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BÉZOUT RINGS AND WEAKLY BÉZOUT RINGS

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Abstract. In this paper, we study some properties of Bézout and weakly Bézout rings. Then, we investigate the transfer of these notions to trivial ring extensions and amalgamated algebras along an ideal. Also, in the context of domains we show that the amalgamated is a Bézout ring if and only if it is a weakly Bézout ring. All along the paper, we put the new results to enrich the current literature with new families of examples of non-Bézout weakly Bézout rings.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$ and that all modules are unital. If R is a ring, then $U(R)$ denotes the set of units of R; $Z(R)$ the set of zero-divisors of R; $Reg(R) := R - Z(R)$ the set of regular elements of R; and $tq(R) = R_{R-Z(R)}$ the total quotient ring of R. A ring R is called a total ring of quotients if $R = tq(R)$, that is, every element of R is invertible or zero-divisor.

A ring R is said to be a Bézout ring if the sum of two principal ideals is again principal. By induction it follows that every finitely generated ideal is principal. The class of Bézout rings includes strictly the classes of Hermite rings, elementary divisor rings, and valuation rings. For more details, see [14, 20, 21]. During the past three decades, several notions grew out of Bézout (e.g., Almost Bézout property, ϕ-Bézout, and weakly Bézout (i.e., every finitely generated ideal of R contained in a principal proper ideal of R is itself principal)). See for instance [3, 5, 6].

For a ring A and an A-module E, the trivial ring extension of A by E is the ring $R := A \ltimes E$, where the underlying group is $A \times E$ and the multiplication is defined by $(a, e)(b, f) = (ab, af + be)$. The ring R is also called the *idealization* of E over A and is denoted by $A(+)E$. This construction was first introduced, in 1962, by Nagata [23] in order to facilitate interaction between rings and their modules and also to provide various families of examples of commutative rings containing zero-divisors. The literature abounds of papers on trivial extensions

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dealing with the transfer of ring-theoretic notions in various settings of these constructions (see, for instance, [1,15,16,18]). For more details on commutative trivial extensions (*or idealizations*), we refer the reader to Glaz's and Huckaba's respective books [15, 16], and also D. D. Anderson and M. Winders relatively recent and comprehensive survey paper [4].

The *amalgamation* algebras along an ideal, introduced and studied by D'Anna, Finocchiaro and Fontana in [8–10] and defined as follows:

Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) : a \in A,\ j \in J\}$$

called the amalgamation of A and B along J with respect to f. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [11, 12]). See for instance [2, 8–10, 19, 22]. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [8, Examples 2.5 and 2.6].

In [18], the authors studied the transfer of the Bézout property to the trivial ring extensions of any domain by its quotient field. In [17], the author established necessary and sufficient conditions for classical $D + M$ constructions to inherit Bézout property. In [13], the authors examined the transfer of the weakly Bézout proprety to the trivial ring extensions, and after observing that each weakly Bézout ring contains a non-invertible regular element (that is, is not a total ring of quotients) is a Bézout ring. The purpose of this paper is to study the possible transfer of the Bézout properties and weakly Bézout rings to trivial ring extension and amalgamated algebras along an ideal. Also, in the context of domains we show that the amalgamated is a Bézout ring if and only if it is a weakly Bézout ring, and we give new results to enrich the current literature with new families of examples of non-Bézout weakly Bézout rings.

2. Main results

The first theorem of this paper develops a result of the transfer of weakly Bézout property to trivial ring extensions. Recall that a module over a ring is divisible if each element of the module is divisible by every regular element of the ring. Recall also that a ring R is weakly Bézout if every finitely generated ideal of R contained in a principal proper ideal of R is itself principal.

Theorem 2.1. *Let A be a domain, E a nonzero divisible A-module, and $R := A \ltimes E$. Then, the following statements hold*:

(1) *If R is weakly Bézout, then so is A.*

(2) *Assume that $A \subseteq E$ is an extension of domains. Then, R is a weakly Bézout ring if and only if A is a weakly Bézout ring and $E = qf(A)$.*

Proof. (1) Let $I := \sum_{i=1}^{i=n} Aa_i$ for some positive integer n and $J := Ab$ be two proper ideals of A such that $I \subseteq J$. Then, $I \ltimes E$ contained in $J \ltimes E := Ab \ltimes E = R(b, 0)$, by [1, Lemma 2.3]. So, $I \ltimes E = R(a, e)$ for some element (a, e) of R since R is a weakly Bézout ring. Thus $I \ltimes E = Aa \ltimes E$ by [1, Lemma 2.3] and hence A is a weakly Bézout ring.

