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$*$-Ricci Soliton on $(\kappa < 0, \mu)$-almost Cosymplectic Manifolds

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Abstract. We study $*$-Ricci solitons on non-cosymplectic (κ, μ)-acs (almost cosymplectic) manifolds M. We find $*$-solitons that are steady, and such that both the scalar curvature and the divergence of the potential field is negative. Further, we study concurrent, concircular, torse forming and torqued vector fields on M admitting Ricci and $*$-Ricci solitons. Also, we provide some examples.

1. Introduction

In the framework of Riemannian geometry, Blair et al. [2] introduced a (κ, μ)-space in contact geometry as contact manifold M whose curvature tensor R satisfies

$$R(U, W)\xi = \kappa(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW), \tag{1.1}$$

for any U and $W \in TM$ and for h a symmetric operator given by $h = \frac{1}{2}\mathcal{L}_\xi \psi$, where ψ is a $(1, 1)$ tensor field and κ, μ are constants. If $\kappa = 1$ and $h = 0$, then (κ, μ)-spaces reduces to the Sasakian manifolds. Non-Sasakian manifolds have proven to be more interesting in this context. The unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure is an example of non-Sasakian spaces of this type. Moreover, this type of spaces is invariant under D-homothetic transformations. These factors drive the study of this type of manifold. Boeckx proved that a non-Sasakian contact metric manifold satisfying (1.1) is completely determined locally by its dimension for the constant values of κ, μ [3].

If κ, μ are functions, then a contact metric manifold satisfying (1.1) is called a generalized (κ, μ)-space [16]. Koufogiorgos et al. introduced a (κ, μ, ν)-contact metric manifold which satisfies [17]

$$\begin{aligned} R(U, W)\xi = {} & \kappa(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW) \\ & + \nu(\eta(W)\psi hU - \eta(U)\psi hW), \end{aligned} \tag{1.2}$$

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for smooth functions κ, μ, ν on M^{2n+1} and they proved that it reduces to a (κ,μ)-manifold in dimension $2n+1 \geq 5$. Later, the *generalized* (κ,μ)*-space with divided* R_5 was introduced in [5]. This further generalizes generalized (κ,μ)-spaces. Here $R_5 = R_{5,1} - R_{5,2}$ is divided into $R_{5,1}$ and $R_{5,2}$ such that

$$\begin{aligned} R_{5,1}(U,W)X &= g(hW,X)hU - g(hU,X)hW, \\ R_{5,2}(U,W)X &= g(\psi hW,X)\psi hU - g(\psi hU,X)\psi hW, \end{aligned}$$

for any U, W, $X \in TM$. Sharma and Vrancken [19] studied (κ,μ)-contact manifolds with non-Killing conformal vector fields. Chen [7] examined a closed Einstein-Weyl structure and two Einstein-Weyl structures on an acs $(\kappa < 0, \mu)$-manifold. Ghosh and Sharma [12] investigated a (κ,μ)-contact manifold with a divergence free Cotton tensor. De and Sardar [10], studied Bach-flat (κ,μ)-almost co-Kähler manifolds.

In the last few decades there has been extensive study about Ricci solitons and $*$-Ricci solitons on manifolds. The notion of Ricci solitons was introduced by Hamilton as a natural generalization of Einstein metrics.

A Ricci soliton on a Riemannian manifold satisfies the following equation [8]

$$Ric + \frac{1}{2}\mathcal{L}_Y g = \rho g, \tag{1.3}$$

where $\mathcal{L}_Y$ is the Lie-derivative along a smooth potential field Y, g is the Riemannian metric, ρ is a real scalar and *Ric* is the Ricci tensor. Ricci solitons serve as solutions to the Ricci flow of Hamilton [14], which evolve along the symmetries of the flow. The soliton is steady, expanding or shrinking if $\rho = 0$, < 0 or > 0, respectively.

In 1959, Tachibana [21] introduced the notion of $*$-Ricci tensors on almost Hermitian manifolds. Later, Hamada [13] defined $*$-Ricci tensors of real hypersurfaces in non-flat complex space forms, and then Kaimakamis et al. [15] introduced the notion of $*$-Ricci solitons in non-flat complex space forms.

