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THE KÄHLER DIFFERENT OF A SET OF POINTS IN $\mathbb{P}^m \times \mathbb{P}^n$

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Abstract. Given an ACM set $\mathbb{X}$ of points in a multiprojective space $\mathbb{P}^m \times \mathbb{P}^n$ over a field of characteristic zero, we are interested in studying the Kähler different and the Cayley-Bacharach property for $\mathbb{X}$. In $\mathbb{P}^1 \times \mathbb{P}^1$, the Cayley-Bacharach property agrees with the complete intersection property and it is characterized by using the Kähler different. However, this result fails to hold in $\mathbb{P}^m \times \mathbb{P}^n$ for $n > 1$ or $m > 1$. In this paper we start an investigation of the Kähler different and its Hilbert function and then prove that $\mathbb{X}$ is a complete intersection of type $(d_1, \ldots, d_m, d'_1, \ldots, d'_n)$ if and only if it has the Cayley-Bacharach property and the Kähler different is non-zero at a certain degree. We characterize the Cayley-Bacharach property of $\mathbb{X}$ under certain assumptions.

1. Introduction

Let $\mathbb{X}$ be a finite set of points in the multiprojective space $\mathbb{P}^m \times \mathbb{P}^n$ over a field K of characteristic zero, let $I_{\mathbb{X}} \subseteq S := K[X_0, \ldots, X_m, Y_0, \ldots, Y_n]$ be the bihomogeneous vanishing ideal of $\mathbb{X}$, and let $R_{\mathbb{X}} = S/I_{\mathbb{X}}$ be the bigraded coordinate ring of $\mathbb{X}$. The set $\mathbb{X}$ is called *arithmetically Cohen-Macaulay* (*ACM*) if $R_{\mathbb{X}}$ is a Cohen-Macaulay ring, and $\mathbb{X}$ is called a *complete intersection of type* $(d_1, \ldots, d_m, d'_1, \ldots, d'_n)$ if $I_{\mathbb{X}}$ is generated by a bihomogeneous regular sequence $\{F_1, \ldots, F_m, G_1, \ldots, G_n\}$ with $\deg(F_i) = (d_i, 0)$ for $i = 1, \ldots, m$ and $\deg(G_j) = (0, d'_j)$ for $j = 1, \ldots, n$. The study of special classes of finite sets of points such

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as ACM sets of points, complete intersections, etc. in a multiprojective space is a very active field of research and has been attracted by many authors. For instance, the work on finding a classification of ACM set of points includes [3, 7–9, 18, 23] and the work on complete intersections includes [2, 5, 6, 12].

Obviously, every complete intersection of type $(d_1, \dots, d_m, d'_1, \dots, d'_n)$ is ACM. It is a subject of research to understand when $\mathbb{X}$ is a complete intersection of type $(d_1, \dots, d_m, d'_1, \dots, d'_n)$. One of the classical tools for studying the complete intersection property is the Kähler different (see [12, 16, 19]). When $\mathbb{X}$ is ACM, we may assume that $R_o := K[X_0, Y_0]$ is a Noetherian normalization of $R_{\mathbb{X}}$ and define the *Kähler different* $\vartheta_{\mathbb{X}}$ of $\mathbb{X}$ or of the bigraded algebra $R_{\mathbb{X}}/R_o$ which is known as the initial Fitting ideal of the Kähler differential module of $R_{\mathbb{X}}/R_o$. In the case $m = n = 1$, [5, Proposition 7.3] shows that an ACM set $\mathbb{X}$ is a complete intersection of type (d_1, d'_1) if and only if $\vartheta_{\mathbb{X}}$ contains no separators for $\mathbb{X}$ of degree less than $(2r_{\mathbb{X}_1}, 2r_{\mathbb{X}_2})$, where $\mathbb{X}_i = \pi_i(\mathbb{X})$ and $r_{\mathbb{X}_i}$ is the regularity index of the Hilbert function of $\mathbb{X}_i$ for $i = 1, 2$ and $\pi_1 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ and $\pi_2 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ are the canonical projections, which in turn is equivalent to the condition that $\mathbb{X}$ has the Cayley-Bacharach property. Here, we say that $\mathbb{X}$ has the *Cayley-Bacharach property* if the Hilbert function of $\mathbb{X} \setminus \{p\}$ is independent of the choice of $p \in \mathbb{X}$. A nice history about the study of the Cayley-Bacharach property of a finite set of points in the projective space can be found in [13]. Notice that the above result of [5] does not hold true in general, for instance when $m > 1$ or $n > 1$ as Example 4.6 shows. But if $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is a complete intersection of type $(d_1, \dots, d_m, d'_1, \dots, d'_n)$, then it still has the Cayley-Bacharach property and $\vartheta_{\mathbb{X}}$ contains no separators for $\mathbb{X}$ of degree less than $(2r_{\mathbb{X}_1}, 2r_{\mathbb{X}_2})$. It is natural to ask which additional conditions make an ACM set of points $\mathbb{X}$ with Cayley-Bacharach property being a complete intersection of type $(d_1, \dots, d_m, d'_1, \dots, d'_n)$.

Working on this question, in this paper we prove the following result.

Theorem 1.1 (Theorem 4.7). *For a set $\mathbb{X}$ of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$, the following are equivalent.*

(a) $\mathbb{X} = CI(d_1, \dots, d_m, d'_1, \dots, d'_n)$ *for some positive integers* $d_i, d'_j \geq 1$.
(b) $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ *has the Cayley-Bacharach property and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}_1}, r_{\mathbb{X}_2}) \neq 0$.

Also, when $\mathbb{X}$ satisfies the $(\star)$-property (see [11, Definition 3.19]), we look closely at the Cayley-Bacharach property for $\mathbb{X}$. If we write $\mathbb{X}_1 = \pi_1(\mathbb{X}) = \{q_1, \dots, q_{s_1}\} \subseteq \mathbb{P}^m$ and $\mathbb{X}_2 = \pi_2(\mathbb{X}) = \{q'_1, \dots, q'_{s_2}\} \subseteq \mathbb{P}^n$ and put

$$W_i := \pi_2(\pi_1^{-1}(q_i) \cap \mathbb{X}) \subseteq \mathbb{X}_2, \quad V_j := \pi_1(\pi_2^{-1}(q'_j) \cap \mathbb{X}) \subseteq \mathbb{X}_1$$

for $i = 1, \dots, s_1$ and $j = 1, \dots, s_2$, then we obtain the following characterization of the Cayley-Bacharach property for $\mathbb{X}$.

Theorem 1.2 (Theorem 5.2). *Suppose that $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ has the $(\star)$-property. Then $\mathbb{X}$ has the Cayley-Bacharach property if and only if the following conditions are satisfied*:

(a) $V_1, \dots, V_{s_2}$ *are Cayley-Bacharach schemes in* $\mathbb{P}^m$ *and* $r_{V_1} = \cdots = r_{V_{s_2}}$;
(b) $W_1, \dots, W_{s_1}$ *are Cayley-Bacharach schemes in* $\mathbb{P}^n$ *and* $r_{W_1} = \cdots = r_{W_{s_1}}$.

Using Theorem 1.2, in $\mathbb{P}^1 \times \mathbb{P}^n$ we can drop the condition $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ in part (b) of Theorem 1.1 and get the following consequence.

Theorem 1.3 (Corollary 5.6). *Suppose that* $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ *has the* $(\star)$*-property. Then* $\mathbb{X} = CI(d_1, d_1', \dots, d_n')$ *for some positive integers* $d_1, d_1', \dots, d_n' \geq 1$ *if and only if* $\mathbb{X}$ *has the Cayley-Bacharach property and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(d_1 - 1, r_{\mathbb{X}_2}) \neq 0$.

The paper is organized as follows. In Section 2 we fix the notation and recall the definitions of the border of the Hilbert function of $\mathbb{X}$ and the Kähler differential modules $\Omega^1_{R_{\mathbb{X}}/K}$ and $\Omega^1_{R_{\mathbb{X}}/R_o}$. In particular, we use a presentation of $\Omega^1_{R_{\mathbb{X}}/K}$ (see Theorem 2.5) and its relation with $\Omega^1_{R_{\mathbb{X}}/R_o}$ to give a formula for the Hilbert function of $\Omega^1_{R_{\mathbb{X}}/R_o}$ when $\mathbb{X}$ is ACM (see Proposition 2.7). In Section 3 we take a closed look at the Kähler different $\vartheta_{\mathbb{X}}$ of an ACM set of points $\mathbb{X}$ in $\mathbb{P}^m \times \mathbb{P}^n$. We provide several basic properties of the Hilbert function of $\vartheta_{\mathbb{X}}$ and its border. Section 4 contains the first main result (Theorem 4.7) which characterize $\mathbb{X} = CI(d_1, \dots, d_m, d_1', \dots, d_n')$ using the Kähler different and the Cayley-Bacharach property. In this special case we describe explicitly the Hilbert function of $\vartheta_{\mathbb{X}}$ and its border (see Proposition 4.3 and Corollary 4.4). In the final section, we restrict our attention to the finite sets of points in $\mathbb{P}^m \times \mathbb{P}^n$ having the $(\star)$-property. In this setting, we relate the degree of a point $q_i \times q_j' \in \mathbb{X}$ to degrees of points in W_i and V_j (see Proposition 5.1). This enables us to prove a characterization of the Cayley-Bacharach property of $\mathbb{X}$ (see Theorem 5.2) and derive some consequences in $\mathbb{P}^1 \times \mathbb{P}^n$ (see Proposition 5.5 and Corollary 5.6). All examples in this paper were calculated using the computer algebra system ApCoCoA [21].

2. The Kähler differential modules

Let K be a field of characteristic zero, let $m, n \geq 1$ be positive integers. For $(i_1, j_1), (i_2, j_2) \in \mathbb{Z}^2$, we write $(i_1, j_1) \preceq (i_2, j_2)$ if $i_1 \leq i_2$ and $j_1 \leq j_2$. The bigraded coordinate ring of $\mathbb{P}^m \times \mathbb{P}^n$ is the polynomial ring $S = K[X_0, \dots, X_m, Y_0, \dots, Y_n]$ equipped with the $\mathbb{Z}^2$-grading defined by $\deg(X_0) = \cdots = \deg(X_m) = (1, 0)$ and $\deg(Y_0) = \cdots = \deg(Y_n) = (0, 1)$. For $(i, j) \in \mathbb{Z}^2$, we let $S_{i,j}$ be the bihomogeneous component of degree (i, j) of S, i.e., the K-vector space with basis

$$\{X_0^{\alpha_0} \cdots X_m^{\alpha_m} \cdot Y_0^{\beta_0} \cdots Y_n^{\beta_n} \mid \textstyle\sum_{k=0}^m \alpha_k = i, \sum_{k=0}^n \beta_k = j, \alpha_k, \beta_k \in \mathbb{N}\}.$$

Given an ideal $I \subseteq S$, we set $I_{i,j} := I \cap S_{i,j}$ for all $(i, j) \in \mathbb{Z}^2$. The ideal I is called *bihomogeneous* if $I = \bigoplus_{(i,j) \in \mathbb{Z}^2} I_{i,j}$. If I is a bihomogeneous ideal of S, then the quotient ring S/I also inherits the structure of a bigraded ring via $(S/I)_{i,j} := S_{i,j}/I_{i,j}$ for all $(i, j) \in \mathbb{Z}^2$.

A finitely generated S-module M is a *bigraded S-module* if it has a direct sum decomposition

$$M = \bigoplus_{(i,j)\in\mathbb{Z}^2} M_{i,j}$$

with the property that $S_{(i_1,j_1)}M_{(i_2,j_2)} \subseteq M_{i_1+i_2,j_1+j_2}$ for all $(i_1,j_1),(i_2,j_2) \in \mathbb{Z}^2$.

