{5k+1}$ for all k.
- EVM: $F(x) = x - x^0$, $\alpha_k = \frac{1}{5k+1}$, $S = I$, where I is identify mapping.

The obtained numerical results in this test are shown in Table 2. In the light of this table, we present the following simple observation:

Our algorithm in general works well and has competitive advantages over other known ones. Specifically, Algorithm 3.2 has a smaller number of iterations and computation time than ($HSEM$) and (HSM). The average number of iterations in (EVM) is smaller than in Algorithm 3.2 but has a more expensive computation time. Reasons to explain the above statements are that our algorithm uses only one projection and self-adaptive step size at each iteration, while the rest of the algorithms use constant step sizes, in addition, two projections are used in the algorithms ($HSEM$) and (EVM).

Example 5.3. We consider the multivalued variational inequality problem ($MVIPs$) with the multivalued mapping F is defined by

$$F = \{h(t)Mx : t \in D\}, \quad \forall x \in \mathbb{R}^n,$$

TABLE 2. The comparative results where the stopping criterion is $\|x^{k+1} - x^k\| \le 10^{-3}$.

	$m = 50$							
	Alg. 3.2		$HSEM$		EVM		HSM	
n	Iter.	CPU-times	Iter.	CPU-times	Iter.	CPU-times	Iter.	CPU-times
2	19	50.3727	25	67.9540	20	105.7999	92	92.6334
5	44	64.8028	57	87.1422	34	99.4974	106	128.9378
10	59	59.1400	77	77.7353	60	110.0587	143	146.1085
20	132	84.6461	140	111.1663	108	123.4280	166	182.3783
	$m = 100$							
	Alg. 3.2		$HSEM$		EVM		HSM	
n	Iter.	CPU-times	Iter.	CPU-times	Iter.	CPU-times	Iter.	CPU-times
2	27	83.6009	32	96.8454	22	134.6913	119	145.1121
5	46	96.1377	60	105.0667	43	145.1589	127	163.3624
10	68	77.5481	64	118.0629	60	167.0157	176	232.4332
20	194	168.8243	371	337.1650	101	227.1357	268	248.7158

where $M = AA^\top + B + Q$, A is a matrix of order n, B is an $n \times n$ skew-symmetric matrix, Q is an $n \times n$ positive diagonal matrix and h is a continuous mapping from a nonempty compact subset D of $\mathbb{R}$ to $\mathbb{R}$ such that $h(t) > 0$ for all $t \in D$; It is proved in [3] that F is pseudomonotone and $b\|M\|$-Lipschitz Hausdorff continuous.

Test 4. Let all entries A, B, Q and x^0 be randomly generated by using the commands $A = 2*n*rand(n,n) - n; B = skewdec(n,1); Q = diag(1:n); x^0 = rand(n,1)$. This test performs computational experiments to compare Algorithm 4.1 with the Halpern projection method (HPM) in [3] and the cutting hyperplane method (CHM) in [2]. The function $f(t)$, feasible set C and parameters of the algorithms are chosen as follows:

- $h(t) = 3t^2 - 2t + 1,\ D = [0,1]$,

  $$C := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_k \le n\ \forall k = 1, \dots, n, \|x\| \le 2\}.$$

- Alg. 4.1: $\lambda_0 = 0.5$, $\nu = 0.5$, $\overline{L} = 2\|M\| + 1$, $t_k = \frac{1}{3k+2000}$, $\rho_k = \frac{1}{k^2+1}$, $\epsilon_k = \eta_k = 0$ for all k.
- HPM: $\alpha_k = \frac{1}{3k+2000}$, $\bar{L} = 2\|M\| + 1$, $\lambda_k = \frac{1}{8\|M\|+5}$.
- CHM: $\sigma = 5$, $c = \frac{1}{2\sigma+1000}$ and $\gamma = 1$.

Figure 2 shows the test results with $n = 10$ and Table 3 shows the test results with the choice of different dimensions n. From Figure 2 and Table 3, we see that the CPU time and the number iterations of our algorithm are less than of the algorithms (HPM) and (CHM). This assertion is reasonable because the algorithms (HPM) and (CHM) use projections with constant step sizes and at each iteration Algorithm (CHM) must perform two projections on C.

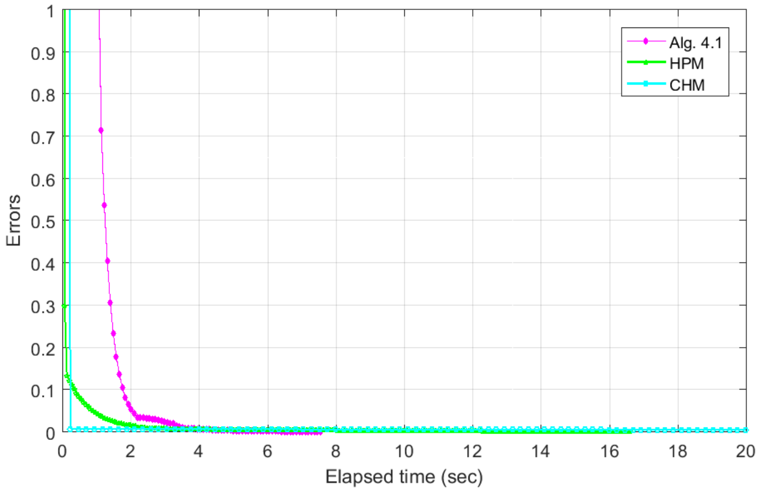

FIGURE 2. Convergence of Algorithm 4.1 with the tolerance 10^{-3}.

TABLE 3. The comparative results for Test 3.

Dim. n	Iter			CPU-times		
	Alg. 4.1	HPM	CHM	Alg. 4.1	HPM	CHM
5	62	195	275	3.7128	9.6253	37.6274
10	59	185	512	5.9748	13.4005	113.0227
15	100	309	1028	11.8249	27.8930	309.4748
20	111	462	1346	17.8777	50.0607	599.2466
25	122	369	1065	24.4298	46.6911	611.0247
30	121	399	1239	28.2050	57.7360	773.9366
35	99	365	620	25.1162	60.3724	405.1034
40	138	453	888	37.1282	87.4854	583.8493
50	127	363	941	41.2467	85.9410	722.2513
70	123	384	812	79.1705	193.1604	875.0264

Acknowledgments. The authors would like to thank the Editor and the referees for their comments on the manuscript which helped in improving earlier version of this paper.

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Nguyen Minh Khoa
Electric Power University
Hanoi, Vietnam
Email address: khoanm@epu.edu.vn

Tran Van Thang
Electric Power University
Hanoi, Vietnam
Email address: thangtv@epu.edu.vn