The $*$-Ricci tensor on an almost contact metric (a.c.m) manifold M ([13]) is defined by

$$Ric^*(U,W) = \frac{1}{2}\,\mathrm{trace}(X \mapsto R(U,\psi W)\psi X),\ \forall\, U, W \in TM, \tag{1.4}$$

where ψ is a $(1,1)$-tensor field and R is a Riemann curvature tensor.

A $*$-Ricci soliton on a Riemannian manifold (M,g) is a generalisation of the $*$-Einstein manifold and defined as [15]:

$$Ric^* + \frac{1}{2}\mathcal{L}_Y g = \rho g. \tag{1.5}$$

Many authors have studied solitons on a.c.m manifolds: Sharma initiated the study of Ricci solitons in contact geometry as a K-contact and (κ,μ)-contact metric [18]. Suh et al. studied Ricci solitons on almost co-Kähler manifolds [20]. Dai [9] investigated $*$-Ricci soliton on a $(\kappa < 0, \mu)$-acs manifold and proved that there do not exist $*$-Ricci soliton on a $(\kappa < 0, \mu)$-acs manifold.

In view of the above, we study the existence of $*$-Ricci solitons on a non-cosymplectic (κ, μ)-acs manifold. Contrary to the non-existence of $*$-Ricci soliton in [9] we show that there exists a steady $*$-Ricci soliton. Further, we study the existence/non-existence of some particular type of potential vector field Y on a non-cosymplectic (κ, μ)-acs manifold admitting Ricci solitons and $*$-Ricci solitons.

This paper is organised as follows: in Section 2, we give some background which is necessary to understand the subsequent sections. In Section 3, we study $*$-Ricci solitons on non-cosymplectic (κ, μ)-almost cosymplectic manifolds. Section 4 deals with the study of a potential vector field Y as concurrent, concircular, torse forming and torqued vector field. In Section 5, we give some examples.

2. Preliminaries

A smooth Riemannian manifold M^{2n+1} is called an a.c.m manifold if there exists structure tensors (ψ, ξ, η, g) satisfying [1]

$$\psi^2 = -id + \eta \otimes \xi, \eta \; o \; \psi = 0, \psi\xi = 0, \eta(\xi) = 1, \tag{2.1}$$

$$g(\psi U, \psi W) = g(U, W) - \eta(U)\eta(W), \tag{2.2}$$

for any U, $W \in TM$, where ψ is a tensor field of type $(1, 1)$, ξ a global vector field and η a 1-form. We denote by Φ the fundamental 2-form which is defined as $\Phi(U, W) = g(U, \psi W)$. An a.c.m manifold M^{2n+1} with $d\eta = \Phi$ is called contact manifold. An a.c.m manifold with η and Φ closed is called an almost cosymplectic manifold. A normal almost cosymplectic manifold is called cosymplectic manifold.

On an acs manifold, we have [4]

$$h = \frac{1}{2}\mathcal{L}_\xi \psi, \; h' = h \, o \, \psi, \tag{2.3}$$

$$h\xi = 0, \; h\psi + \psi h = 0, \; trh = trh' = 0, \tag{2.4}$$

$$\psi l \psi - l = 2h^2, \tag{2.5}$$

$$\nabla_U \xi = h'U, \tag{2.6}$$

where $l = R(., \xi)\xi$ and both h, h' are symmetric operators with respect to metric g.

Also, on an acs manifold, we have [11]

$$R(U, W)\xi = (\nabla_W \psi h)U - (\nabla_U \psi h)W, \tag{2.7}$$

for any U, $W \in TM$.

The (κ, μ)-nullity distribution of an acs manifold M^{2n+1} for $(\kappa, \mu) \in R^2$ is a distribution [11]

$$\begin{aligned} N(\kappa, \mu) : p \rightarrow N_p(\kappa, \mu) \;\; = \;\; & \{Z \in T_p(M) | R(U, W)X = \kappa(g(W, X)U \\ & -g(U, X)W) + \mu(g(W, X)hU - g(U, X)hW)\}. \end{aligned}$$

Endo [11] introduced (κ, μ)-acs manifolds with the Reeb vector ξ in the (κ, μ)-nullity distribution, which satisfy (1.1). If ξ is in the $(\kappa \neq 0, \mu)$-nullity distribution, then such manifolds are called non-cosymplectic (κ, μ)-acs manifolds. For more work about non-cosymplectic (κ, μ)-acs manifolds, please see [11].