Definition. Let M be a finitely generated bigraded S-module. The *Hilbert function* of M is the numerical function $\mathrm{HF}_M : \mathbb{Z}^2 \to \mathbb{N}$ defined by

$$\mathrm{HF}_M(i,j) := \dim_K M_{i,j} \quad \text{for all } (i,j) \in \mathbb{Z}^2.$$

In particular, for a bihomogeneous ideal I of S, the Hilbert function of S/I satisfies

$$\mathrm{HF}_{S/I}(i,j) := \dim_k (S/I)_{i,j} = \dim_k S_{i,j} - \dim_k I_{i,j} \quad \text{for all } (i,j) \in \mathbb{Z}^2.$$

If M is a finitely generated bigraded S-module such that $\mathrm{HF}_M(i,j) = 0$ for $(i,j) \not\geq (0,0)$, we write the Hilbert function of M as an infinite matrix, where the initial row and column are indexed by 0.

A point in the space $\mathbb{P}^m \times \mathbb{P}^n$ has the form

$$p = [a_0 : a_1 : \cdots : a_m] \times [b_0 : b_1 : \cdots : b_n] \in \mathbb{P}^m \times \mathbb{P}^n,$$

where $[a_0 : a_1 : \cdots : a_m] \in \mathbb{P}^m$ and $[b_0 : b_1 : \cdots : b_n] \in \mathbb{P}^n$. Its vanishing ideal is the bihomogeneous prime ideal of the form

$$I_p = \langle \ell_1, \ldots, \ell_m, \ell'_1, \ldots, \ell'_n \rangle \subseteq S,$$

where $\deg(\ell_i) = (1,0)$ and $\deg(\ell'_j) = (0,1)$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Definition. Let $s \geq 1$ and let $\mathbb{X} = \{p_1, \ldots, p_s\}$ be a set of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$. The *bihomogeneous vanishing ideal* of $\mathbb{X}$ is given by $I_{\mathbb{X}} = I_{p_1} \cap \cdots \cap I_{p_s}$ and its *bigraded coordinate ring* is $R_{\mathbb{X}} = S/I_{\mathbb{X}}$.

In what follows, let $\mathbb{X} = \{p_1, \ldots, p_s\}$ be a set of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$, and let x_i and y_j denote the images of X_i and Y_j in $R_{\mathbb{X}}$ for $0 \leq i \leq m$ and $0 \leq j \leq n$. We write $\mathrm{HF}_{\mathbb{X}}$ for the Hilbert function of $R_{\mathbb{X}}$ and call it the Hilbert function of $\mathbb{X}$. It is worth to noting here that a bihomogeneous element is a zerodivisor of $R_{\mathbb{X}}$ if and only if it vanishes at some points of $\mathbb{X}$.

Convention 2.1. Given the canonical projections $\pi_1 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ and $\pi_2 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$, we let $\mathbb{X}_1 = \pi_1(\mathbb{X})$, $s_1 = |\mathbb{X}_1|$, $\mathbb{X}_2 = \pi_2(\mathbb{X})$, and $s_2 = |\mathbb{X}_2|$. The set $\mathbb{X}_1$ has its homogeneous vanishing ideal $I_{\mathbb{X}_1} \subseteq K[X_0, \ldots, X_m]$ and its homogeneous coordinate ring $R_{\mathbb{X}_1} = K[X_0, \ldots, X_m]/I_{\mathbb{X}_1}$. Similarly, $\mathbb{X}_2$ has its homogeneous vanishing ideal $I_{\mathbb{X}_2} \subseteq K[Y_0, \ldots, Y_n]$ and its homogeneous coordinate ring $R_{\mathbb{X}_2} = K[Y_0, \ldots, Y_n]/I_{\mathbb{X}_2}$.

Notice that there exists a linear form $\ell \in K[X_0, \ldots, X_m]$ such that ℓ does not vanish at any point of $\mathbb{X}_1$. Analogously, we find a linear form $\ell' \in K[Y_0, \ldots, Y_n]$

which does not vanish at any point of $\mathbb{X}_2$. It follows that $\overline{\ell}, \overline{\ell}' \in R_{\mathbb{X}}$ are non-zerodivisors (see also e.g. [7, Lemma 1.2]). As a consequence of this fact and [20, Proposition 1.9] and [22, Proposition 4.6], we get several basis properties of the Hilbert function of $\mathbb{X}$.

Proposition 2.2. *Let* $(i,j) \in \mathbb{Z}^2$ *with* $(i,j) \succeq (0,0)$.

(a) *We have* $\mathrm{HF}_{\mathbb{X}}(i,j) \le \min\{\mathrm{HF}_{\mathbb{X}}(i+1,j), \mathrm{HF}_{\mathbb{X}}(i,j+1)\} \le s$.

(b) *If* $\mathrm{HF}_{\mathbb{X}}(i,j) = \mathrm{HF}_{\mathbb{X}}(i+1,j)$, *then* $\mathrm{HF}_{\mathbb{X}}(i,j) = \mathrm{HF}_{\mathbb{X}}(i+2,j)$. *Also,* $\mathrm{HF}_{\mathbb{X}}(i,j) = \mathrm{HF}_{\mathbb{X}}(s_1-1,j)$ *for* $i \ge s_1 - 1$ *and* $j < s_2 - 1$.

(c) *If* $\mathrm{HF}_{\mathbb{X}}(i,j) = \mathrm{HF}_{\mathbb{X}}(i,j+1)$, *then* $\mathrm{HF}_{\mathbb{X}}(i,j) = \mathrm{HF}_{\mathbb{X}}(i,j+2)$. *Also,* $\mathrm{HF}_{\mathbb{X}}(i,j) = \mathrm{HF}_{\mathbb{X}}(i,s_2-1)$ *for* $i < s_1 - 1$ *and* $j \ge s_2 - 1$.

(d) *We have* $\mathrm{HF}_{\mathbb{X}}(i,j) = s$ *for all* $(i,j) \succeq (s_1-1, s_2-1)$.

For $k,l \in \mathbb{N}$ set $\nu_k := \min\{i \in \mathbb{N} \mid \mathrm{HF}_{\mathbb{X}}(i,k) = \mathrm{HF}_{\mathbb{X}}(i+1,k)\}$ and $\varrho_l := \min\{j \in \mathbb{N} \mid \mathrm{HF}_{\mathbb{X}}(l,j) = \mathrm{HF}_{\mathbb{X}}(l,j+1)\}$. Let $\nu := \sup\{\nu_k \mid k \in \mathbb{N}\}$ and $\varrho := \sup\{\varrho_l \mid l \in \mathbb{N}\}$. In view of Proposition 2.2, we have $(\nu,\varrho) \preceq (s_1-1, s_2-1)$. Especially, $(\nu,\varrho) = (s_1-1, s_2-1)$ if $m = n = 1$. Moreover, the tuple (ν,ϱ) can be described by the following lemma.

Lemma 2.3. *Let* $k,l \in \mathbb{N}$. *If* $\mathrm{HF}_{\mathbb{X}}(i,k) = \mathrm{HF}_{\mathbb{X}}(i+1,k)$, *then* $\mathrm{HF}_{\mathbb{X}}(i,k+1) = \mathrm{HF}_{\mathbb{X}}(i+1,k+1)$; *and if* $\mathrm{HF}_{\mathbb{X}}(l,j) = \mathrm{HF}_{\mathbb{X}}(l,j+1)$, *then* $\mathrm{HF}_{\mathbb{X}}(l+1,j) = \mathrm{HF}_{\mathbb{X}}(l+1,j+1)$. *In particular, we have* $(\nu,\varrho) = (r_{\mathbb{X}_1}, r_{\mathbb{X}_2})$, *where* $r_{\mathbb{X}_k}$ *is the regularity index of* $\mathrm{HF}_{\mathbb{X}_k}$ *for* $k = 1,2$.

Proof. As in the argument before Proposition 2.2, we find $\ell \in S_{1,0}$ and $\ell' \in S_{0,1}$ such that their images $\bar{\ell}, \bar{\ell}'$ in $R_{\mathbb{X}}$ are non-zerodivisors. Then we have

$$\begin{aligned}\mathrm{HF}_{\mathbb{X}}(i,k+1) &= \dim_K((R_{\mathbb{X}})_{i,k} \cdot (R_{\mathbb{X}})_{0,1}) = \dim_K(\bar{\ell} \cdot (R_{\mathbb{X}})_{i,k} \cdot (R_{\mathbb{X}})_{0,1}) \\ &= \dim_K((R_{\mathbb{X}})_{i+1,k} \cdot (R_{\mathbb{X}})_{0,1}) = \mathrm{HF}_{\mathbb{X}}(i+1,k+1),\end{aligned}$$

where the second equality follows from the fact that $\bar{\ell} \in (R_{\mathbb{X}})_{1,0}$ is a non-zerodivisor of $R_{\mathbb{X}}$ and the third equality induces by assumption that $\mathrm{HF}_{\mathbb{X}}(i,k) = \mathrm{HF}_{\mathbb{X}}(i+1,k)$. Analogously, by using the non-zerodivisor $\bar{\ell}' \in (R_{\mathbb{X}})_{0,1}$, we have $\mathrm{HF}_{\mathbb{X}}(l+1,j) = \mathrm{HF}_{\mathbb{X}}(l+1,j+1)$ when $\mathrm{HF}_{\mathbb{X}}(l,j) = \mathrm{HF}_{\mathbb{X}}(l,j+1)$. Consequently, we get $\nu_k \ge \nu_{k+1}$ for all $k \in \mathbb{N}$ and $\varrho_l \ge \varrho_{l+1}$ for all $l \in \mathbb{N}$, and hence $\nu = \nu_0 = r_{\mathbb{X}_1}$ and $\varrho = \varrho_0 = r_{\mathbb{X}_2}$. □

The lemma leads us to the following definition, which agrees with [22, Definition 4.9] if $(\nu,\varrho) = (s_1-1, s_2-1)$.

Definition. Let $r_{\mathbb{X}_1}$, $r_{\mathbb{X}_2}$ be regularity indices of $\mathrm{HF}_{\mathbb{X}_1}$ and $\mathrm{HF}_{\mathbb{X}_2}$, respectively. The pair $B_{\mathbb{X}} = (B_C, B_R)$, where

$$B_C = (\mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}_1},0), \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}_1},1), \dots, \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}_1}, r_{\mathbb{X}_2}))$$

and

$$B_R = (\mathrm{HF}_{\mathbb{X}}(0,r_{\mathbb{X}_2}), \mathrm{HF}_{\mathbb{X}}(1,r_{\mathbb{X}_2}), \dots, \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}_1}, r_{\mathbb{X}_2})),$$

is called the *border of the Hilbert function* of $\mathbb{X}$.

Example 2.4. Let $K = \mathbb{Q}$, let $\mathbb{X} = \{p_1, \dots, p_9\}$ be a set of nine points in $\mathbb{P}^2 \times \mathbb{P}^2$ given by $p_1 = q_1 \times q_1$, $p_2 = q_1 \times q_2$, $p_3 = q_1 \times q_3$, $p_4 = q_1 \times q_4$, $p_5 = q_2 \times q_1$, $p_6 = q_2 \times q_2$, $p_7 = q_2 \times q_3$, $p_8 = q_3 \times q_1$ and $p_9 = q_3 \times q_2$, where $q_1 = (1 : 0 : 0)$, $q_2 = (1 : 1 : 0)$, $q_3 = (1 : 0 : 1)$, $q_4 = (1 : 1 : 1)$ in $\mathbb{P}^2$. Then $\mathbb{X}_1 = \{q_1, q_2, q_3\}$, $s_1 = 3$, $\mathbb{X}_2 = \{q_1, q_2, q_3, q_4\}$ and $s_2 = 4$. The Hilbert function of $\mathbb{X}$ is given by

$$\mathrm{HF}_{\mathbb{X}} = \begin{bmatrix} 1 & 3 & 4 & 4 & \cdots \\ 3 & 8 & 9 & 9 & \cdots \\ 3 & 8 & 9 & 9 & \cdots \\ 3 & 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and so $r_{\mathbb{X}_1} = 1$ and $r_{\mathbb{X}_2} = 2$. The border of the Hilbert function of $\mathbb{X}$ is given by $B_{\mathbb{X}} = ((3, 8, 9), (4, 9))$. In this case we have $r_{\mathbb{X}_1} < 2 = s_1 - 1$ or $r_{\mathbb{X}_2} < 3 = s_2 - 1$, and $\mathrm{HF}_{\mathbb{X}}(i, j) = s = 9$ for all $(i, j) \succeq (r_{\mathbb{X}_1}, r_{\mathbb{X}_2})$.

