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Pseudohermitian Curvatures on Bounded Strictly Pseudoconvex Domains in $\mathbb{C}^2$

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Abstract. In this paper, we present a formula for pseudohermitian curvatures on bounded strictly pseudoconvex domains in $\mathbb{C}^2$ with respect to the coefficients of adapted frames given by Graham and Lee in [3] and their structure equations. As an application, we will show that the pseudohermitian curvatures on strictly plurisubharmonic exhaustions of Thullen domains diverges when the points converge to a weakly pseudoconvex boundary point of the domain.

1. Introduction

Let (M, θ) be a pseudohermitian manifold of real dimension $2n+1$, i.e. M is an integrable, nondegenerate, real hypersurface in $\mathbb{C}^{n+1}$ with CR structure $HM \subset TM$ so that θ is a 1-form satisfying $\theta(X) = 0$ for any $X \in HM$ and $d\theta$ can be expressed by

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$$

where $(h_{\alpha\bar{\beta}})$ is a positive definite $n \times n$ matrix with respect to a local basis $\theta^1, \ldots, \theta^n$ for HM. The equivalence problem for such pseudohermitian structures was studied first by Chern using Cartan's method and later it turned out that it was related to the pseudoconformal invariants analyzed by Chern-Moser [1] and Tanaka [4]. In [5], Webster showed that there exists a natural connection in the bundle $H^{1,0}M$ adapted to a pseudohermitian structure, where $H^{1,0}M$ denotes the eigenspace of the endomorphism $J: HM \otimes \mathbb{C} \to HM \otimes \mathbb{C}$ satisfying $J \circ J = -I$, which defines the CR structure of M, with an eigenvalue $\sqrt{-1}$. Moreover he gave an expression of pseudoconformal curvature tensor in terms of pseudohermitian curvature tensor.

In this paper, we present an expression of pseudohermitian curvatures on bounded strictly pseudoconvex domains in $\mathbb{C}^2$ with respect to the coefficients of adapted frames given by Graham-Lee in [3] and their structure equations. As an

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application, we will show that the pseudohermitian curvatures on strictly plurisubharmonic exhaustions of Thullen domains diverge when the point converges to the weakly pseudoconvex boundary point of the domain. Our main result is

Main Theorem. *For a Thullen domain*

$$\Omega_m := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}$$

with $m > 1$*, the pseudohermitian curvatures of the exhaustions*

$$\Omega_{m,\epsilon} := \{(z_1, z_2) \in \mathbb{C}^2 : \phi(z_1, z_2) = \epsilon\}$$

of Ω_m *with* $\phi = -K(z,z)^{-m/(2m+1)}$ *diverge as the point tends to weakly pseudoconvex boundary points along the complex line* $\{z_1 = 0\}$ *in* Ω_m*.*

2. Pseudohermitian Curvatures on Strictly Pseudoconvex Domains

In this section, we calculate the pseudohermitian curvature explicitly in terms of the coefficient of an adapted coframe for a strictly pseudoconvex bounded domain $\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : \phi(z_1, z_2) < 0\}$ where ϕ is a defining function of Ω. Let $\{\theta^0, \theta^1\}$ be the adapted coframe and $\{X^0, X^1\}$ its dual frame given by Graham-Lee ([3]) satisfying the following.

$$\begin{aligned}
d\theta^0 &= -\theta^1 \wedge \overline{\theta^1} - r\theta^0 \wedge \overline{\theta^0} \\
d\theta^1 &= \theta^1 \wedge \omega - iA\theta^0 \wedge \overline{\theta^1} + \frac{r}{2}\theta^0 \wedge \theta^1 + \frac{r}{2}\overline{\theta^0} \wedge \theta^1 - X^1 r\theta^0 \wedge \overline{\theta^0} \\
d\omega &= R\theta^1 \wedge \theta^{\bar 1} + \left(i\left(X^{\bar 1}A - 2\omega(X^{\bar 1}\bar A)\right) - \frac{3}{2}(X^1 r)\right)\theta^1 \wedge \theta^{\bar 0} \\
&\quad - \left(i\left(X^1 \bar A - 2\omega(X^1 A)\right) + \frac{3}{2}(X^{\bar 1} r)\right)\theta^0 \wedge \theta^{\bar 1} + \frac{3}{2}(X^1 r)\theta^0 \wedge \theta^1 \\
&\quad - \frac{3}{2}(X^{\bar 1} r)\theta^{\bar 0} \wedge \theta^{\bar 1} + \left(\frac{1}{2}\Delta_b r - |A|^2\right)\theta^0 \wedge \theta^{\bar 0}
\end{aligned} \tag{2.1}$$