(2) If R is a weakly Bézout ring, then A is a weakly Bézout ring by (1). It remains to prove that $qf(A) \subseteq E$. Let a be a nonzero element of A, set $I := (a, 0)R + (a, 1)R$ and $J := Aa \ltimes E = R(a, 0)$. It is easily seen that I contained in J, and so it is principal, that is, there exists an element (c, b) in $A \ltimes E$ such that $I = A \ltimes E(c, b)$. Then, there exist (a_1, b_1), (a_2, b_2), (a_3, b_3) and (a_4, b_4) in $A \ltimes E$ such that

$$\begin{aligned}(a, 0) &= (a_1, b_1)(c, b) = (a_1c, a_1b + cb_1),\\ (a, 1) &= (a_2, b_2)(c, b) = (a_2c, a_2b + cb_2),\\ (c, b) &= (a_3, b_3)(a, 0) + (a_4, b_4)(a, 1).\end{aligned}$$

Then, we obtain the following equations:

(i) $a = a_1c$,
(ii) $0 = a_1b + cb_1$,
(iii) $a = a_2c$,
(iv) $1 = a_2b + cb_2$,
(v) $c = a_3a + a_4a$.

Hence, from the first and third equation, we have $a_1c = a_2c$. So, $0 \neq c$ since $0 \neq a$, and so $a_1 = a_2$. Also from the fifth equation, we obtain $c = a(a_3 + a_4) = \alpha c(a_3 + a_4)$. Then, $(a_3 + a_4)\alpha = 1$, and so $\alpha \in U(A)$. Finally, from the second and fourth equation, we have $0 = \alpha b + cb_1$ and $1 = \alpha b + cb_2$. Thus, $1 = c(b_2 - b_1) = \alpha^{-1}a(b_2 - b_1) \in aE$. Hence, $a \in U(E)$. Therefore, $qf(A) \subseteq E$. Clearly, $qf(A) = E$. Deny, $\dim_{qf(A)}(E) \geq 2$, a contradiction by [13, Proposition 2.9]. The converse of (2) is an immediate consequence of [13, Proposition 2.2] and [18, Proposition 3.5]. $\square$

For the special case of trivial extensions of domains by their quotient fields, we obtain the following result.

Corollary 2.1. *Let A be a domain, $K := qf(A)$ and $R := A \ltimes K$ be the trivial ring extension of A by K. Then, R is a weakly Bézout ring if and only if A is a weakly Bézout.*

In this part, we give characterization of weakly Bézout rings.

Proposition 2.1. *Let R be a ring. Consider the following properties*:

(1) *For each $a \in Rb$, where b is a nonunit element of R, then Ra is a direct summand of R.*

(2) *For each* $a \in Rb$, *where* b *is a nonunit element of* R, *then* $a \in Ra^2$.
(3) R *is a weakly Bézout ring.*

Then, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *In particular, if* a *is an idempotent element of* R, *then* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let $a \in Rb$, where b is a nonunit element of R and let I be an ideal of R such that

$$(*) \qquad I \oplus Ra = R.$$

We can write $1 = u + v$ for some $u \in I$ and $v \in Ra$. Multiplying the above equality by u (resp., v) we get that $u^2 = u$ (resp., $v^2 = v$). Thus $I = Ru$ and $Ra = Rv$, therefore $a = au + av = av$ by $(*)$, hence $a = a^2x$ for some $x \in R$.

(2) $\Rightarrow$ (3) Let J be a principal ideal generated by a nonunit element b of R, and let I be a finitely generated ideal of R contained in J. It suffices to prove that if $I = (e, f)$, then there exists an element c in R such that $I = Rc$. Since $e \in J = Rb$ then $e \in Re^2$, also $f \in Rf^2$. Let $u = ex$ and $v = fy$, where $e^2x = e$ and $f^2y = f$. Then, the element $c = u + v - uv$ has the required property. Indeed, we have $c = ex + fy - exfy$. So, $ec = e + efy - efy$, hence $e \in Rc$; by symmetry $f \in Rc$.