In the bigraded enveloping algebra $R_{\mathbb{X}} \otimes_K R_{\mathbb{X}}$ we have the bihomogeneous ideal $J = \mathrm{Ker}(\mu)$, where $\mu : R_{\mathbb{X}} \otimes_K R_{\mathbb{X}} \to R_{\mathbb{X}}$ is the bihomogeneous $R_{\mathbb{X}}$-linear map given by $\mu(f \otimes g) = fg$. The bigraded $R_{\mathbb{X}}$-module $\Omega^1_{R_{\mathbb{X}}/K} = J/J^2$ is called the *module of Kähler differentials* of $R_{\mathbb{X}}/K$. The bihomogeneous K-linear map $d_{R_{\mathbb{X}}/K} : R_{\mathbb{X}} \to \Omega^1_{R_{\mathbb{X}}/K}$ given by $f \mapsto f \otimes 1 - 1 \otimes f + J^2$ satisfies the universal property. We call d the *universal derivation* of $R_{\mathbb{X}}/K$. More generally, for any bigraded K-algebra T/R we can define in the same way the Kähler differential module $\Omega^1_{T/R}$, and the universal derivation of T/R (cf. [16, Section 2]). Note that

$$\Omega^1_{S/K} = \bigoplus_{i=0}^{m} S dX_i \oplus \bigoplus_{j=0}^{n} S dY_j \cong S^{m+1}(-1, 0) \oplus S^{n+1}(0, -1)$$

and $\Omega^1_{R_{\mathbb{X}}/K} = \langle dx_i, dy_j \mid 0 \le i \le m, 0 \le j \le n \rangle_{R_{\mathbb{X}}}$. Especially, the Hilbert function of $\Omega^1_{R_{\mathbb{X}}/K}$ can be computed by using the following theorem (see [6, Theorem 3.5]).

Theorem 2.5. *Let $\mathbb{Y}$ be the subscheme of $\mathbb{P}^m \times \mathbb{P}^n$ defined by the bihomogeneous ideal $I_{\mathbb{Y}} = I_{p_1}^2 \cap \cdots \cap I_{p_s}^2$. There is an exact sequence of bigraded $R_{\mathbb{X}}$-modules*

$$0 \longrightarrow I_{\mathbb{X}}/I_{\mathbb{Y}} \longrightarrow R_{\mathbb{X}}^{m+1}(-1, 0) \oplus R_{\mathbb{X}}^{n+1}(0, -1) \longrightarrow \Omega^1_{R_{\mathbb{X}}/K} \longrightarrow 0.$$

In particular, for $(i, j) \in \mathbb{Z}^2$, we have

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}(i, j) = (m+1)\,\mathrm{HF}_{\mathbb{X}}(i-1, j) + (n+1)\,\mathrm{HF}_{\mathbb{X}}(i, j-1) + \mathrm{HF}_{\mathbb{X}}(i, j) - \mathrm{HF}_{\mathbb{Y}}(i, j).$$

Notice that $R_{\mathbb{X}}$ has the Krull dimension 2, but $1 \le \mathrm{depth}(R_{\mathbb{X}}) \le 2$ (see [23, Section 2]). In case $\mathrm{depth}(R_{\mathbb{X}})$ attains the maximal value, we have the following notion.

Definition. We say that $\mathbb{X}$ is *arithmetically Cohen-Macaulay* (*ACM*) if we have $\mathrm{depth}(R_{\mathbb{X}}) = 2$.

When $\mathbb{X}$ is ACM, then there exist two linear forms $\ell \in S_{1,0}$, $\ell' \in S_{0,1}$ such that $\overline{\ell}$ and $\overline{\ell}'$ give rise to a regular sequence in $R_\mathbb{X}$ (see [23, Proposition 3.2]). After a change of coordinates, we can assume that $\ell = X_0$ and $\ell' = Y_0$, so that x_0, y_0 form a regular sequence in $R_\mathbb{X}$. In this case we set $R_o := K[x_0, y_0]$. Then

$$R_\mathbb{X} = S/I_\mathbb{X} = R_o[x_1, \dots, x_m, y_1, \dots, y_n]$$

is a finitely generated, bigraded R_o-module, and the monomorphism $R_o \hookrightarrow R_\mathbb{X}$ defines a Noetherian normalization.

Remark 2.6. The *Euler derivation* of $R_\mathbb{X}/K$ is given by $\epsilon : R_\mathbb{X} \to R_\mathbb{X}, f \mapsto (i+j)f$ for $f \in (R_\mathbb{X})_{i,j}$ (see [16, Section 1]). Set $\mathfrak{m} := \langle x_0, \dots, x_m, y_0, \dots, y_n \rangle_{R_\mathbb{X}}$. By the universal property of $\Omega^1_{R_\mathbb{X}/K}$, this induces a bihomogeneous surjective $R_\mathbb{X}$-linear map $\gamma : \Omega^1_{R_\mathbb{X}/K} \to \mathfrak{m}$ with $\gamma(dx_i) = x_i$ and $\gamma(dy_j) = y_j$ for all i, j. In particular, $\mathrm{Ann}_{R_\mathbb{X}}(dx_0) = \mathrm{Ann}_{R_\mathbb{X}}(dy_0) = \langle 0 \rangle$, since x_0, y_0 are non-zerodivisors of $R_\mathbb{X}$.

There are relations between $\Omega^1_{R_\mathbb{X}/K}$ and $\Omega^1_{R_\mathbb{X}/R_o}$ as follows.

Proposition 2.7. *Let $\mathbb{X}$ be an ACM set of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$. There exists an exact sequence of bigraded $R_\mathbb{X}$-modules*

$$0 \to R_\mathbb{X} dx_0 \oplus R_\mathbb{X} dy_0 \hookrightarrow \Omega^1_{R_\mathbb{X}/K} \xrightarrow{\psi} \Omega^1_{R_\mathbb{X}/R_o} \to 0,$$

where $\psi(gdf) = g d_{R_\mathbb{X}/R_o} f$ for $f, g \in R_\mathbb{X}$. In particular, we have

$$\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/R_o}}(i,j) = m\,\mathrm{HF}_\mathbb{X}(i-1,j) + n\,\mathrm{HF}_\mathbb{X}(i,j-1) + \mathrm{HF}_\mathbb{X}(i,j) - \mathrm{HF}_\mathbb{Y}(i,j)$$

for all $(i,j) \in \mathbb{N}^2$, where $\mathbb{Y}$ is the subscheme of $\mathbb{P}^m \times \mathbb{P}^n$ defined by $I_\mathbb{Y} = I_{p_1}^2 \cap \cdots \cap I_{p_s}^2$.

Proof. By [16, Proposition 3.24], we have an exact sequence of bigraded $R_\mathbb{X}$-modules

$$R_\mathbb{X} \otimes_{R_o} \Omega^1_{R_o/K} \xrightarrow{\varphi} \Omega^1_{R_\mathbb{X}/K} \xrightarrow{\psi} \Omega^1_{R_\mathbb{X}/R_o} \to 0,$$

where $\Omega^1_{R_o/K} \cong R_o dx_0 \oplus R_o dy_0$ and $\varphi(f \otimes (f_1 dx_0 + f_2 dy_0)) = f f_1 dx_0 + f f_2 dy_0$. Hence the claimed exact sequence follows from $\mathrm{Im}(\varphi) = R_\mathbb{X} dx_0 \oplus R_\mathbb{X} dy_0$. Furthermore, the Hilbert function of $\Omega^1_{R_\mathbb{X}/R_o}$ satisfies

$$\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/R_o}}(i,j) = \mathrm{HF}_{\Omega^1_{R_\mathbb{X}/K}}(i,j) - \mathrm{HF}_\mathbb{X}(i-1,j) - \mathrm{HF}_\mathbb{X}(i,j-1).$$

An application of Theorem 2.5 gives the desired formula for $\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/R_o}}$. □

3. The Kähler different

Let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ be an ACM set of points, suppose that $\{x_0, y_0\}$ is a regular sequence in $R_\mathbb{X}$, and let $R_o = K[x_0, y_0]$. Further, let

$\{F_1,\dots,F_r\}$, $r \ge n+m$, be a bihomogeneous system of generators of $I_{\mathbb{X}}$. By [16, Corollary 2.14], $\Omega^1_{R_{\mathbb{X}}/R_o}$ has the following presentation

$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{m} R_{\mathbb{X}} dX_i \oplus \bigoplus_{j=1}^{n} R_{\mathbb{X}} dY_j \to \Omega^1_{R_{\mathbb{X}}/R_o} \to 0,$$

where the bigraded $R_{\mathbb{X}}$-module $\mathcal{K}$ is generated by the elements $\sum_{i=1}^{m} \frac{\partial F_k}{\partial x_i} dX_i + \sum_{j=1}^{n} \frac{\partial F_k}{\partial y_j} dY_j$ for $k = 1,\dots,r$. The Jacobian matrix

$$\mathcal{J} := \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_r}{\partial x_1} & \cdots & \frac{\partial F_r}{\partial x_m} & \frac{\partial F_r}{\partial y_1} & \cdots & \frac{\partial F_r}{\partial y_n} \end{pmatrix}$$

is a relation matrix of $\Omega^1_{R_{\mathbb{X}}/R_o}$ with respect to $\{dx_1,\dots,dx_m,dy_1,\dots,dy_n\}$. It is easy to see that every $m+n$-minors of $\mathcal{J}$ is a bihomogeneous element of $R_{\mathbb{X}}$.

Definition. The bihomogeneous ideal of $R_{\mathbb{X}}$ generated by all $m+n$-minors of the Jacobian matrix $\mathcal{J}$ is called the *Kähler different* of $\mathbb{X}$ and is denoted by $\vartheta_{\mathbb{X}}$.

In the same way as above, we can define the Kähler different $\vartheta_{\mathbb{X}_1}$ of $\mathbb{X}_1 = \pi_1(\mathbb{X})$ (or of the graded algebra $R_{\mathbb{X}_1}/K[x_0]$). Similarly, we get the Kähler different $\vartheta_{\mathbb{X}_2}$ of $\mathbb{X}_2 = \pi_2(\mathbb{X})$ (or of the graded algebra $R_{\mathbb{X}_2}/K[y_0]$). When $|\mathbb{X}| = 1$, we see that $\vartheta_{\mathbb{X}} = \langle 1 \rangle = \vartheta_{\mathbb{X}_1} R_{\mathbb{X}} \cdot \vartheta_{\mathbb{X}_2} R_{\mathbb{X}}$. In general, we have the following relation.