Here R denotes the pseudohermitian curvature and $\Delta_b := -{r_\alpha}^\alpha - {r^\alpha}_\alpha$ denotes the sublaplacian of the pseudohermitian structure. In coordinates of $\mathbb{C}^2$, let

$$\theta^0 := A_1 dz_1 + A_2 dz_2, \quad \theta^1 := B_1 dz_1 + B_2 dz_2,$$

$$\omega := C_1 dz_1 + C_2 dz_2 - \overline{C_1} d\bar z_1 - \overline{C_2} d\bar z_2, \tag{2.2}$$

and

$$X^0 = \frac{1}{D}\left(B_2 \frac{\partial}{\partial z_1} - B_1 \frac{\partial}{\partial z_2}\right), \quad X^1 = \frac{1}{D}\left(-A_2 \frac{\partial}{\partial z_1} + A_1 \frac{\partial}{\partial z_2}\right)$$

where $D := A_1 B_2 - A_2 B_1$ and $A_j := \frac{\partial \phi}{\partial z_j}$ for each $j = 1, 2$.

Proposition 2.1. *Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \phi(z_1, z_2) < 0\}$ be a strictly pseudoconvex domain with a defining function ϕ. Then the pseudohermitian curvature is given by*

$$R = \frac{1}{|B_1|^2}\left(2Re\left(\left(i(X^1\bar{A} - 2\omega(X^{\bar{1}}\bar{A})) + \frac{3}{2}(X^{\bar{1}}r)\right)A_1\overline{B_1} - \frac{\partial C_1}{\partial \bar{z}_1}\right) - \left(\frac{1}{2}\Delta_b r - |A|^2\right)|A_1|^2\right).$$

Proof. By (2.1), we have

$$\begin{aligned} d\theta^0 &= -\theta^1 \wedge \overline{\theta^1} - r\theta^0 \wedge \overline{\theta^0} \\ &= -(B_1dz_1 + B_2dz_2) \wedge (\overline{B_1dz_1 + B_2dz_2}) \\ &\quad - r(A_1dz_1 + A_2dz_2) \wedge (\overline{A_1dz_1 + A_2dz_2}) \\ &= -(|B_1|^2 + r|A_1|^2)dz_1 \wedge d\overline{z}_1 - (B_1\overline{B}_2 + rA_1\overline{A}_2)dz_1 \wedge d\overline{z}_2 \\ &\quad - (B_2\overline{B}_1 + rA_2\overline{A}_1)dz_2 \wedge d\overline{z}_1 - (|B_2|^2 + r|A_2|^2)dz_2 \wedge d\overline{z}_2. \end{aligned} \tag{2.3}$$

On the other hand, since

$$\begin{aligned} d\theta^0 &= d(A_1dz_1 + A_2dz_2) \\ &= -\frac{\partial A_1}{\partial \overline{z}_1}dz_1 \wedge d\overline{z}_1 - \frac{\partial A_1}{\partial \overline{z}_2}dz_1 \wedge d\overline{z}_2 - \frac{\partial A_2}{\partial \overline{z}_1}dz_2 \wedge d\overline{z}_1 - \frac{\partial A_2}{\partial \overline{z}_2}dz_2 \wedge d\overline{z}_2, \end{aligned} \tag{2.4}$$

by comparing equations (2.3) and (2.4) we have

$$|B_1|^2 + r|A_1|^2 = \frac{\partial A_1}{\partial \overline{z}_1}, \tag{2.5}$$

$$B_1\overline{B}_2 + rA_1\overline{A}_2 = \frac{\partial A_1}{\partial \overline{z}_2}, \tag{2.6}$$