(3) $\Rightarrow$ (1) Let a be an idempotent element of R and b be a nonunit element of R such that $a \in Rb$. Since b is nonunit we have the containments $Ra \subseteq Rb \subsetneq R$. From the definition of a weakly Bézout ring, we can write $Ra = Rc$ for some element $c \in R$. It follows that $Ra \oplus R(1-c) = R$. $\square$

Now, we turn our attention to the transfer of the Bézout property to amalgamation of rings $A \bowtie^f J$. It is easy to see that, if $J = 0$, then $A \bowtie^f J \cong A$, and so $A \bowtie^f J$ is a Bézout ring if and only if so is A. If $J = B$, then $A \bowtie^f J = A \times B$ is a Bézout ring if and only if so is A and B by [19, Lemma 2.8]. We assume now that J is a nonzero proper ideal of B.

Theorem 2.2. *Let* $f : A \to B$ *be a ring homomorphism and let* J *be a proper ideal of* B.

(1) *Assume that* J *is a finitely generated ideal of* $f(A) + J$. *If* $A \bowtie^f J$ *is Bézout, then so is* A.
(2) *Assume that* $f^{-1}(J)$ *is a finitely generated ideal of* A. *If* $A \bowtie^f J$ *is Bézout, then so is* $f(A) + J$.
In particular, if $f^{-1}(J) = 0$, *then* $A \bowtie^f J$ *is a Bézout ring if and only if* $f(A) + J$ *is a Bézout ring.*

Proof. (1) Assume that J is a finitely generated ideal of $f(A) + J$. By applying condition (1) of [2, Lemma 2.3] we get that $(0) \times J$ is finitely generated of $A \bowtie^f J$, and since $\frac{A \bowtie^f J}{\{0\} \times J} \cong A$ by [8, Proposition 5.1(3)], then A is Bézout by [7, Proposition 2.8].

(2) Assume that $f^{-1}(J)$ is a finitely generated ideal of A. By applying condition (1) of [2, Lemma 2.3] we get that $f^{-1}(J) \times (0)$ is finitely generated

of $A \bowtie^f J$, and since $\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J$ by [8, Proposition 5.1(3)], then $f(A) + J$ is Bézout by [7, Proposition 2.8]. $\square$

Next, we study the weakly Bézout property in the amalgamated algebra along an ideal J with respect to a ring homomorphism $f : A \to B$. Note that J can be regarded as an A-module with the A-module structure naturally induced by f in the following way:

$$r \cdot j = f(r)j.$$

Theorem 2.3. *Let $f : A \to B$ be a ring homomorphism, J a proper ideal of B and $R := A \bowtie^f J$.*

(1) *Suppose that A is a domain and J is a divisible A-module (e.g., $B := qf(A)[X]$ and $J := (X)$). If R is weakly Bézout, then A is weakly Bézout.*
(2) *Let A be a local ring with maximal ideal M and $f(M)J = 0$.*
   (a) *If R is a weakly Bézout ring. Then, A is a weakly Bézout ring.*
   (b) *If A is a weakly Bézout ring and $J^2 = 0$. Then, R is a weakly Bézout ring.*

The proof of this theorem requires the following lemma.

Lemma 2.1. *Let A be a domain. Under the hypothesis of Theorem 2.3, the following statements are equivalent*:

(1) *J is a divisible A-module.*
(2) *$R(a, f(a)) = Aa \bowtie^f J$ for all $a \in A \smallsetminus \{0\}$.*

Proof. (1) $\Rightarrow$ (2) Assume that J is a divisible A-module and let a be a nonzero element in A. Our aim is to show that $R(a, f(a)) = Aa \bowtie^f J$. Indeed, let $\alpha \in A$ and $e \in J$. Then, by divisibility $e = a \cdot j = f(a)j$ for some $j \in J$ and, hence, $(\alpha a, f(\alpha a)+e) = (\alpha, f(\alpha))(a, f(a))+(0, e) = (\alpha, f(\alpha))(a, f(a))+(0, j)(a, f(a))$.

(2) $\Rightarrow$ (1) Assume that $R(a, f(a)) = Aa \bowtie^f J$ for each nonzero element in A. Then, $Aa \bowtie^f aJ = R(a, f(a)) = Aa \bowtie^f J$, so $J = aJ$. $\square$

Proof of Theorem 2.3. (1) Let $I := \sum_{i=1}^{i=n} Aa_i$ with $a_i \in A$ for $i = 1, \dots, n$ and $K := Ab$ be two proper ideals of A such that $I \subseteq K$. Then, $I \bowtie^f J := \sum_{i=1}^{i=n} Aa_i \bowtie^f J = Aa_1 \bowtie^f J + Aa_2 \bowtie^f J + \cdots + Aa_n \bowtie^f J = \sum_{i=1}^{i=n} R(a_i, f(a_i))$ contained in $K \bowtie^f J := Ab \bowtie^f J = R(b, f(b))$, by Lemma 2.1. Therefore, $I \bowtie^f J = R(a, f(a)+k)$ for some element $(a, f(a)+k)$ of R since R is a weakly Bézout ring. Hence, $I = Aa$, and therefore A is a weakly Bézout ring.