Lemma 3.1. (a) *We have* $\vartheta_{\mathbb{X}_1} R_{\mathbb{X}} \cdot \vartheta_{\mathbb{X}_2} R_{\mathbb{X}} \subseteq \vartheta_{\mathbb{X}}$.
(b) $\vartheta_{\mathbb{X}}$ *contains a bihomogeneous non-zerodivisor.*

Proof. Obviously, we have $I_{\mathbb{X}_1} S \subseteq I_{\mathbb{X}}$ and $I_{\mathbb{X}_2} S \subseteq I_{\mathbb{X}}$. For any $G_{11},\dots,G_{1m} \in I_{\mathbb{X}_1}$ and $G_{21},\dots,G_{2n} \in I_{\mathbb{X}_2}$, we have $\{G_{11},\dots,G_{1m},G_{21},\dots,G_{2n}\} \subseteq I_{\mathbb{X}}$, and so

$$\det \begin{pmatrix} \frac{\partial G_{11}}{\partial x_1} & \cdots & \frac{\partial G_{11}}{\partial x_m} & \frac{\partial G_{11}}{\partial y_1} & \cdots & \frac{\partial G_{11}}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial G_{2n}}{\partial x_1} & \cdots & \frac{\partial G_{2n}}{\partial x_m} & \frac{\partial G_{2n}}{\partial y_1} & \cdots & \frac{\partial G_{2n}}{\partial y_n} \end{pmatrix}$$
$$= \frac{\partial(G_{11},\dots,G_{1m})}{\partial(x_1,\dots,x_m)} \cdot \frac{\partial(G_{21},\dots,G_{2n})}{\partial(y_1,\dots,y_n)} \in \vartheta_{\mathbb{X}},$$

where $\frac{\partial(G_{11},\dots,G_{1m})}{\partial(x_1,\dots,x_m)}$ denotes the image of the Jacobian determinant $\frac{\partial(G_{11},\dots,G_{1m})}{\partial(X_1,\dots,X_m)}$ in $R_{\mathbb{X}}$ (similarly for $\frac{\partial(G_{21},\dots,G_{2n})}{\partial(y_1,\dots,y_n)}$). Moreover, $\vartheta_{\mathbb{X}_1} R_{\mathbb{X}}$ is generated by elements of the form $\frac{\partial(G_{11},\dots,G_{1m})}{\partial(x_1,\dots,x_m)}$, and $\vartheta_{\mathbb{X}_2} R_{\mathbb{X}}$ is generated by elements of the form $\frac{\partial(G_{21},\dots,G_{2n})}{\partial(y_1,\dots,y_n)}$, and therefore $\vartheta_{\mathbb{X}_1} R_{\mathbb{X}} \cdot \vartheta_{\mathbb{X}_2} R_{\mathbb{X}} \subseteq \vartheta_{\mathbb{X}}$ and (a) follows.

To prove (b), observe that $x_0^i y_0^j \in R_{\mathbb{X}}$ is a bihomogeneous non-zerodivisor for every $i,j \ge 0$. By [15, Proposition 3.5], there are $k,l \in \mathbb{N}$ such that $x_0^k \in \vartheta_{\mathbb{X}_1}$ and $y_0^l \in \vartheta_{\mathbb{X}_2}$. Hence the non-zerodivisor $x_0^k y_0^l$ belongs to $\vartheta_{\mathbb{X}}$ by (a). ∎

Some fundamental properties of the Hilbert function of $\vartheta_{\mathbb{X}}$ are given in the following proposition.

Proposition 3.2. *Let $s_1 = |\mathbb{X}_1|$ and $s_2 = |\mathbb{X}_2|$.*

(a) *For all $(i,j) \in \mathbb{N}^2$, we have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) \le \min\{\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1,j), \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j+1)\}$.*

(b) *For all $i,j \in \mathbb{N}$, we have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,0) \le \mathrm{HF}_{\mathbb{X}_1}(i)$ and $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(0,j) \le \mathrm{HF}_{\mathbb{X}_2}(j)$.*

(c) *If $s_1 = 1$, then $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}_2}}(j)$ for all $(i,j) \in \mathbb{N}^2$; and if $s_2 = 1$, then $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}_1}}(i)$ for all $(i,j) \in \mathbb{N}^2$.*

(d) *For all $(i,j) \in \mathbb{N}^2$, we have*

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) \le \mathrm{HF}_{\mathbb{X}}(i,j) \le \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+(m+1)(s_1-1), j+(n+1)(s_2-1)).$$

Proof. Claim (a) follows by the fact that x_0, y_0 are non-zerodivisors of $R_{\mathbb{X}}$ and $\vartheta_{\mathbb{X}}$ is a bihomogeneous ideal of $R_{\mathbb{X}}$. Note that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,0) \le \mathrm{HF}_{\mathbb{X}}(i,0)$ and $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(0,j) \le \mathrm{HF}_{\mathbb{X}}(0,j)$ for all $i,j \in \mathbb{N}$. So, claim (b) follows from [22, Proposition 3.2].

To prove (c), it suffices to consider the case $s_1 = 1$. In this case we may assume $q_1 = [1:0:\cdots:0] \in \mathbb{P}^m$ and $\mathbb{X} = \{q_1 \times q'_1, \dots, q_1 \times q'_s\} \subseteq \mathbb{P}^m \times \mathbb{P}^n$. We claim that $I_{\mathbb{X}} = \langle X_1, \dots, X_m \rangle + I_{\mathbb{X}_2}S$. Clearly, $\langle X_1, \dots, X_m\rangle + I_{\mathbb{X}_2}S \subseteq I_{\mathbb{X}}$. Now let $F \in I_{\mathbb{X}}$ be bihomogeneous of degree (i,j). Using the Division Algorithm (see e.g. [14, Proposition 1.6.4]), we may present $F = \sum_{k=1}^m H_k X_k + X_0^i G$ with $H_k \in S_{i-1,j}$ and $G \in K[Y_0, \dots, Y_n]$ of degree $(0,j)$. Then

$$G(q_1 \times q'_l) = (X_0^i G)(q_1 \times q'_l) = (F - \sum_{k=1}^m H_k X_k)(q_1 \times q'_l) = 0$$

for all $l = 1, \dots, s$. This implies $G \in I_{\mathbb{X}_2}$, and hence $F \in \langle X_1, \dots, X_m \rangle + I_{\mathbb{X}_2}S$.

Consequently, the ideal $I_{\mathbb{X}}$ has a bihomogeneous system of generators of the form $\{X_1, \dots, X_m, G_1, \dots, G_t\}$, where $\{G_1, \dots, G_t\}$ is a homogeneous system of generators of $I_{\mathbb{X}_2} \subseteq K[Y_0, \dots, Y_n]$. Observe that $\vartheta_{\mathbb{X}_1} = \langle 1 \rangle$ and $\vartheta_{\mathbb{X}}$ is generated by elements $\frac{\partial(X_1,\dots,X_m,G_{k_1},\dots,G_{k_n})}{\partial(x_1,\dots,x_m,y_1,\dots,y_n)} = \frac{\partial(G_{k_1},\dots,G_{k_n})}{\partial(y_1,\dots,y_n)}$ with $\{k_1, \dots, k_n\} \subseteq \{1, \dots, t\}$. By Lemma 3.1(a), $\vartheta_{\mathbb{X}} = \vartheta_{\mathbb{X}_2} R_{\mathbb{X}}$. Moreover,

$$R_{\mathbb{X}} \cong K[X_0, Y_0, \dots, Y_n]/I_{\mathbb{X}_2} \cong R_{\mathbb{X}_2}[x_0].$$

Since x_0 is a non-zerodivisor of $R_{\mathbb{X}}$, we have

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}_2} R_{\mathbb{X}}}(i,j) = \dim_K((\vartheta_{\mathbb{X}_2})_j x_0^i) = \mathrm{HF}_{\vartheta_{\mathbb{X}_2}}(j)$$

for all $(i,j) \in \mathbb{N}^2$.

For (d), it suffices to demonstrate the inequality

$$\mathrm{HF}_{\mathbb{X}}(i,j) \le \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+(m+1)(s_1-1), j+(n+1)(s_2-1)).$$

In the proof of Lemma 3.1(b), there exist $k,l \in \mathbb{N}$ such that $h := x_0^k y_0^l \in \vartheta_{\mathbb{X}}$. In particular, we may choose $k = (m+1)(s_1-1)$ and $l = (n+1)(s_2-1)$ by [15, Proposition 3.5]. So, the multiplication map $(R_{\mathbb{X}})_{i,j} \stackrel{\times h}{\to} (\vartheta_{\mathbb{X}})_{(i+k,j+l)}$ is injective as K-vector spaces. This yields that $\mathrm{HF}_{\mathbb{X}}(i,j) \le \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+k, j+l)$. $\square$

The following corollary is a direct consequence of Propositions 2.2(d) and 3.2(d).

Corollary 3.3. *In the setting of Proposition 3.2, we have* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = s$ *for all* $(i,j) \succeq ((s_1-1)(m+2),(s_2-1)(n+2))$.

Lemma 3.4. *Let* $\{h_1,\dots,h_t\}$ *be a bihomogeneous minimal system of generators of* $\vartheta_{\mathbb{X}}$*, write* $\deg(h_k) = (i_k, j_k)$ *for* $k = 1,\dots,t$ *and set*

$$i_{\max} := \max\{i_k \mid k = 1,\dots,t\}, \quad j_{\max} := \max\{j_k \mid k = 1,\dots,t\},$$

and let $(i,j) \in \mathbb{N}^2$.

(a) *If* $i \geq i_{\max}$ *and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1,j)$*, then* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+2,j)$.
(b) *If* $j \geq j_{\max}$ *and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j+1)$*, then* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j+2)$.

Proof. It suffices to prove (a), since (b) is analogous. For $i \geq i_{\max}$, consider the multiplication map $\mu_{x_0,i} : (\vartheta_{\mathbb{X}})_{(i,j)} \to (\vartheta_{\mathbb{X}})_{(i+1,j)}, h \mapsto x_0 h$. Since $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1,j)$, $\mu_{x_0,i}$ is an isomorphism of K-vector spaces. So, we have $(\vartheta_{\mathbb{X}})_{(i+1,j)} = x_0 \cdot (\vartheta_{\mathbb{X}})_{(i,j)}$. We need to show that $\mu_{x_0,i+1} : (\vartheta_{\mathbb{X}})_{(i+1,j)} \to (\vartheta_{\mathbb{X}})_{(i+2,j)}$ is also an isomorphism of K-vector spaces. Clearly, $\mu_{x_0,i+1}$ is injective, as x_0 is a non-zerodivisor. Now we check that $\mu_{x_0,i+1}$ is surjective. Let $h \in (\vartheta_{\mathbb{X}})_{(i+2,j)} \setminus \{0\}$. Because $i \geq i_{\max}$, we may write $h = \sum_{k=0}^{m} x_k g_k$ where $g_k \in (\vartheta_{\mathbb{X}})_{i+1,j}$. For each $k \in \{0,\dots,m\}$, we write $g_k = x_0 g'_k$ for some $g'_k \in (\vartheta_{\mathbb{X}})_{(i,j)}$, and hence

$$h = x_0 g_0 + \cdots + x_m g_m = x_0(x_0 g'_0 + \cdots + x_m g'_m) \in x_0 \cdot (\vartheta_{\mathbb{X}})_{(i+1,j)}.$$

Therefore $\mu_{x_0,i+1}$ is surjective, as wanted to show. □

From the lemma and the fact that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) \leq s$ for all $(i,j) \in \mathbb{N}^2$, we have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i_{\max}+s,j)$ for all $i \geq i_{\max}+s$ and $j \in \mathbb{N}$ and $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j_{\max}+s)$ for all $j \geq j_{\max}+s$ and $i \in \mathbb{N}$.