and

$$|B_2|^2 + r|A_2|^2 = \frac{\partial A_2}{\partial \overline{z}_2}. \tag{2.7}$$

Since we have

$$\begin{aligned} |B_1|^2|B_2|^2 &= \left(\frac{\partial A_1}{\partial \overline{z}_1} - r|A_1|^2\right)\left(\frac{\partial A_2}{\partial \overline{z}_2} - r|A_2|^2\right) \\ &= \left(\frac{\partial A_1}{\partial \overline{z}_2} - rA_1\overline{A}_2\right)\left(\frac{\partial \overline{A_1}}{\partial z_2} - rA_2\overline{A}_1\right) \end{aligned}$$

by (2.5), (2.7), (2.6), we obtain

$$r = \frac{\left|\frac{\partial A_1}{\partial \overline{z}_2}\right|^2 - \frac{\partial A_1}{\partial \overline{z}_1}\frac{\partial A_2}{\partial \overline{z}_2}}{A_1\overline{A}_2\frac{\partial \overline{A_1}}{\partial z_2} + A_2\overline{A}_1\frac{\partial A_1}{\partial \overline{z}_2} - |A_2|^2\frac{\partial A_1}{\partial \overline{z}_1} - |A_1|^2\frac{\partial A_2}{\partial \overline{z}_2}}. \tag{2.8}$$

By substituting (2.8) into (2.5) and (2.7), we have

$$|B_1|^2 = \frac{A_1\overline{A_2}\frac{\partial A_1}{\partial \overline{z}_1}\frac{\partial \overline{A_1}}{\partial z_2} + A_2\overline{A_1}\frac{\partial A_1}{\partial \overline{z}_1}\frac{\partial A_1}{\partial \overline{z}_2} - |A_2|^2\left(\frac{\partial A_1}{\partial \overline{z}_1}\right)^2 - |A_1|^2\left|\frac{\partial A_1}{\partial \overline{z}_2}\right|^2}{A_1\overline{A_2}\frac{\partial \overline{A_1}}{\partial z_2} + A_2\overline{A_1}\frac{\partial A_1}{\partial \overline{z}_2} - |A_2|^2\frac{\partial A_1}{\partial \overline{z}_1} - |A_1|^2\frac{\partial A_2}{\partial \overline{z}_2}}, \tag{2.9}$$

and

$$|B_2|^2 = \frac{A_1\overline{A_2}\frac{\partial \overline{A_1}}{\partial z_2}\frac{\partial A_2}{\partial \overline{z}_2} + A_2\overline{A_1}\frac{\partial A_1}{\partial \overline{z}_2}\frac{\partial A_2}{\partial \overline{z}_2} - |A_1|^2\left(\frac{\partial A_2}{\partial \overline{z}_2}\right)^2 - |A_2|^2\left|\frac{\partial A_1}{\partial \overline{z}_2}\right|^2}{A_1\overline{A_2}\frac{\partial \overline{A_1}}{\partial z_2} + A_2\overline{A_1}\frac{\partial A_1}{\partial \overline{z}_2} - |A_2|^2\frac{\partial A_1}{\partial \overline{z}_1} - |A_1|^2\frac{\partial A_2}{\partial \overline{z}_2}}. \tag{2.10}$$