(2) (a) Suppose that R is a weakly Bézout ring. Our aim is to show that A is weakly Bézout. Let K be a principal ideal generated by a nonunit element b of A, and let I be a finitely generated ideal of A contained in K. Then, $I \bowtie^f 0 \subseteq K \bowtie^f 0$ are two finitely generated proper ideals of R. Moreover, $K \bowtie^f 0 = Ab \bowtie^f 0 = R(b, f(b))$ and so $I \bowtie^f 0 \subseteq K \bowtie^f 0$ is a principal ideal of R since R is a weakly Bézout ring that is, $I \bowtie^f 0 = R(a, f(a)) = Aa \bowtie^f 0$ for some element a of A. Hence $I = Aa$.

(b) Assume that A is a weakly Bézout ring and $J^2 = 0$. Our aim is to show that R is a weakly Bézout ring. Let $I := \sum_{i=1}^{i=n} R(a_i, f(a_i) + e_i) \subseteq K := R(b, f(b)+l)$ be two proper ideals of R such that n is a positive integer, $a_i, b \in A$ and $e_i, l \in J$ for each $i \in \{1, \ldots, n\}$, we wish to show that I is principal. Three cases are then possible.
Case 1. Let $b = 0$. Then, $a_i = 0$ for all $i = 1, \ldots, n$; $I := 0 \bowtie^f J_1$ and $K := 0 \bowtie^f J_2$ where J_1 (resp., J_2) is a vector subspace of J generated by the vectors $e_1, \ldots, e_n$ (resp., l). Hence, J_1 is a (A/M)-vector space of rank at most 1 (since $J_1 \subseteq J_2 = (A/M)l$) that is, $J_1 = (A/M)h$, where $h \in J_1$. Therefore, $I := 0 \bowtie^f (A/M)h = R(0, h)$ and so I is a principal ideal of R.
Case 2. Let $b \neq 0$ and $a_i = 0$ for all $i \in \{1, \ldots, n\}$. In this case, $I := 0 \bowtie^f J_1$ and so principal since $K := R(b, f(b) + l) \subseteq Ab \bowtie^f (A/M)l$.
Case 3. Let $b \neq 0$ and $a_i \neq 0$ for some $i \in \{1, \ldots, n\}$. We assume that $((a_i, f(a_i) + e_i)_{i=1}^n)$ is a minimal generating set of I, $I_0 := \sum_{i=1}^{i=n} Aa_i$ and $K_0 := Ab$. Consider the exact sequence of R-modules:

$$0 \to \mathrm{Ker}(u) \to R^n \cong A^n \bowtie^{f^n} J^n \xrightarrow{u} I \to 0,$$

where $u((c_i, f(c_i)+g_i)_{i=1}^n) = \sum_{i=1}^n (c_i, f(c_i)+g_i)(a_i, f(a_i)+e_i)$. But, $\mathrm{Ker}(u) \subseteq (M \bowtie^f J)^n$ by [24, Lemma 4.43, page 134] since R is local by [22, Lemma 2.2]. Hence,

$$\begin{aligned}
&\mathrm{Ker}(u)\\
&= \{((c_i, f(c_i) + g_i))_{i=1}^n \in R^n : \sum_{i=1}^n (c_i, f(c_i) + g_i)(a_i, f(a_i) + e_i) = 0\}\\
&= \{((c_i, f(c_i) + g_i))_{i=1}^n \in R^n : \sum_{i=1}^n c_i a_i = 0\} \quad (\text{since } f(M)J = 0 \text{ and } J^2 = 0)\\
&= V \bowtie^{f^n} J^n,
\end{aligned}$$

where $V := \{(c)_{i=1}^n \in A^n : \sum_{i=1}^n c_i a_i = 0\}$. Also, we have the exact sequence of R-modules:

$$0 \to Ker(w) \to R^n \xrightarrow{w} I_0 \bowtie^f 0 \to 0,$$

where $w((\alpha_i, f(\alpha_i) + k_i)) = \sum_{i=1}^n (\alpha_i, f(\alpha_i) + k_i)(a_i, f(a_i))$. But, $\mathrm{Ker}(w) = \{((\alpha_i, f(\alpha_i) + k_i))_{i=1}^n \in R^n : \sum_{i=1}^n \alpha_i a_i = 0\} = V \bowtie^{f^n} J^n$. Therefore, $I \cong I_0 \bowtie^f 0$ and since $I_0 \subseteq K_0$ (because $I \subseteq K$), then $I_0 = Aa$ for some element $a \in A$ since A is a weakly Bézout ring and so $I_0 \bowtie^f 0 = Aa \bowtie^f 0 = R(a, f(a))$. Hence, I is a principal ideal of R in all cases. So, R is a weakly Bézout ring. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.2. *Let A be a local ring with maximal ideal M, I and J be two proper ideals of A. Let $B = A/I$ and $f : A \to B$ be the canonical homomorphism ($f(x) = \bar{x}$) such that $M\bar{J} = 0$. Then, $A \bowtie^f J$ is weakly Bézout if and only if A is weakly Bézout.*

One may use Theorem 2.2 and Theorem 2.3 to enrich the literature with new examples of weakly Bézout rings which are not Bézout, as shown below.

Example 2.1. Let A be a local Bézout ring with maximal ideal M (e.g., $A := \mathbb{Z}_p$ and $M := p\mathbb{Z}_p$, where p is a prime ideal of $\mathbb{Z}$), $B := A \ltimes (A/M)^2$ and $J := 0 \ltimes (A/M)^2$. Consider the natural injective ring homomorphism $f : A \hookrightarrow B$. Then, the amalgamation $A \bowtie^f J$ is a non-Bézout weakly Bézout ring.

Proof. Notice first that $f(M)J = 0$ and $f^{-1}(J) = 0$. Further, B is weakly Bézout by [13, Theorem 2.4 (1)]. So, R is weakly Bézout by Theorem 2.3(2)(b) as $J^2 = 0$. However, R is not Bézout by Theorem 2.2(2) since $f(A) + J = B$ which is not Bézout by [13, Theorem 2.4(2)]. $\square$

Proposition 2.2. *Let $f : A \to B$ be a ring homomorphism, J a proper nonzero ideal of B and $R := A \bowtie^f J$. If A contains a non-invertible regular element (that is, A is not a total ring of quotients) such that $f(a)J = 0$ and $J^2 = 0$. Then, R is never a Bézout ring.*

Proof. Suppose the result is false, i.e., R is a Bézout ring. Let $0 \neq e \in J$, and consider the elements $(a, f(a))$ and $(0, e)$ of R. Then, the ideal $K := R(a, f(a)) + R(0, e)$ of R is principal. Set $K = (b, f(b) + j)$ for some $b \in A$ and $j \in J$. Hence, $Aa = Ab$. Then, there exists a nonzero element $\alpha \in A$ such that $a = \alpha b$, thus b is a regular element of A since a is regular, and so $b = ua$ for some invertible element u of A. Therefore, $R(b, f(b) + j) = R(ua, f(u)f(a) + j) = R(u, f(u))(a, f(a) + f(u^{-1})j) = R(a, f(a) + f(u^{-1})j)$. Then, $K := R(a, f(a) + f(u^{-1})j)$. On the other hand $(a, f(a)) \in K$, so there exists $(c, f(c)+l) \in R$ such that $(a, f(a)) = (c, f(c) + l)(a, f(a) + f(u^{-1})j) = (ca, f(ca) + f(cu^{-1})j)$ since $f(a)J = 0$ and $J^2 = 0$. Hence, $ac = a$ and $f(ca)+f(cu^{-1})j = f(a)$. Thus $c = 1$, $j = 0$; and so $R(a, f(a)) + R(0, e) = K = R(b, f(b)) = R(a, f(a)) = Aa \bowtie^f 0$. Therefore, $(0, e) \in Aa \bowtie^f 0$, which is a contradiction since $0 \neq e$. It follows that R is not a Bézout ring. $\square$

By Theorem 2.2 and Proposition 2.2, we have the following example.