For $k,l \in \mathbb{N}$ set $\nu_k := \min\{i \in \mathbb{N} \mid \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,k) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i_{\max}+s,k)\}$ and $\varrho_l := \min\{j \in \mathbb{N} \mid \mathrm{HF}_{\vartheta_{\mathbb{X}}}(l,j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(l,j_{\max}+s)\}$ and $\nu_{\vartheta_{\mathbb{X}}} := \sup\{\nu_k \mid k \in \mathbb{N}\}$ and $\varrho_{\vartheta_{\mathbb{X}}} := \sup\{\varrho_l \mid l \in \mathbb{N}\}$. Then $(\nu_{\vartheta_{\mathbb{X}}}, \varrho_{\vartheta_{\mathbb{X}}}) \leq (i_{\max}+s, j_{\max}+s)$ and if the values of $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j)$ for finite tuples $(0,0) \preceq (i,j) \preceq (\nu_{\vartheta_{\mathbb{X}}}, \varrho_{\vartheta_{\mathbb{X}}})$ are computed, then we know all values of $\mathrm{HF}_{\vartheta_{\mathbb{X}}}$. This leads us to the following notion.

Definition. Let $(\nu,\varrho) := (\nu_{\vartheta_{\mathbb{X}}}, \varrho_{\vartheta_{\mathbb{X}}})$. The pair $B_{\vartheta_{\mathbb{X}}} = (B_{C,\vartheta_{\mathbb{X}}}, B_{R,\vartheta_{\mathbb{X}}})$, where

$$B_{C,\vartheta_{\mathbb{X}}} = (\mathrm{HF}_{\vartheta_{\mathbb{X}}}(\nu,0), \mathrm{HF}_{\vartheta_{\mathbb{X}}}(\nu,1), \dots, \mathrm{HF}_{\vartheta_{\mathbb{X}}}(\nu,\varrho))$$

and

$$B_{R,\vartheta_{\mathbb{X}}} = (\mathrm{HF}_{\vartheta_{\mathbb{X}}}(0,\varrho), \mathrm{HF}_{\vartheta_{\mathbb{X}}}(1,\varrho), \dots, \mathrm{HF}_{\vartheta_{\mathbb{X}}}(\nu,\varrho)),$$

is called the *border of the Hilbert function* of $\vartheta_{\mathbb{X}}$.

Example 3.5. Consider the set of nine points $\mathbb{X} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ given in Example 2.4. We know that $s_1 = 3$, $s_2 = 4$, $r_{\mathbb{X}_1} = 1$, and $r_{\mathbb{X}_2} = 2$. Also, the set $\mathbb{X}$ is ACM. Then a bihomogeneous minimal system of generators of $\vartheta_{\mathbb{X}}$ consists of 8 elements with degrees in $\{(1,3),(2,2),(3,1),(0,5),(3,2)\}$. This implies $i_{\max} = 3$ and $j_{\max} = 5$. The Hilbert function of $\vartheta_{\mathbb{X}}$ is computed by

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & \cdots \\ 0 & 0 & 3 & 8 & 9 & 9 & 9 & \cdots \\ 0 & 1 & 6 & 8 & 9 & 9 & 9 & \cdots \\ 0 & 1 & 6 & 8 & 9 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It follows that $\nu_{\vartheta_{\mathbb{X}}} = i_{\max} = 3$ and $\varrho_{\vartheta_{\mathbb{X}}} = j_{\max} = 5$ and the border of $\mathrm{HF}_{\vartheta_{\mathbb{X}}}$ is $B_{\vartheta_{\mathbb{X}}} = ((0,1,6,8,9,9),(1,2,9,9))$.

If a bihomogeneous minimal system of $\vartheta_{\mathbb{X}}$ is given, we can compute the tuple $(\nu_{\vartheta_{\mathbb{X}}}, \varrho_{\vartheta_{\mathbb{X}}})$ using the following lemma.

Lemma 3.6. *Let* $\{h_1,\dots,h_t\}$ *be a bihomogeneous minimal system of generators of* $\vartheta_{\mathbb{X}}$ *with* $\deg(h_k) = (i_k, j_k)$ *for* $k = 1,\dots,t$. *Put*

$$i_{\min} := \min\{i_k \mid k = 1,\dots,t\}, \quad j_{\min} := \min\{j_k \mid k = 1,\dots,t\}.$$

Then $\nu_{\vartheta_{\mathbb{X}}} = \max\{\nu_{j_{\min}},\dots,\nu_{j_{\max}}\}$ *and* $\varrho_{\vartheta_{\mathbb{X}}} = \max\{\varrho_{i_{\min}},\dots,\varrho_{i_{\max}}\}$.

Proof. For $(i,j) \in \mathbb{N}^2$ with $i < i_{\min}$ or $j < j_{\min}$, it is clearly true that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,j) = 0$. By the definition of ν_j and $\nu_{\vartheta_{\mathbb{X}}}$, we have $\nu_j = 0$ if $j < j_{\min}$ and $\nu_{\vartheta_{\mathbb{X}}} \geq \nu_k$ for $k \geq 0$. It suffices to show that $\nu_{j_{\max}} \geq \nu_k$ for all $k \geq j_{\max}$.

When $k = j_{\max}$ and $i \geq \nu_{j_{\max}}$, we have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,k) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1,k)$. So, $x_0(\vartheta_{\mathbb{X}})_{i,k} = (\vartheta_{\mathbb{X}})_{i+1,k}$, since x_0 is a non-zerodivisor of $R_{\mathbb{X}}$. Also, for any $l \geq 0$, $(\vartheta_{\mathbb{X}})_{l,k+1}$ contains no minimal generators, and hence $(\vartheta_{\mathbb{X}})_{l,k+1} = (\vartheta_{\mathbb{X}})_{l,k} \cdot (R_{\mathbb{X}})_{0,1}$. This implies $(\vartheta_{\mathbb{X}})_{i+1,k+1} = (\vartheta_{\mathbb{X}})_{i+1,k} \cdot (R_{\mathbb{X}})_{0,1} = x_0(\vartheta_{\mathbb{X}})_{i,k} \cdot (R_{\mathbb{X}})_{0,1} = x_0(\vartheta_{\mathbb{X}})_{i,k+1}$. Thus $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i,k+1) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1,k+1)$ for any $i \geq \nu_{j_{\max}}$, and so $\nu_k \geq \nu_{k+1}$. By induction on k, we get $\nu_{j_{\max}} \geq \nu_k$ for all $k \geq j_{\max}$, and this completes the proof of the equality for $\nu_{\vartheta_{\mathbb{X}}}$. The equality for $\varrho_{\vartheta_{\mathbb{X}}}$ can be achieved similarly using the non-zerodivisor $y_0 \in (R_{\mathbb{X}})_{0,1}$. $\square$

As a consequence of the lemma, when $\vartheta_{\mathbb{X}}$ is a principal ideal then $\nu_{\vartheta_{\mathbb{X}}} = \nu_{j_{\min}} = \nu_{j_{\max}}$ and $\varrho_{\vartheta_{\mathbb{X}}} = \varrho_{i_{\min}} = \varrho_{i_{\max}}$.

4. Special ACM sets

In this section we look at finite sets of points in $\mathbb{P}^m \times \mathbb{P}^n$ having the complete intersection or Cayley-Bacharach properties. As before, we let $\mathbb{X} = \{p_1,\dots,p_s\}$ be a set of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$.

Definition. (a) $\mathbb{X}$ is called a *complete intersection* if its bihomogeneous ideal $I_{\mathbb{X}}$ is generated by a bihomogeneous regular sequence.

(b) If $I_{\mathbb{X}}$ is generated by $\{F_1, \dots, F_m, G_1, \dots, G_n\}$ which forms a bihomogeneous regular sequence with $F_i \in S_{d_i,0}$ and $G_j \in S_{0,d'_j}$ for $1 \le i \le m$ and $1 \le j \le n$, we say that $\mathbb{X}$ is a *complete intersection of type* $(d_1, \dots, d_m, d'_1, \dots, d'_n)$ and write $CI(d_1, \dots, d_m, d'_1, \dots, d'_n)$.

It is worth noticing that every complete intersection $\mathbb{X}$ is ACM. When $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$, where $\mathbb{X}_k = \pi_k(\mathbb{X})$ for $k = 1, 2$ (see Convention 2.1), we also have the following property.

Lemma 4.1. *Let $I_{\mathbb{X}_1}$, $I_{\mathbb{X}_2}$ be the homogeneous vanishing ideals of $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively. If $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$, then $\mathbb{X}$ is ACM with $I_{\mathbb{X}} = I_{\mathbb{X}_1}S + I_{\mathbb{X}_2}S$ and*

$$\mathrm{HF}_{\mathbb{X}}(i, j) = \mathrm{HF}_{\mathbb{X}_1}(i) \cdot \mathrm{HF}_{\mathbb{X}_2}(j)$$

for all $(i, j) \in \mathbb{Z}^2$.

Proof. The ACM property of $\mathbb{X}$ and the equality $I_{\mathbb{X}_1}S + I_{\mathbb{X}_2}S = I_{\mathbb{X}}$ follow from [1, Theorem 2.1] and [3, Lemma 3.5]. Moreover, we have $R_{\mathbb{X}} \cong R_{\mathbb{X}_1} \otimes_K R_{\mathbb{X}_2}$ by [17, G.2], where $R_{\mathbb{X}_1} = K[x_0, \dots, x_m]/I_{\mathbb{X}_1}$ is the homogeneous coordinate ring of $\mathbb{X}_1 \subseteq \mathbb{P}^m$ and $R_{\mathbb{X}_2} = K[y_0, \dots, y_n]/I_{\mathbb{X}_2}$ is the homogeneous coordinate ring of $\mathbb{X}_2 \subseteq \mathbb{P}^n$. Therefore we get the equality $\mathrm{HF}_{\mathbb{X}}(i, j) = \mathrm{HF}_{\mathbb{X}_1}(i) \cdot \mathrm{HF}_{\mathbb{X}_2}(j)$ for all $(i, j) \in \mathbb{Z}^2$. □

As a direct consequence of the lemma, we get the following shape of the border of the Hilbert function of $\mathbb{X}$ for this case.

Corollary 4.2. *In the setting of Lemma 4.1, let $s_k = |\mathbb{X}_k|$ and let $r_{\mathbb{X}_k}$ be the regularity index of $\mathrm{HF}_{\mathbb{X}_k}$ for $k = 1, 2$. The border $B_{\mathbb{X}} = (B_C, B_R)$ of the Hilbert function of $\mathbb{X}$ is given by*

$$B_C = (s_1, s_1 \,\mathrm{HF}_{\mathbb{X}_2}(1), \dots, s_1 \,\mathrm{HF}_{\mathbb{X}_2}(r_{\mathbb{X}_2}) = s_1 s_2)$$

and

$$B_R = (s_2, s_2 \,\mathrm{HF}_{\mathbb{X}_1}(1), \dots, s_2 \,\mathrm{HF}_{\mathbb{X}_1}(r_{\mathbb{X}_1}) = s_1 s_2).$$

Notice that if $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$, then it is also ACM by Lemma 4.1, so that the Kähler different of $\mathbb{X}$ exists.

Proposition 4.3. *If $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$, then the Kähler different $\vartheta_{\mathbb{X}}$ satisfies*

$$\vartheta_{\mathbb{X}} = \vartheta_{\mathbb{X}_1} R_{\mathbb{X}} \cdot \vartheta_{\mathbb{X}_2} R_{\mathbb{X}}.$$

In addition, if $\mathbb{X} = CI(d_1, \dots, d_m, d'_1, \dots, d'_n)$, then $\vartheta_{\mathbb{X}}$ is a bihomogeneous principal ideal and has Hilbert function

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}_1} + i, r_{\mathbb{X}_2} + j) = \mathrm{HF}_{\mathbb{X}}(i, j)$$

for all $(i, j) \in \mathbb{N}^2$, where $r_{\mathbb{X}_1} = \sum_{k=1}^{m} d_k - m$ and $r_{\mathbb{X}_2} = \sum_{l=1}^{n} d'_l - n$.