By (2.1) and a straightforward calculation we obtain

$$\begin{aligned}
d\theta^1 &= \theta^1 \wedge \omega - iA\theta^0 \wedge \overline{\theta^1} + \frac{r}{2}\theta^0 \wedge \theta^1 + \frac{r}{2}\overline{\theta^0} \wedge \theta^1 - X^1 r\theta^0 \wedge \overline{\theta^0} \\
&= (B_1dz_1 + B_2dz_2) \wedge (C_1dz_1 + C_2dz_2 - \overline{C_1}d\overline{z}_1 - \overline{C_2}d\overline{z}_2) \\
&\quad - iA(A_1dz_1 + A_2dz_2) \wedge \overline{(B_1dz_1 + B_2dz_2)} \\
&\quad + \frac{r}{2}(A_1dz_1 + A_2dz_2) \wedge (B_1dz_1 + B_2dz_2) \\
&\quad + \frac{r}{2}\overline{(A_1dz_1 + A_2dz_2)} \wedge (B_1dz_1 + B_2dz_2) \\
&\quad - X^1 r(A_1dz_1 + A_2dz_2) \wedge \overline{(A_1dz_1 + A_2dz_2)} \\
&= \Big(B_1C_2 - B_2C_1 + \frac{r}{2}A_1B_2 - \frac{r}{2}A_2B_1\Big)\, dz_1 \wedge dz_2 \\
&\quad + \Big(-B_1\overline{C_1} - iAA_1\overline{B_1} - \frac{r}{2}\overline{A_1}B_1 - (X^1r)|A_1|^2\Big)\, dz_1 \wedge d\overline{z}_1 \\
&\quad + \Big(-B_1\overline{C_2} - iAA_1\overline{B_2} - \frac{r}{2}\overline{A_2}B_1 - (X^1r)A_1\overline{A_2}\Big)\, dz_1 \wedge d\overline{z}_2 \\
&\quad + \Big(-B_2\overline{C_1} - iAA_2\overline{B_1} - \frac{r}{2}\overline{A_1}B_2 - (X^1r)A_2\overline{A_1}\Big)\, dz_2 \wedge d\overline{z}_1 \\
&\quad + \Big(-B_2\overline{C_2} - iAA_2\overline{B_2} - \frac{r}{2}\overline{A_2}B_2 - (X^1r)|A_2|^2\Big)\, dz_2 \wedge d\overline{z}_2.
\end{aligned} \tag{2.11}$$

On the other hand, since

$$\begin{aligned}
d\theta^1 &= d(B_1dz_1 + B_2dz_2) \\
&= -\frac{\partial B_1}{\partial \overline{z}_1}dz_1 \wedge d\overline{z}_1 - \frac{\partial B_1}{\partial z_2}dz_1 \wedge dz_2 - \frac{\partial B_1}{\partial \overline{z}_2}dz_1 \wedge d\overline{z}_2 \\
&\quad - \frac{\partial B_2}{\partial z_1}dz_2 v dz_1 - \frac{\partial B_2}{\partial \overline{z}_1}dz_2 \wedge d\overline{z}_1 - \frac{\partial B_2}{\partial \overline{z}_2}dz_2 \wedge d\overline{z}_2,
\end{aligned} \tag{2.12}$$

by comparing equations (2.11) and (2.12), we obtain

$$B_1C_2 - B_2C_1 + \frac{r}{2}A_1B_2 - \frac{r}{2}A_2B_1 = -\frac{\partial B_1}{\partial z_2} + \frac{\partial B_2}{\partial z_1}, \tag{2.13}$$

$$B_1\overline{C_1}+iAA_1\overline{B_1}+\frac{r}{2}\overline{A_1}B_1+(X^1r)|A_1|^2=\frac{\partial B_1}{\partial\overline{z}_1}, \tag{2.14}$$

$$B_1\overline{C_2}+iAA_1\overline{B_2}+\frac{r}{2}\overline{A_2}B_1+(X^1r)A_1\overline{A_2}=\frac{\partial B_1}{\partial\overline{z}_2},$$

$$B_2\overline{C_1}+iAA_2\overline{B_1}+\frac{r}{2}\overline{A_1}B_2+(X^1r)A_2\overline{A_1}=\frac{\partial B_2}{\partial\overline{z}_1},$$

and

$$B_2\overline{C_2}+iAA_2\overline{B_2}+\frac{r}{2}\overline{A_2}B_2+(X^1r)|A_2|^2=\frac{\partial B_2}{\partial\overline{z}_2}. \tag{2.15}$$

By considering $B_2\times(2.14)-B_1\times(2.15)$, we have

$$A=\frac{1}{i\overline{B_1}D}\left(\frac{\partial B_1}{\partial\overline{z}_1}B_2-\frac{\partial B_2}{\partial\overline{z}_1}B_1-(X^1r)\overline{A_1}D\right). \tag{2.16}$$

By substituting (2.16) into (2.14), we obtain

$$\overline{C_1}=\frac{1}{B_1}\frac{\partial B_1}{\partial\overline{z}_1}-\frac{1}{D}\frac{A_1}{B_1}\left(\frac{\partial B_1}{\partial\overline{z}_1}B_2-\frac{\partial B_2}{\partial\overline{z}_1}B_1\right)-\frac{r}{2}\overline{A_1}.$$