On $(\kappa < 0, \mu)$-acs manifold using (1.1) and (2.7), we have

$$(\nabla_W \psi h)U - (\nabla_U \psi h)W = \kappa(g(W,X)U - g(U,X)W) + \mu(g(W,X)hU - g(U,X)hW). \tag{2.8}$$

Also, on a Riemannian manifold M we have following [23]:

$$(\nabla_X \mathcal{L}_Y g)(U,W) = g((\mathcal{L}_Y \nabla)(X,U),W) + g((\mathcal{L}_Y \nabla)(X,W),U), \tag{2.9}$$

$$(\mathcal{L}_Y R)(U,W)X = (\nabla_U \mathcal{L}_Y \nabla)(W,X) - (\nabla_W \mathcal{L}_Y \nabla)(U,X), \tag{2.10}$$

$\forall\ U,\ W,\ X \in TM$.

3. $*$-Ricci Solitons on $(\kappa < 0, \mu)$-acs Manifolds

Let M^{2n+1} be a (κ, μ)-acs manifold. Then, from (1.1), we obtain

$$l = -\kappa\psi^2 + \mu h. \tag{3.1}$$

Using (3.1) in (2.5), we get

$$h^2 = \kappa\psi^2. \tag{3.2}$$

Let U be an eigenvector of h for eigenvalue θ with $U \perp \xi$, then using (2.3), (2.4) and (3.2), we get

$$\theta^2 = -\kappa. \tag{3.3}$$

From (3.3), we find that $\kappa \leq 0$. However, $\kappa = 0$ if and only if $h = 0$. Here, we study non-cosymplectic $(\kappa < 0, \mu)$-acs manifolds.

On an acs manifold $M^{2n+1}(\kappa < 0, \mu)$, we have([4, 9])

$$Q = \mu h + 2n\kappa\eta \otimes \xi, \tag{3.4}$$

$$Ric^*(U,W) = -\kappa g(\psi U, \psi W), \tag{3.5}$$

where Q denotes Ricci operator and $U,\ W \in TM$.

Lemma 3.1. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.5), then*

$$\begin{aligned}(\mathcal{L}_Y R)(U,W)X &= 2\kappa^2(\eta(U)g(W,X) - \eta(W)g(U,X))\xi \\ &+ 2\kappa\mu(\eta(U)g(hW,X) - \eta(W)g(hU,X))\xi \\ &+ 2\kappa(g(h'U,X)h'W - g(h'W,X)h'U),\end{aligned} \tag{3.6}$$

for any $U, W, X \in TM$.

Proof. Using (3.5) in (1.5), we obtain

$$(\mathcal{L}_Y g)(W, X) = 2\rho g(W, X) + 2\kappa g(\psi W, \psi X). \tag{3.7}$$

Differentiating (3.7) with respect to U on M and using (2.6), we find

$$(\nabla_U \mathcal{L}_Y g)(W, X) = -2\kappa\Big(g(h'U, W)\eta(X) + g(h'U, X)\eta(W)\Big). \tag{3.8}$$

Using (3.8) in (2.9), we obtain

$$\begin{aligned} g((\mathcal{L}_Y \nabla)(U, W), X) + g((\mathcal{L}_Y \nabla)(U, X), W) = -2\kappa(g(h'U, W)\eta(X) \\ +g(h'U, X)\eta(W)). \end{aligned} \tag{3.9}$$

Similarly, we get

$$\begin{aligned} g((\mathcal{L}_Y \nabla)(W, X), U) + g((\mathcal{L}_Y \nabla)(W, U), X) = -2\kappa(g(h'W, X)\eta(U) \\ +g(h'W, U)\eta(X)), \end{aligned} \tag{3.10}$$

$$\begin{aligned} g((\mathcal{L}_Y \nabla)(X, U), W) + g((\mathcal{L}_Y \nabla)(X, W), U) = -2\kappa(g(h'X, U)\eta(W) \\ +g(h'X, W)\eta(U)). \end{aligned} \tag{3.11}$$

Adding (3.9) and (3.10) and subtracting (3.11), we get

$$g((\mathcal{L}_Y \nabla)(U, W), X) = -2\kappa g(h'U, W)\eta(X),$$

which gives

$$(\mathcal{L}_Y \nabla)(W, X) = -2\kappa g(h'W, X)\xi. \tag{3.12}$$

Differentiating (3.12) along U on M, we find

$$(\nabla_U \mathcal{L}_Y \nabla)(W, X) = -2\kappa\Big(g((\nabla_U h')W, X)\xi + g(h'W, X)h'U\Big). \tag{3.13}$$