Proof. Suppose that $\{F_1, \dots, F_r\}$ is a homogeneous system of generators of $I_{\mathbb{X}_1}$ and $\{G_1, \dots, G_t\}$ is a homogeneous system of generators of $I_{\mathbb{X}_2}$. Then

Lemma 4.1 yields that the relation matrix of Ω^1_{R/R_o} with respect to $\{dx_1, \ldots, dx_m, dy_1, \ldots, dy_n\}$ is

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_r}{\partial x_1} & \cdots & \frac{\partial F_r}{\partial x_m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\partial G_t}{\partial y_1} & \cdots & \frac{\partial G_t}{\partial y_n} \end{pmatrix}.$$

Because $\frac{\partial(F_{i_1},\ldots,F_{i_k},G_{i_{k+1}},\ldots,G_{i_{n+m}})}{\partial(x_1,\ldots,x_m,y_1,\ldots,y_n)} = 0$ if $k \neq m$, it follows that $\vartheta_{\mathbb{X}}$ is generated by elements of the form $\frac{\partial(F_{i_1},\ldots,F_{i_m},G_{j_1},\ldots,G_{j_n})}{\partial(x_1,\ldots,x_m,y_1,\ldots,y_n)}$ where $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, r\}$ and $\{j_1, \ldots, j_n\} \subseteq \{1, \ldots, t\}$. But this element can be written as

$$\frac{\partial(F_{i_1},\ldots,F_{i_m},G_{j_1},\ldots,G_{j_n})}{\partial(x_1,\ldots,x_m,y_1,\ldots,y_n)} = \frac{\partial(F_{i_1},\ldots,F_{i_m})}{\partial(x_1,\ldots,x_m)} \cdot \frac{\partial(G_{j_1},\ldots,G_{j_n})}{\partial(y_1,\ldots,y_n)}.$$

Hence we get $\vartheta_{\mathbb{X}} = \vartheta_{\mathbb{X}_1} R_{\mathbb{X}} \cdot \vartheta_{\mathbb{X}_2} R_{\mathbb{X}}$. If $\mathbb{X} = CI(d_1, \ldots, d_m, d'_1, \ldots, d'_n) = \mathbb{X}_1 \times \mathbb{X}_2$, then $\mathbb{X}_1$ and $\mathbb{X}_2$ are complete intersections. By [15, Corollary 2.6], $\vartheta_{\mathbb{X}_1}$ is a principal ideal generated by a homogeneous non-zerodivisor of degree $r_{\mathbb{X}_1}$ and $\vartheta_{\mathbb{X}_2}$ is a principal ideal generated by a homogeneous non-zerodivisor of degree $r_{\mathbb{X}_2}$, and hence $\vartheta_{\mathbb{X}}$ is a principal ideal generated by a homogeneous non-zerodivisor of degree $(r_{\mathbb{X}_1}, r_{\mathbb{X}_2})$. This also implies the claimed formula for $\mathrm{HF}_{\vartheta_{\mathbb{X}}}$. □

Corollary 4.4. *If $\mathbb{X} = CI(d_1, \ldots, d_m, d'_1, \ldots, d'_n)$ and $B_{\mathbb{X}} = (B_C, B_R)$, then we have $(\nu_{\vartheta_{\mathbb{X}}}, \varrho_{\vartheta_{\mathbb{X}}}) = (2r_{\mathbb{X}_1}, 2r_{\mathbb{X}_2})$ and the border of the Hilbert function $\vartheta_{\mathbb{X}}$ is given by*

$$B_{\vartheta_{\mathbb{X}}} = ((\underbrace{0, \ldots, 0}_{r_{\mathbb{X}_2}}, B_C), (\underbrace{0, \ldots, 0}_{r_{\mathbb{X}_1}}, B_R)).$$

Recall that for a finite set $\mathbb{X}$ of points in $\mathbb{P}^m$ and $p \in \mathbb{X}$, a *minimal separator* of p is a homogeneous element $F \in K[X_0, \ldots, X_m]$ of minimal degree such that $F(p) \neq 0$ and $F(p') = 0$ for all $p' \in \mathbb{X} \setminus \{p\}$. The *degree* $\deg_{\mathbb{X}}(p)$ of p in $\mathbb{X}$ is the degree of a minimal separator of p. We have $\deg_{\mathbb{X}}(p) \leq r_{\mathbb{X}}$ for every point $p \in \mathbb{X}$, where $r_{\mathbb{X}}$ is the regularity index of $\mathrm{HF}_{\mathbb{X}}$ (see [4, Lemma 2.4]). We say that $\mathbb{X}$ is a *Cayley-Bacharach scheme* if all points of $\mathbb{X}$ have the same degree $r_{\mathbb{X}}$. For many interesting results and more information about these notions in the standard case, see [4, 13].

Now we look at the generalization of these notions for a (not necessary ACM) set $\mathbb{X}$ of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$. In the same manner as above, for each $p \in \mathbb{X}$, a bihomogeneous form $F \in S$ is a *separator* of p in $\mathbb{X}$ if $F(p) \neq 0$ and $F(p') = 0$ for all $p' \in \mathbb{X} \setminus \{p\}$, and a separator $F \in S$ of p in $\mathbb{X}$ is *minimal* if there does not exist a separator G of p with $\deg(G) \prec \deg(F)$. For the existence of a finite set of minimal separators of any point in $\mathbb{X}$ and their properties, see e.g. [8, 9, 18].

Definition. The *degree* of a point $p \in \mathbb{X}$ is the set

$$\deg_{\mathbb{X}}(p) = \{\deg(F) \mid F \text{ is a minimal separator of } p\}.$$

For any $(i,j) \in \mathbb{N}^2$, we define $D_{(i,j)} := \{(k,l) \in \mathbb{N}^2 \mid (i,j) \preceq (k,l)\}$ and for a finite set $\Sigma = \{(i_1,j_1),\dots,(i_t,j_t)\} \subseteq \mathbb{N}^2$ we put $D_\Sigma := \bigcup_{k=1}^{t} D_{(i_k,j_k)}$. Clearly, for every $(i,j) \in D_{\deg_{\mathbb{X}}(p)}$, there exists a separator F of p with $\deg(F) = (i,j)$. In the following we collect several useful properties of degrees of points in $\mathbb{X}$ (see [8, Theorem 5.7] and [9, Theorem 2.2]).

Theorem 4.5. *Let $p \in \mathbb{X}$ and $\mathbb{Y} = \mathbb{X} \setminus \{p\}$.*

(a) *If $\{F_1,\dots,F_t\}$ is a set of minimal separators of p, then $I_{\mathbb{Y}} = I_{\mathbb{X}} + \langle F_1,\dots,F_t\rangle$.*

(b) *We have*

$$\mathrm{HF}_{\mathbb{Y}}(i,j) = \begin{cases} \mathrm{HF}_{\mathbb{X}}(i,j) & \text{if } (i,j) \notin D_{\deg_{\mathbb{X}}(p)}, \\ \mathrm{HF}_{\mathbb{X}}(i,j) - 1 & \text{if } (i,j) \in D_{\deg_{\mathbb{X}}(p)}. \end{cases}$$

(c) *If $\mathbb{X}$ is ACM, then $|\deg_{\mathbb{X}}(p)| = 1$ for every $p \in \mathbb{X}$.*

The converse of Theorem 4.5(c) holds true for $n = m = 1$ by [10, Theorem 8] or [18, Theorem 6.7], but it fails to hold in general (see [8, Example 5.10] for an example in $\mathbb{P}^2 \times \mathbb{P}^2$). When $\mathbb{X}$ is ACM, we write $\deg_{\mathbb{X}}(p) = (i,j)$ instead of $\deg_{\mathbb{X}}(p) = \{(i,j)\}$.

Definition. The set $\mathbb{X}$ is said to have the *Cayley-Bacharach property* if the Hilbert function of $\mathbb{X}\setminus\{p\}$ is independent of the choice of $p \in \mathbb{X}$, or equivalently, if all of its points have the same degree.

In [5, Proposition 7.3], we know that $\mathbb{X} = CI(d_1, d_1')$ if and only if $\mathbb{X}$ has the Cayley-Bacharach property. However, it fails to hold in general as the following example shows.

Example 4.6. In $\mathbb{P}^1 \times \mathbb{P}^2$, consider the set $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ of six points, where $\mathbb{X}_1 = \{q_1, q_2\} \subseteq \mathbb{P}^1$ with $q_1 = (1:0)$, $q_2 = (1:1)$, and $\mathbb{X}_2 = \{q_1', q_2', q_3'\} \subseteq \mathbb{P}^2$ with $q_1' = (1:0:0)$, $q_2' = (1:1:0)$, $q_3' = (1:1:1)$. Then $I_{\mathbb{X}}$ has a bihomogeneous minimal system of generators given by

$$\{\, x_0x_1 - x_1^2, y_0y_1 - y_1^2, y_1y_2 - y_2^2, y_0y_2 - y_2^2 \,\},$$

so $\mathbb{X}$ is not a complete intersection. On the other hand, $\mathbb{X}_1 \subseteq \mathbb{P}^1$ is a complete intersection with $r_{\mathbb{X}_1} = 1$, and hence $\mathbb{X}_1$ is a Cayley-Bacharach scheme, and $\mathbb{X}_2 = \{q_1', q_2', q_3'\} \subseteq \mathbb{P}^2$ is also a Cayley-Bacharach scheme with $r_{\mathbb{X}_2} = 1$. Using ApCoCoA we can check that $\deg(q_i \times q_j') = (1,1)$ for all $i = 1,2$ and $j = 1,2,3$. Thus $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ has the Cayley-Bacharach property (this also follows by

Proposition 4.9). In this case the Kähler different has its Hilbert function

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 3 & \cdots \\ 0 & 0 & 6 & 6 & \cdots \\ 0 & 0 & 6 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}_1}, r_{\mathbb{X}_2}) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(1,1) = 0$.

Using the Kähler different, we give a characterization of complete intersections of type $(d_1, \dots, d_m, d'_1, \dots, d'_n)$ as follows.

Theorem 4.7. *For a set $\mathbb{X}$ of s distinct points in $\mathbb{P}^m \times \mathbb{P}^n$, the following statements are equivalent.*

(a) $\mathbb{X} = CI(d_1, \dots, d_m, d'_1, \dots, d'_n)$ *for some positive integers* $d_i, d'_j \geq 1$.
(b) $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ *and* $\mathbb{X}_1 \subseteq \mathbb{P}^m$ *is a complete intersection of type* $(d_1, \dots, d_m)$ *and* $\mathbb{X}_2 \subseteq \mathbb{P}^n$ *is a complete intersection of type* $(d'_1, \dots, d'_n)$.
(c) $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ *has the Cayley-Bacharach property and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}_1}, r_{\mathbb{X}_2}) \neq 0$.

In the proof of this theorem, we use the following properties.