Example 2.2. Let A be a local Bézout ring with maximal ideal M (e.g., $A := K[[X]]$ denote the ring of formal power series over the field K in an indeterminate X and $M := (X)$), $B := A/(X^2)$ and $J := (X)/(X^2)$. Consider the canonical ring homomorphism $f : A \to B$ ($f(x) = \bar{x}$). Then, the amalgamation $A \bowtie^f J$ is a non-Bézout weakly Bézout ring. Indeed, it is easily seen that $J^2 = 0$. So, $A \bowtie^f J$ is weakly Bézout by Theorem 2.3(2)(b) since $f(M)J = 0$, but not Bézout by Proposition 2.2 since $f(X)J = 0$.

Finally, in the context of domains we show that the amalgamated is a Bézout ring if and only it is a weakly Bézout ring. Thus, one may provide new examples of weakly Bézout rings.

Theorem 2.4. *Let A and B be a pair of domains, $f : A \to B$ a ring homomorphism and let J be an ideal of B and divisible A-module.*

(1) *Assume that f is injective. Then the following statements are equivalent:*
   (i) *$A \bowtie^f J$ is a Bézout ring.*
   (ii) *$A \bowtie^f J$ is a weakly Bézout ring.*
   (iii) *One of the following conditions holds:*
      - *$J = B$, A and B are weakly Bézout rings.*
      - *$J \neq B$, $f(A) \cap J = 0$ and $f(A) + J$ is a weakly Bézout ring.*

(2) *Assume that f is not injective. Then the following statements are equivalent:*
   (i) *$A \bowtie^f J$ is a Bézout ring.*
   (ii) *$A \bowtie^f J$ is a weakly Bézout ring.*
   (iii) *$J = 0$, and A is a weakly Bézout ring.*

The proof of this theorem needs the following lemma.

Lemma 2.2. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B and divisible A-module.*

(1) *Assume that $f(Reg(A)) \subseteq Reg(B)$. If J is proper, then $f(Reg(A)) \cap J = 0$.*
(2) *Assume that A is domain. If f is not injective, then $J = 0$.*

Proof. (1) Suppose that $f(Reg(A)) \cap J \neq 0$. Then, there exists $a \in Reg(A)$ such that $f(a) \in J \setminus \{0\}$, and since J is a divisible A-module, then $f(a) = a \cdot e$ for some $e \in J$ and hence $J = B$ since $f(a) \in Reg(B)$.

(2) Let $e \in J$. Since f is not injective there exists $a \in \mathrm{Ker}(f) \setminus \{0\}$ such that $f(a) = 0$. On the other hand, J is a divisible A-module, then there exists $j \in J$ such that $e = a \cdot j = f(a)j = 0$. Hence, $J = 0$. $\square$

Proof of Theorem 2.4. (1) (ii)$\Rightarrow$(iii) Assume that $J \neq B$. By using the condition (1) of Lemma 2.2 , we get that $f(A) \cap J = 0$. Then, the natural projection $p_B : A \bowtie^f J \to f(A) + J$ ($p_B(a, f(a) + j) = f(a) + j$)) is a ring isomorphism. Indeed, $f(a) + j = 0$ implies that $f(a) = -j$, which is in J. Hence $f(a)$ belongs $f(A)$ intersection $J = 0$. Consequently, $f(a) = j = 0$, which tends by the injectivity of f to $a = 0$. The conclusion is now straightforward.

(iii)$\Rightarrow$(i) If $J = B$, then A and B are Bézout rings since every weakly Bézout domain is a Bézout ring by [13, Proposition 2.2]. Hence $A \bowtie^f J = A \times B$ is a Bézout ring [19, Lemma 2.8(1)]. Now we assume that $J \neq B$. Then $A \bowtie^f J = f(A) + J$ and so $A \bowtie^f J$ is a weakly Bézout domain. This completes the proof of Theorem 2.4.

(2) It is an immediate consequence of Lemma 2.2(2) and [13, Proposition 2.2]. $\square$

It is worth to mention that in case $A = B$, $J = I$ is a nonzero ideal of A, and f is the identity homomorphism on A, the weakly Bézout ring property on $A \bowtie I$ forces I to be the non-proper as it is shown by the following corollary.

Corollary 2.3. *Let A be a domain and let I be a nonzero ideal of A. Then $A \bowtie I$ is a weakly Bézout ring if and only if so is A and $I = A$.*

Example 2.3. Let $T := \mathbb{R}[X]_{(X)} = \mathbb{R} + M$, where X is an indeterminate over $\mathbb{R}$ and $M := XT$ the maximal ideal of T. Then, $\mathbb{R} \bowtie^i M$, where i is the inclusion map of $\mathbb{R}$ into T; is a weakly Bézout ring by Theorem 2.4(1).

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