Using (3.13) in (2.10), we obtain

$$\begin{aligned} (\mathcal{L}_Y R)(U, W)X \quad = \quad & -2\kappa\Big(g((\nabla_W \psi h)U, X)\xi - g(\psi hU, X)\psi hW\Big) \\ & +2\kappa\Big(g((\nabla_U \psi h)W, X)\xi - g(\psi hW, X)\psi hU\Big). \end{aligned} \tag{3.14}$$

Using (2.8) in (3.14), we get (3.6). Hence, the proof of Lemma is complete. □

Theorem 3.2. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.5). Then $*$-soliton is steady.*

Proof. Let $\{e_i\}_{i=0}^{2n}$ be a local orthonormal basis for TM. From (3.6), we obtain

$$\begin{aligned} g((\mathcal{L}_Y R)(U, W)X, U) = 2\kappa^2\Big(\eta(U)g(W, X) - \eta(W)g(U, X)\Big)\eta(U) \\ +2\kappa\mu\Big(\eta(U)g(hW, X) - \eta(W)g(hU, X)\Big)\eta(U) \\ +2\kappa\Big(g(h'U, X)g(h'W, U) - g(h'W, X)g(h'U, U)\Big), \end{aligned} \tag{3.15}$$

for $U, W, X \in TM$.

Contracting (3.15) over U, we get

$$(\mathcal{L}_Y S)(W, X) = 2\kappa^2 g(\psi W, \psi X) + 2\kappa\mu g(hW, X) + 2\kappa g(h'X, h'W),$$

which gives

$$(\mathcal{L}_Y g)(QW, X) + g((\mathcal{L}_Y Q)W, X) = 2\kappa^2 g(\psi W, \psi X) + 2\kappa\mu g(hW, X) + 2\kappa g(\psi hX, \psi hW). \tag{3.16}$$

Using (3.7) in (3.16), we obtain

$$\begin{aligned} 2(\kappa + \rho)g(QW, X) - 2\kappa\eta(QW)\eta(X) + g((\mathcal{L}_Y Q)W, X) \\ = 2\kappa^2 g(\psi W, \psi X) + 2\kappa\mu g(hW, X) + 2\kappa g(hW, hX). \end{aligned} \tag{3.17}$$

Using (3.2) and (3.4) in (3.17), we find

$$(\mathcal{L}_Y Q)W = -2\mu\rho hW - 4n\kappa\rho\eta(W)\xi. \tag{3.18}$$

From (3.4), we get

$$QW = \mu hW + 2n\kappa\eta(W)\xi. \tag{3.19}$$

Lie-derivative of (3.19) along Y gives

$$(\mathcal{L}_Y Q)W = \mu(\mathcal{L}_Y h)W + 2n\kappa g(\nabla_W Y, \xi)\xi + 2n\kappa g(W, h'Y)\xi + 2n\kappa\eta(W)\mathcal{L}_Y\xi. \tag{3.20}$$

Comparing (3.18) and (3.20), we obtain

$$\begin{aligned} -2\mu\rho hW - 4n\kappa\rho\eta(W)\xi = \mu(\mathcal{L}_Y h)W + 2n\kappa g(\nabla_W Y, \xi)\xi \\ +2n\kappa g(W, h'Y)\xi + 2n\kappa\eta(W)\mathcal{L}_Y\xi. \end{aligned} \tag{3.21}$$

Putting $W = \xi$ in (3.21), we get

$$-4n\kappa\rho\xi = -\mu h\mathcal{L}_Y\xi + 2n\kappa g(\nabla_\xi Y, \xi)\xi + 2n\kappa\mathcal{L}_Y\xi. \tag{3.22}$$

Taking inner product of (3.22) with ξ, we get

$$\kappa\rho = 0,$$

which implies $\rho = 0$ as $\kappa < 0$. Hence the proof of the Theorem. □

Using (1.5), (2.4), (3.4), (3.5), and Theorem 3.2, we get $r = 2\kappa n$, and $div\, Y = 2\kappa n$, where r and div denote the scalar curvature and divergence, respectively. Therefore, we have