Lemma 4.8. *For an ACM set of s points $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$, if $q \times q' \in \mathbb{X}$, then*

$$\deg_{\mathbb{X}}(q \times q') \preceq (\deg_{\mathbb{X}_1}(q), \deg_{\mathbb{X}_2}(q')) \preceq (r_{\mathbb{X}_1}, r_{\mathbb{X}_2}).$$

Proof. Since $\mathbb{X}$ is ACM, and so each point of $\mathbb{X}$ has exactly one degree. The claim follows from the fact that if F_k is a separator of q in $\mathbb{X}_1$ and G_l is a separator of q' in $\mathbb{X}_2$, then $F_k G_l$ is also a separator of $q \times q'$ in $\mathbb{X}$. □

Proposition 4.9. *Suppose $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \subseteq \mathbb{P}^m \times \mathbb{P}^n$. Then $\mathbb{X}$ has the Cayley-Bacharach property if and only if $\mathbb{X}_1$ and $\mathbb{X}_2$ are Cayley-Bacharach schemes.*

Proof. Note that $\mathbb{X}$ is ACM. Let us write $\mathbb{X}_1 = \{q_1, \dots, q_{s_1}\} \subseteq \mathbb{P}^m$ and $\mathbb{X}_2 = \{q'_1, \dots, q'_{s_2}\} \subseteq \mathbb{P}^n$. Firstly, we prove that

$$\deg_{\mathbb{X}}(q_k \times q'_l) = (\deg_{\mathbb{X}_1}(q_k), \deg_{\mathbb{X}_2}(q'_l))$$

for all $1 \leq k \leq s_1, 1 \leq l \leq s_2$. According to Lemma 4.8, it suffices to show that $\deg_{\mathbb{X}}(q_k \times q'_l) \succeq (\deg_{\mathbb{X}_1}(q_k), \deg_{\mathbb{X}_2}(q'_l))$. Suppose $\deg_{\mathbb{X}}(q_k \times q'_l) = (i,j)$. Let $F \in S_{i,j}$ be a minimal separator of the point $q_k \times q'_l$. Then $F = \sum_u G_u H_u$ with $G_u \in S_{i,0}$ and $H_u \in S_{0,j}$. Let $T_1, \dots, T_{m_i} \in S_{i,0}$ (resp. $T'_1, \dots, T'_{n_j} \in S_{0,j}$) be terms whose residue classes form a K-basis of $S_{i,0}/(I_{\mathbb{X}_1}S)_{i,0}$ (resp. $S_{0,j}/(I_{\mathbb{X}_2}S)_{0,j}$). This enables us to write $G_u = a_{u1}T_1 + \cdots + a_{um_i}T_{m_i} + G'_u$ with $G'_u \in (I_{\mathbb{X}_1}S)_{i,0}$ and $a_{ur} \in K$, $H_u = b_{u1}T'_1 + \cdots + b_{un_j}T'_{n_j} + H'_u$ with $H'_u \in (I_{\mathbb{X}_2}S)_{0,j}$ and $b_{ut} \in K$. Since $I_{\mathbb{X}} = I_{\mathbb{X}_1}S + I_{\mathbb{X}_2}S$, we have

$$F = \sum_u G_u H_u = \sum_{1 \leq r \leq m_i, 1 \leq t \leq n_j} c_{rt} T_r T'_t \pmod{I_{\mathbb{X}}}, \text{ with } c_{rt} = \sum_u a_{ur} b_{ut}.$$

Put $F_k := \sum_{rt} c_{rt}T'_t(q'_l)T_r \in S_{i,0}$. Since $F(q_k \times q'_l) \neq 0$, we have $F_k(q_k) \neq 0$. Moreover, $F_k(q_{k'}) = F(q_{k'} \times q'_l) = 0$ for $k' \neq k$. So, F_k is a separator of q_k in $\mathbb{X}_1$, and this yields $i \geq \deg_{\mathbb{X}_1}(q_k)$. Analogously, the element $G_l := \sum_{rt} c_{rt}T_r(q_k)T'_t \in S_{0,j}$ is a separator of q'_l in $\mathbb{X}_2$, and hence $j \geq \deg_{\mathbb{X}_2}(q'_l)$. Thus, $(i,j) \succeq (\deg_{\mathbb{X}_1}(q_k), \deg_{\mathbb{X}_2}(q'_l))$, and therefore we get $\deg_{\mathbb{X}}(q_k \times q'_l) = (\deg_{\mathbb{X}_1}(q_k), \deg_{\mathbb{X}_2}(q'_l))$ for all k, l.

If $\mathbb{X}_1$ and $\mathbb{X}_2$ are Cayley-Bacharach schemes, then

$$\deg_{\mathbb{X}}(q_k \times q'_l) = (\deg_{\mathbb{X}_1}(q_k), \deg_{\mathbb{X}_2}(q'_l)) = (r_{\mathbb{X}_1}, r_{\mathbb{X}_2})$$

for all $1 \leq k \leq s_1$ and $1 \leq l \leq s_2$, and hence $\mathbb{X}$ has the Cayley-Bacharach property. Conversely, suppose that $\mathbb{X}$ has the Cayley-Bacharach property, but $\mathbb{X}_1$ is not a Cayley-Bacharach-scheme. Then there is a point $q_k \in \mathbb{X}_1$ such that $\deg_{\mathbb{X}_1}(q_k) < r_{\mathbb{X}_1}$. By [4, Proposition 1.14], we find $q_{k'} \in \mathbb{X}_1$ such that $\deg_{\mathbb{X}_1}(q_{k'}) = r_{\mathbb{X}_1}$ and $q'_l \in \mathbb{X}_2$ such that $\deg_{\mathbb{X}_2}(q'_l) = r_{\mathbb{X}_2}$. This implies

$$\deg_{\mathbb{X}}(q_k \times q'_l) \preceq (r_{\mathbb{X}_1} - 1, r_{\mathbb{X}_2}) \prec (r_{\mathbb{X}_1}, r_{\mathbb{X}_2}) = \deg_{\mathbb{X}}(q_{k'} \times q_l),$$

and thus $\mathbb{X}$ does not have the Cayley-Bacharach property, a contradiction. Therefore, $\mathbb{X}_1$ is a CB-scheme, so is $\mathbb{X}_2$. □

Proof of Theorem 4.7. The implication "(b)⇒(a)" follows from Lemma 4.1. To prove "(a)⇒(b)", suppose that $\mathbb{X} = CI(d_1, \ldots, d_m, d'_1, \ldots, d'_n)$ for some positive integers $d_i, d'_j \geq 1$. Then $I_{\mathbb{X}} = \langle F_1, \ldots, F_m, G_1, \ldots, G_n \rangle_S$ with $\deg(F_i) = (d_i, 0)$ and $\deg(G_j) = (0, d'_j)$, particularly, $I_{\mathbb{X}_1} = \langle F_1, \ldots, F_m \rangle$ is a saturated homogeneous ideal of $K[X_0, \ldots, X_m]$ defining a complete intersection $\mathbb{X}_1 \subseteq \mathbb{P}^m$ and $I_{\mathbb{X}_2} = \langle G_1, \ldots, G_n \rangle$ is a saturated homogeneous ideal of $K[Y_0, \ldots, Y_n]$ defining a complete intersection $\mathbb{X}_2 \subseteq \mathbb{P}^n$. Moreover, it is not hard to verify that $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$.

The implication "(b)⇒(c)" holds true by Proposition 4.3 and Proposition 4.9 and the fact that a complete intersection set of s points in $\mathbb{P}^m$ is always a Cayley-Bacharach scheme.

Now we prove "(c)⇒(b)". It suffice to show that $\mathbb{X}_1$ is a complete intersection in $\mathbb{P}^m$ (similarly for $\mathbb{X}_2 \subseteq \mathbb{P}^n$). By assumption, $\mathbb{X}$ has the Cayley-Bacharach property, then $\mathbb{X}_1$ and $\mathbb{X}_2$ are Cayley-Bacharach schemes by Proposition 4.9. According to Proposition 4.3, we have $\vartheta_{\mathbb{X}} = \vartheta_{\mathbb{X}_1} R_{\mathbb{X}} \cdot \vartheta_{\mathbb{X}_2} R_{\mathbb{X}}$, and so $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}_1}, r_{\mathbb{X}_2}) \neq 0$ implies $\mathrm{HF}_{\vartheta_{\mathbb{X}_1}}(r_{\mathbb{X}_1}) \neq 0$. By [12, Theorem 5.6], $\mathbb{X}_1$ is a complete intersection, as desired. □

Lemma 4.10. *If $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ and for every point $p \in \mathbb{X}$ the Kähler different $\vartheta_{\mathbb{X}}$ contains no separator of p of degree $\prec (mr_{\mathbb{X}_1}, nr_{\mathbb{X}_2})$, then $\mathbb{X}$ has the Cayley-Bacharach property.*

Proof. Suppose that $\mathbb{X}$ does not have the Cayley-Bacharach property. By Proposition 4.9, $\mathbb{X}_1$ or $\mathbb{X}_2$ is not a Cayley-Bacharach scheme. Assume that $\mathbb{X}_1$ is not a Cayley-Bacharach scheme. There is $i \in \{1, \ldots, s_1\}$ such that $\deg_{\mathbb{X}_1}(q_i) \leq r_{\mathbb{X}_1} - 1$. Let $F_i \in K[X_0, \ldots, X_m]$ be a minimal separator of q_i in

$\mathbb{X}_1$ and $G_1 \in K[Y_0, \dots, Y_n]$ be a minimal separator of q'_1 in $\mathbb{X}_2$. By [15, Corollary 2.6], the image of F_i^m in $R_{\mathbb{X}_1}$ belongs to $\vartheta_{\mathbb{X}_1}$ and the image of G_1^n in $R_{\mathbb{X}_2}$ belongs to $\vartheta_{\mathbb{X}_2}$. So, the image of $F_i^m G_1^n$ in $R_{\mathbb{X}}$ is contained in $\vartheta_{\mathbb{X}}$. Moreover, $F_i^m G_1^n$ is a separator of $q_i \times q'_1$ in $\mathbb{X}$ of degree $\preceq (m(r_{\mathbb{X}_1} - 1), nr_{\mathbb{X}_2})$. This contradicts to the assumption. $\square$

5. Finite Sets with the ($\star$)-property

Now we investigate the Cayley-Bacharach property for a finite set $\mathbb{X}$ of points in $\mathbb{P}^m \times \mathbb{P}^n$ which satisfies the ($\star$)-property. According to [8, Definition 4.2], the set $\mathbb{X}$ is said to have the *($\star$)-property* if whenever $q_1 \times q'_1$ and $q_2 \times q'_2$ are two points in $\mathbb{X}$ with $q_1 \neq q_2$ and $q'_1 \neq q'_2$, then either $q_1 \times q'_2$ or $q_2 \times q'_1$ (or both) is also in $\mathbb{X}$. By [3, Theorem 3.7], if $\mathbb{X}$ has the ($\star$)-property, then $\mathbb{X}$ is ACM. Except for the case $m = n = 1$, the converse of this result does not hold true in general (see [8, Theorem 4.3 and Example 4.9] and [3, Example 4.2]). As before, for an ACM set $\mathbb{X}$ we always assume that x_0, y_0 form a regular sequence in $R_{\mathbb{X}}$.

Write $\mathbb{X}_1 = \pi_1(\mathbb{X}) = \{q_1, \dots, q_{s_1}\} \subseteq \mathbb{P}^m$ and $\mathbb{X}_2 = \pi_2(\mathbb{X}) = \{q'_1, \dots, q'_{s_2}\} \subseteq \mathbb{P}^n$. For $i = 1, \dots, s_1$ and $j = 1, \dots, s_2$, put

$$W_i := \pi_2(\pi_1^{-1}(q_i) \cap \mathbb{X}) \subseteq \mathbb{X}_2, \quad V_j := \pi_1(\pi_2^{-1}(q'_j) \cap \mathbb{X}) \subseteq \mathbb{X}_1.$$

After renaming, we can always assume that $|W_{s_1}| \leq \cdots \leq |W_1| \leq s_2$ and $|V_{s_2}| \leq \cdots \leq |V_1| \leq s_1$. When $\mathbb{X}$ has the ($\star$)-property, we may assume $\mathbb{X}_1 = V_1 \supseteq \cdots \supseteq V_{s_2}$ and $\mathbb{X}_2 = W_1 \supseteq \cdots \supseteq W_{s_1}$ (see e.g. [3, Lemma 3.4]).