Corollary 3.3. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying* (1.5)*, then $div\, Y = r < 0$.*

Corollary 3.4. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying* (1.5)*, then Y cannot be ξ.*

Proof. Suppose $Y = \xi$. Then, putting potential vector field $Y = \xi$ in (1.5) and using (2.6), (3.5) and Theorem 3.2, we get

$$g(h'U, W) = \kappa g(\psi U, \psi W) \tag{3.23}$$

Taking trace of (3.23) and using (2.4), we obtain $2\kappa n = 0$, which is not possible as $\kappa < 0$. Hence the result. □

Remark 3.5. In [9] the author proved that there do not exist $*$-Ricci soliton on non-cosymplectic (κ, μ)-acs manifolds. Unfortunately, there is a crucial error in their proofs. In page 4 of [9], the equation (3.5) should be corrected as

$$\begin{aligned} (\mathcal{L}_V R)(X, Y)Z &= 2\kappa^2(\eta(X)g(Y, Z) - \eta(Y)g(X, Z))\xi \\ &+ 2\kappa\mu(\eta(X)g(hY, Z) - \eta(Y)g(hX, Z))\xi \\ &+ 2\kappa(g(h'X, Z)h'Y - g(h'Y, Z)h'X). \end{aligned}$$

In page 5 of [9], the equation (3.12) should be corrected as

$$(\mathcal{L}_V Q)X = -2\mu\lambda hX - 4n\kappa\lambda\eta(X)\xi.$$

Thereafter, the argument given in [9] is not useful to obtain correct result.

4. $*$-Ricci Solitons on $(\kappa < 0, \mu)$-acs Manifolds with Some Particular Potential Vector Fields

In 1944, Yano [22] introduced a torse-forming vector field as a generalization of concircular, concurrent and parallel vector fields.

Definition 4.1. A vector field $\mathcal{V}$ is called torse forming if

$$\nabla_U \mathcal{V} = fU + \omega(U)\mathcal{V}, \tag{4.1}$$

where $f \in C^\infty(M)$, $U \in TM$ and ω is a 1-form.

The vector field $\mathcal{V}$ is called concircular if ω in (4.1) vanishes identically. Concircular vector field is also known as geodesic vector field as its integral curve forms geodesics. It has interesting applications in general relativity and in the theory of conformal and projective transformation.

If $\mathcal{V}$ satisfies

$$\nabla_U \mathcal{V} = U, \ \forall U \in TM, \tag{4.2}$$

then $\mathcal{V}$ is called concurrent. If $\mathcal{V}$ satisfies (4.1) with $f = 0$, then $\mathcal{V}$ is called recurrent. Also, if $f = \omega = 0$ in (4.1), then $\mathcal{V}$ is called parallel.

Recently, in 2017, Chen [6] introduced a torqued vector field. If a non-vanishing $\mathcal{V}$ satisfies (4.1) with $\omega(\mathcal{V}) = 0$, then $\mathcal{V}$ is called torqued, f the torqued function and ω torqued form of $\mathcal{V}$. For more details about these vector fields (please see [6, 22]) and references therein.

Now, we have

Theorem 4.2. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.3). Then the potential vector field Y cannot be concurrent.*

Proof. Suppose Y be a concurrent field, then using (3.4) and (4.2) in (1.3), we have

$$\mu g(hU, W) + 2n\kappa\eta(U)\eta(W) = (\rho - 1)g(U, W). \tag{4.3}$$

Contracting (4.3) over U and W, we obtain

$$2n\kappa = (\rho - 1)(2n + 1). \tag{4.4}$$

From (4.4), we find that

$$\kappa = \frac{(\rho - 1)(2n + 1)}{2n}. \tag{4.5}$$

Using $\rho = 2n\kappa$ (cf. [20], Theorem 5.1) in (4.5), we get $\kappa = \frac{2n+1}{4n^2} > 0$, a contradiction as $\kappa < 0$. Thus proof is complete. □

Theorem 4.3. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.5), then the potential vector field Y cannot be concurrent.*

Proof. Suppose Y be a concurrent field. Using (3.5), (4.2) and Theorem 3.2 in (1.5), we get

$$(1 - \kappa)g(U, W) + \kappa\eta(U)\eta(W) = 0. \tag{4.6}$$

Contracting (4.6) over U and W, we obtain $\kappa = \frac{2n+1}{2n} > 0$, which is not possible as $\kappa < 0$. Hence the result. □

Theorem 4.4. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.5). If Y be a torse forming vector field, then*

$$f + \frac{\omega(Y)}{2n + 1} = \frac{2n\kappa}{2n + 1}, \tag{4.7}$$

where $f \in C^\infty(M)$ satisfying (4.1).

Proof. Let Y be a torse forming field, then using (3.5), (4.1) and Theorem 3.2 in (1.5), we get

$$\omega(U)g(Y, W) + \omega(W)g(Y, U) + (2f - 2\kappa)g(U, W) = -2\kappa\eta(U)\eta(W). \tag{4.8}$$

Putting $U = e_i$ and $W = e_i$ and tracing $i = 1$ to $2n + 1$, we obtain (4.7). □

Corollary 4.5. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.5). If Y be a torqued vector field, then torqued function f is a constant and ξ cannot be torqued.*

Proof. Suppose Y be a torqued field then $\omega(Y) = 0$. Using this condition in (4.7) we obtain $f = \frac{2n\kappa}{2n+1}$. Whereby, we get f is a constant. As $\omega(\xi) \neq 0$ so ξ cannot be torqued. □

Theorem 4.6. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.5). If Y be a concircular vector field, then f is a constant given by*

$$f = \frac{2n\kappa}{2n+1}. \tag{4.9}$$

Proof. Let Y be a concircular field, then

$$\nabla_U Y = fU, f \in C^\infty(M). \tag{4.10}$$

Using (3.5), (4.10) and Theorem 3.2 in (1.4), we get

$$(f - \kappa)g(U, W) + \kappa\eta(U)\eta(W) = 0. \tag{4.11}$$

Contracting (4.11) over U and W, we obtain (4.9). □

Theorem 4.7. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.3). If Y be a torse forming vector field, then*

$$\rho = f + \frac{\omega(Y)}{2n+1} + \frac{2n\kappa}{2n+1}. \tag{4.12}$$

Proof. Let Y be a torse forming field. Using (4.1) and (3.4) in (1.3), we get

$$\begin{aligned}\omega(U)g(Y, W) + \omega(W)g(Y, U) + 2\mu g(hU, W) + 4n\kappa\eta(U)\eta(W)& \\ = 2(\rho - f)g(U, W).&\end{aligned} \tag{4.13}$$

Taking trace of (4.13) over U and W, we obtain (4.12). □

Theorem 4.8. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.3). If Y be a concircular vector field, then f is a constant given by*

$$f = \rho - \frac{2n\kappa}{2n+1}. \tag{4.14}$$

Proof. Let Y be a concircular field. Using (3.4) and (4.10) in (1.3), we get

$$(f - \rho)g(U, W) + \mu g(hU, W) + 2n\kappa\eta(U)\eta(W) = 0. \tag{4.15}$$

Contracting (4.15) over U and W, we obtain (4.14). □

Corollary 4.9. *Let $M^{2n+1}(\kappa < 0, \mu)$ be an acs manifold satisfying (1.3). If Y be a torqued vector field, then torqued function f is a constant.*

Proof. Suppose Y be a torqued field then $\omega(Y) = 0$. Using this condition in (4.12) we obtain $f = \rho - \frac{2n\kappa}{2n+1}$. Hence f is a constant. □

5. Examples of $*$-Ricci Soliton on $(\kappa < 0, \mu)$-acs Manifolds

In the following examples, the (1,1)-tensor ψ is defined as

$$\psi(e_1) = e_2,\ \psi(e_2) = -e_1,\ \psi(e_3) = 0.$$

Example 5.1. Consider $M = \{(x, y, z) \in R^3 : x \neq 0\}$ with structure tensors

$$\begin{cases} g = dx \otimes dx + dy \otimes dy - xdy \otimes dz - xdz \otimes dy + (x^2 + 1)dz \otimes dz, \\ e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_3 = \xi = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y},\ \eta = dz, \\ h = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{x}{2} \\ 0 & 0 & 0 \end{pmatrix}. \end{cases} \tag{5.1}$$

From (5.1), we have

$$[e_q, e_2] = 0,\ q = 1, 3;\ [e_1, e_3] = e_2. \tag{5.2}$$

The Koszul's formula with Riemannian connection ∇ is given by

$$\begin{aligned} 2g(\nabla_U W, X) = {} & Ug(W, X) + Wg(X, U) - Xg(U, W) - g(U, [W, X]) \\ & - g(W, [U, X]) + g(X, [U, W]), \end{aligned} \tag{5.3}$$

$\forall\ U,\ W,\ X \in TM$.