Proposition 5.1. *If $\mathbb{X}$ has the ($\star$)-property, then for $q_i \times q'_j \in \mathbb{X}$ we have*

$$\deg_{\mathbb{X}}(q_i \times q'_j) = (\deg_{V_j}(q_i), \deg_{W_i}(q'_j)).$$

Proof. Since $\mathbb{X}$ is ACM, we have $\deg_{\mathbb{X}}(q_i \times q'_j) = (r, t)$ for some $(r, t) \in \mathbb{N}^2$. Clearly, $q_i \in V_j$ and $q'_j \in W_i$. Let $G \in (K[X_0, \dots, X_m])_{\deg_{V_j}(q_i)}$ be a minimal separator of q_i in V_j and $G' \in (K[Y_0, \dots, Y_n])_{\deg_{W_i}(q'_j)}$ be a minimal separator of q'_j in W_i. Set $F := GG' \in S$. Observe that $F(q_i \times q'_j) = G(q_i)G'(q'_j) \neq 0$. Let $q \times q' \in \mathbb{X} \setminus \{q_i \times q'_j\}$. If $q \in V_j \setminus \{q_i\}$ or $q' \in W_i \setminus \{q'_j\}$, then $G(q) = 0$ or $G'(q') = 0$, and so $F(q \times q') = 0$. Now consider the case $q \notin V_j \setminus \{q_i\}$ and $q' \notin W_i \setminus \{q'_j\}$. There are the following three cases:

- If $q = q_i$ and $q' \neq q'_j$, then $q' \in W_i \setminus \{q'_j\}$, a contradiction.
- If $q' = q'_j$ and $q \neq q_i$, then $q \in V_j \setminus \{q_i\}$, a contradiction.
- If $q \neq q_i$ and $q' \neq q'_j$, then the ($\star$)-property of $\mathbb{X}$ implies $q \times q'_j$ or $q_i \times q' \in \mathbb{X}$. It follows that $q \in V_j \setminus \{q_i\}$ or $q' \in W_i \setminus \{q'_j\}$. This is again a contradiction.

Altogether, $F(q_i \times q'_j) \neq 0$ and $F(q \times q') = 0$ for all $q \times q' \in \mathbb{X} \setminus \{q_i \times q'_j\}$. Hence F is a separator of $q_i \times q'_j$ with $\deg(F) = (\deg_{V_j}(q_i), \deg_{W_i}(q'_j))$, and so $(r, t) \preceq (\deg_{V_j}(q_i), \deg_{W_i}(q'_j))$.

Furthermore, if $(r,t) \prec (\deg_{V_j}(q_i), \deg_{W_i}(q'_j))$, then there is a minimal separator $\tilde{F} \neq 0$ of $q_i \times q'_j$ with $\deg(\tilde{F}) = (r,t)$ and $r < \deg_{V_j}(q_i)$ or $t < \deg_{W_i}(q'_j)$. Suppose that $r < \deg_{V_j}(q_i)$ (a similar argument for the case $t < \deg_{W_i}(q'_j)$). Set $\mathbb{Y} := V_j \times \{q'_j\} \subseteq \mathbb{X}$. Then $\tilde{F}$ is also a separator of $q_i \times q'_j$ in $\mathbb{Y}$. As in the proof of Proposition 4.9, we have $\deg_{\mathbb{Y}}(q_i \times q'_j) = (\deg_{V_j}(q_i), 0)$. This implies $(\deg_{V_j}(q_i), 0) \preceq \deg(\tilde{F}) = (r,t)$, in particularly, we get $\deg_{V_j}(q_i) \leq r < \deg_{V_j}(q_i)$, a contradiction. Therefore it must be $(r,t) = (\deg_{V_j}(q_i), \deg_{W_i}(q'_j))$. □

Theorem 5.2. *Let $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ have the $(\star)$-property. Then $\mathbb{X}$ has the Cayley-Bacharach property if and only if the following conditions are satisfied:*

(a) *$V_1, \dots, V_{s_2}$ are Cayley-Bacharach schemes in $\mathbb{P}^m$ and $r_{V_1} = \cdots = r_{V_{s_2}}$;*
(b) *$W_1, \dots, W_{s_1}$ are Cayley-Bacharach schemes in $\mathbb{P}^n$ and $r_{W_1} = \cdots = r_{W_{s_1}}$.*

Proof. If $\mathbb{X}$ satisfies the conditions (a) and (b), then (a) implies $\deg_{V_j}(q) = r_{V_1}$ for all $q \in V_j$ and for $j = 1, \dots, s_2$, while (b) implies $\deg_{W_i}(q') = r_{W_1}$ for all $q' \in W_i$ and for $i = 1, \dots, s_1$. By Proposition 5.1, we obtain $\deg(q \times q') = (r_{V_1}, r_{W_1})$ for all $q \times q' \in \mathbb{X}$. Therefore $\mathbb{X}$ has the Cayley-Bacharach property.

Conversely, suppose that $\mathbb{X}$ has the Cayley-Bacharach property, i.e., there is $(r,t) \in \mathbb{N}^2$ such that $\deg_{\mathbb{X}}(q \times q') = (r,t)$ for all $q \times q' \in \mathbb{X}$. Note that we may here assume that $\mathbb{X}_1 = V_1 \supseteq \cdots \supseteq V_{s_2}$ and $\mathbb{X}_2 = W_1 \supseteq \cdots \supseteq W_{s_1}$. Especially, $\{q_1\} \times \mathbb{X}_2 \subseteq \mathbb{X}$ and $\mathbb{X}_1 \times \{q'_1\} \subseteq \mathbb{X}$. According to [4, Proposition 1.14], $\mathbb{X}_1$ always contains a point q_i of degree $r_{\mathbb{X}_1}$ and $\mathbb{X}_2$ always contains a point q'_j of degree $r_{\mathbb{X}_2}$. From $\deg(q_1 \times q'_1) = \cdots = \deg(q_{s_1} \times q'_1) = (r,t)$, Proposition 5.1 yields

$$r = \deg_{V_1}(q_1) = \cdots = \deg_{V_1}(q_{s_1}) = \deg_{\mathbb{X}_1}(q_i) = r_{\mathbb{X}_1}.$$

Similarly, it follows from $\deg(q_1 \times q'_1) = \cdots = \deg(q_1 \times q'_{s_2}) = (r,t)$ and Proposition 5.1 that

$$t = \deg_{W_1}(q'_1) = \cdots = \deg_{W_1}(q'_{s_2}) = \deg_{\mathbb{X}_2}(q'_j) = r_{\mathbb{X}_2}.$$

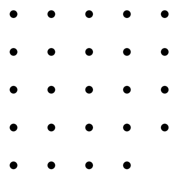
In particular, $\mathbb{X}_1$ and $\mathbb{X}_2$ are Cayley-Bacharach schemes. Moreover, we have $r_{V_{s_2}} \leq \cdots \leq r_{V_1} = r_{\mathbb{X}_1}$ and $r_{W_{s_1}} \leq \cdots \leq r_{W_1} = r_{\mathbb{X}_2}$. Thus $(r_{\mathbb{X}_1}, r_{\mathbb{X}_2}) = \deg_{\mathbb{X}}(q_i \times q'_j) = (\deg_{V_j}(q_i), \deg_{W_i}(q'_j)) \leq (r_{V_j}, r_{W_i})$ for all $q_i \times q'_j \in \mathbb{X}$ implies $r_{V_{s_2}} = \cdots = r_{V_1} = r_{\mathbb{X}_1}$ and $r_{W_{s_1}} = \cdots = r_{W_1} = r_{\mathbb{X}_2}$ and all $V_1, \dots, V_{s_2} \subseteq \mathbb{P}^m$ and $W_1, \dots, W_{s_1} \subseteq \mathbb{P}^n$ are Cayley-Bacharach schemes. □

The next corollary is a direct consequence of Theorem 5.2.

Corollary 5.3. *Let $\mathbb{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ have the $(\star)$-property. If $\mathbb{X}$ has the Cayley-Bacharach property, then $\mathbb{X}_1 \subseteq \mathbb{P}^m$ and $\mathbb{X}_2 \subseteq \mathbb{P}^n$ are Cayley-Bacharach schemes.*

Example 5.4. Let $K = \mathbb{Q}$ and $\mathbb{X}$ be the set of 24 points in $\mathbb{P}^2 \times \mathbb{P}^2$ given by $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \setminus \{q_5 \times q_5\}$, where $\mathbb{X}_1 = \mathbb{X}_2 = \{q_1, \dots, q_5\} \subseteq \mathbb{P}^2$ with $q_1 = (1:0:0)$,

$q_2 = (1 : 1 : 0)$, $q_3 = (1 : 0 : 1)$, $q_4 = (1 : 1 : 1)$ and $q_5 = (1 : 1 : 2)$ (see the figure below).

$\mathbb{X} =$ 

Then we have $V_1 = V_2 = V_3 = V_4 = \mathbb{X}_1$, $V_5 = \mathbb{X}_1 \setminus \{q_5\}$, $W_1 = W_2 = W_3 = W_4 = \mathbb{X}_2$ and $W_5 = \mathbb{X}_2 \setminus \{q_5\}$. Then V_5, W_5 are complete intersections in $\mathbb{P}^2$, and so Cayley-Bacharach schemes. Also, $\mathbb{X}_1$ is a Cayley-Bacharach scheme in $\mathbb{P}^2$ and $r_{\mathbb{X}_1} = 2 = r_{V_5} = r_{W_5}$. So, the conditions (a) and (b) in Theorem 5.2 are satisfied, and therefore $\mathbb{X}$ has the Cayley-Bacharach property.

Proposition 5.5. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ have the $(\star)$-property. Then $\mathbb{X}$ has the Cayley-Bacharach property if and only if $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ and $\mathbb{X}_2 \subseteq \mathbb{P}^n$ is a Cayley-Bacharach scheme.*

Proof. Note that every finite set V in $\mathbb{P}^1$ is a complete intersection and $r_V = |V| - 1$. Suppose that $\mathbb{X}$ has the Cayley-Bacharach property. Then Theorem 5.2 yields $\mathbb{X}_1 = V_1 = \cdots = V_{s_2}$ and $\mathbb{X}_2 = W_1 \supseteq \cdots \supseteq W_{s_1}$ an descending chain of Cayley-Bacharach schemes with $r_{\mathbb{X}_2} = r_{W_1} = \cdots = r_{W_{s_1}}$. For $j = 1, \ldots, s_2$, we have $\pi_1(\pi_2^{-1}(q'_j) \cap \mathbb{X}) = V_j = \{q_1, \ldots, q_{s_1}\}$, and so $\pi_2^{-1}(q'_j) \cap \mathbb{X} = \{q_1 \times q'_j, \ldots, q_{s_1} \times q'_j\} \subseteq \mathbb{X}$. Hence $\mathbb{X}_1 \times \mathbb{X}_2 \subseteq \mathbb{X}$, and therefore $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$. Conversely, assume that $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ and $\mathbb{X}_2$ is a Cayley-Bacharach scheme in $\mathbb{P}^n$. Clearly, $\mathbb{X}_1 \subseteq \mathbb{P}^1$ is a complete intersection, and hence a Cayley-Bacharach scheme. By Proposition 4.9, $\mathbb{X}$ has the Cayley-Bacharach property. □

Corollary 5.6. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ have the $(\star)$-property. Then the following statements are equivalent*:

(a) $\mathbb{X} = CI(d_1, d'_1, \ldots, d'_n)$ *for some positive integers* $d_1, d'_1, \ldots, d'_n \geq 1$.
(b) $\mathbb{X}$ *has the Cayley-Bacharach property and* $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(d_1 - 1, r_{\mathbb{X}_2}) \neq 0$.

Proof. This follows directly from Theorem 4.7 and Proposition 5.5. □

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