From (5.2) and (5.3), we find

$$\begin{cases} \nabla_{e_q} e_q = 0,\ q = 1, 2, 3;\ \nabla_{e_2} e_1 = -\frac{1}{2}e_3 = \nabla_{e_1} e_2, \\ \nabla_{e_3} e_1 = -\frac{1}{2}e_2 = -\nabla_{e_1} e_3,\ \nabla_{e_3} e_2 = \nabla_{e_2} e_3 = \frac{1}{2}e_1. \end{cases} \tag{5.4}$$

Computing Riemann curvature tensors using (5.4), we get

$$R(e_2, e_1)\xi = 0,\ R(e_2, \xi)\xi = \frac{1}{4}e_2,\ R(e_1, \xi)\xi = -\frac{3}{4}e_1. \tag{5.5}$$

Further, from (1.1), we obtain

$$R(e_2, e_1)\xi = 0,\ R(e_q, \xi)\xi = \kappa e_q + \mu h e_q,\ q = 1, 2. \tag{5.6}$$

From (2.4) and (3.3), we find that h on M satisfies

$$he_1 = -\sqrt{-\kappa}e_1,\ he_2 = \sqrt{-\kappa}e_2,\ he_3 = 0. \tag{5.7}$$

Using (5.1), (5.5), (5.6) and (5.7), we find that M is a $(\kappa < 0, \mu)$-acs manifold with $\kappa = \frac{-1}{4}, \mu = 1$.

Further, from (3.5), we obtain

$$\begin{cases} S^*(e_i, e_i) = \frac{1}{4}, i = 1, 2;\ S^*(e_3, e_3) = 0, \\ S^*(e_q, e_p) = 0,\ q \neq p;\ q, p = 1, 2, 3. \end{cases} \tag{5.8}$$

Also, we have

$$Y = (z - \frac{x}{4})\frac{\partial}{\partial x} + (\frac{z^2 - x^2}{2} - \frac{y}{4})\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, [Y, e_1] = (e_1/4) + e_3,\ [Y, e_2] = e_2/4,\ [Y, e_3] = -e_1.$$

Now, we can see that

$$(\mathcal{L}_Y g)(e_p, e_q) + 2S^*(e_p, e_q) = 2\rho g(e_p, e_q), \tag{5.9}$$

for $\rho = 0$ and $p, q = 1, 2, 3$.
Hence M is a non-cosymplectic $(-\frac{1}{4}, 1)$-acs manifold admitting steady $*$-Ricci soliton. Also, $div\, Y = -\frac{1}{2}$.

Example 5.2. Consider $M = \{(x, y, z) \in R^3 : x, y \neq 0\}$ with

$$\begin{cases} g = dx \otimes dx - \frac{y}{2} dx \otimes dz + dy \otimes dy - x dy \otimes dz - \frac{y}{2} dz \otimes dx \\ -x dz \otimes dy + (x^2 + \frac{y^2}{4} + 1) dz \otimes dz, e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y}, \\ e_3 = \xi = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y} + \frac{y}{2}\frac{\partial}{\partial x}, \eta = dz, \\ h = \begin{pmatrix} -\frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{3y}{8} - \frac{3x}{4} \\ 0 & 0 & 0 \end{pmatrix}, Y = -\frac{9x}{16}\frac{\partial}{\partial x} - \frac{9y}{16}\frac{\partial}{\partial y} - 2c_1\frac{\partial}{\partial z}, \end{cases} \tag{5.10}$$

where Y is potential field and c_1 is an arbitrary constant. Then similar to Example 5.1, computation can be done to show that M is a non-cosymplectic $(-\frac{9}{16}, \frac{1}{2})$-acs manifold admitting steady $*$-Ricci soliton.

Example 5.3. Consider $M = \{(x, y, z) \in R^3 : x, y \neq 0\}$ with structure tensors

$$\begin{cases} g = dx \otimes dx - y dx \otimes dz + dy \otimes dy - x dy \otimes dz - y dz \otimes dx \\ \quad -x dz \otimes dy + (x^2 + y^2 + 1) dz \otimes dz,\ \eta = dz,\ e_1 = \frac{\partial}{\partial x}, \\ e_2 = \frac{\partial}{\partial y},\ e_3 = \xi = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},\ h = \begin{pmatrix} -1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 0 \end{pmatrix}. \end{cases} \tag{5.11}$$

Then, M is a non-cosymplectic $(-1, 0)$-acs manifold. Further, M admits steady $*$-Ricci soliton with potential vector field

$$Y = (-x + e^{-z} + e^z)\frac{\partial}{\partial x} + (-y - e^{-z} + e^z)\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

Moreover, M admits expanding Ricci soliton with potential vector field

$$Y = (-2x + e^z + e^{-z})\frac{\partial}{\partial x} + (-2y + e^z - e^{-z})\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

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References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston(2010).

[2] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity distribution*, Israel J. Math., **91**(1995), 189–214.

[3] E. Boeckx, *A full classification of contact metric (κ, μ)-spaces*, Illinois J. Math., **44(1)**(2000), 212–219.

[4] B. Cappelletti-Montano, A. D. Nicola and I. Yudin, *A survey on cosymplectic geometry*, Rev. Math. Phys. **25(10)**(2013), 55 pp.

[5] A. Carriazo and V. Martin-Molina, *Generalized (κ, μ)-space forms and D_a-homothetic deformations*, Balkan J. Geom. Appl., **16(1)**(2011), 37–47.

[6] B. Y. Chen, *Rectifying submanifolds of Riemannian manifolds and torqued vector fields*, Kragujevac J. Math., **41(1)**(2017), 93–103.

[7] X. Chen, *Einstein-Weyl structures on almost cosymplectic manifolds*, Period. Math. Hungar., **79(2)**(2019), 191–203.

[8] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., **61**(2009), 205–212.

[9] X. Dai, *Non-existence of $*$-Ricci solitons on (κ, μ)-almost cosymplectic manifolds*, J. Geom., **110(2)**(2019), 7 pp.

[10] U. C. De and A. Sardar, *Classification of (κ, μ)-almost co-Kähler manifolds with vanishing Bach tensor and divergence free Cotton tensor*, Commum. Korean Math. Soc., **35(4)**(2020), 1245–1254.

[11] H. Endo, *Non-existence of almost cosymplectic manifolds satisfying a certain condition*, Tensor (N.S.), **63(3)**(2002), 272–284.

[12] A. Ghosh and R. Sharma, *Classification of (κ, μ)-contact manifolds with divergence free Cotton tensor and vanishing Bach tensor*, Ann. Polon. Math., **122(2)**(2019), 153–163.

[13] T. Hamada, *Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor*, Tokyo J. Math., **25**(2002), 473–483.

[14] R. S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math., **71**(1988), 237–261.

[15] G. Kaimakamis and K. Panagiotidou, *$*$-Ricci solitons of real hypersurfaces in non-flat complex space forms*, J. Geom. Phys., **86**(2014), 408–413.

[16] T. Koufogiorgos and C. Tsichlias, *On the existence of a new class of contact metric manifolds*, Canad. Math. Bull., **43(4)**(2000), 400–447.

[17] T. Koufogiorgos, M. Markellos and V. J. Papantoniou, *The harmonicity of the Reeb vector fields on contact metric 3-manifolds*, Pacific J. Math., **234(2)**(2008), 325–344.

[18] R. Sharma, *Certain results on K-contact and (κ, μ)-contact manofolds*, J. Geom., **89**(2008), 138-147.

[19] R. Sharma and L. Vrancken, *Conformal classification of (κ, μ)-contact manifolds*, Kodai Math. J., **33**(2010), 267–282.

[20] Y. J. Suh and U. C. De, *Yamabe solitons and Ricci solitons on almost co-Kähler manifolds*, Canad. Math. Bull., **62(3)**(2019), 653–661.

[21] S. Tachibana, *On almost-analytic vectors in almost-Kählerian manifolds*, Tohoku Math. J., **11**(1959), 247–265.

[22] K. Yano, *On torse forming direction in a Riemannian space*, Proc. Imp. Acad. Tokyo, **20**(1944), 340–345.

[23] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York(1970).