Similarly by considering $B_2\times(2.15)-B_1\times(2.15)$ we have

$$A=\frac{1}{i\overline{B_2}D}\left(\frac{\partial B_1}{\partial\overline{z}_2}B_2-\frac{\partial B_2}{\partial\overline{z}_2}B_1-(X^1r)\overline{A_2}D\right) \tag{2.17}$$

and by substituting (2.17) into (2.15), we have

$$\overline{C_2}=\frac{1}{B_1}\frac{\partial B_1}{\partial\overline{z}_2}-\frac{1}{D}\frac{A_1}{B_1}\left(\frac{\partial B_1}{\partial\overline{z}_2}B_2-\frac{\partial B_2}{\partial\overline{z}_2}B_1\right)-\frac{r}{2}\overline{A_2}.$$

Now using the expression (2.1) and (2.2), we obtain

$$\begin{aligned} d\omega &= R(B_1dz_1+B_2dz_2)\wedge\overline{(B_1dz_1+B_2dz_2)}\\ &\quad+\left(i\left(X^{\bar 1}A-2\omega(X^{\bar 1}\bar A)\right)-\frac{3}{2}(X^1r)\right)(B_1dz_1+B_2dz_2)\wedge\overline{(A_1dz_1+A_2dz_2)}\\ &\quad-\left(i\left(X^1\bar A-2\omega(X^1A)\right)+\frac{3}{2}(X^{\bar 1}r)\right)(A_1dz_1+A_2dz_2)\wedge\overline{(B_1dz_1+B_2dz_2)}\\ &\quad+\frac{3}{2}(X^1r)(A_1dz_1+A_2dz_2)\wedge(B_1dz_1+B_2dz_2)\\ &\quad-\frac{3}{2}(X^{\bar 1}r)\overline{(A_1dz_1+A_2dz_2)}\wedge\overline{(B_1dz_1+B_2dz_2)}\\ &\quad+(\Delta_br-|A|^2)(A_1dz_1+A_2dz_2)\wedge\overline{(A_1dz_1+A_2dz_2)}\\ &= d(C_1dz_1+C_2dz_2-\overline{C_1}d\overline{z}_1-\overline{C_2}d\overline{z}_2). \end{aligned} \tag{2.18}$$

By comparing the coefficient of $dz_1 \wedge d\overline{z}_1$ in the equation (2.18), one has

$$-\frac{\partial C_1}{\partial \overline{z}_1} - \frac{\partial \overline{C_1}}{\partial z_1} = R|B_1|^2 + \left(i\left(X^{\bar{1}}A - 2\omega(X^{\bar{1}}\bar{A})\right) - \frac{3}{2}(X^1 r)\right) B_1\overline{A_1} - \left(i\left(X^1\bar{A} - 2\omega(X^1 A)\right) + \frac{3}{2}(X^{\bar{1}} r)\right) A_1\overline{B_1} + \left(\frac{1}{2}\Delta_b r - |A|^2\right)|A_1|^2.$$

As a result, we complete the proof. □

3. Pseudohermitian Curvature of Thullen Domains

For $m \in \mathbb{N}$, let

$$\Omega_m := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}$$

be the Thullen domain in $\mathbb{C}^2$ with $m \geq 1$. The diagonal of the Bergman kernel of Ω_m is given by

$$K(z,z) = \frac{1}{m\pi^2}\frac{(m+1)\left(1-|z_2|^2\right)^{1/m} - (1-m)|z_1|^2}{(1-|z_2|^2)^{2-1/m}\left((1-|z_2|^2)^{1/m} - |z_1|^2\right)^3}.$$

For the detail, see [2]. Then $\phi(z,\bar{z}) := -K(z,z)^{-m/(2m+1)}$ gives a defining function of Ω_m.

For a point $a \in \Delta := \{z \in \mathbb{C} : |z| < 1\}$, denote by ψ_a an automorphism of Ω_m obtained by

$$\psi_a : (z_1, z_2) \mapsto \left(\frac{(1-|a|^2)^{1/2m} z_1}{(1-\overline{a}z_2)^{1/m}}, \frac{z_2 - a}{1 - \overline{a}z_2}\right).$$

Note that ψ_{-a} is the inverse of ψ_a. Then we have

$$\mathrm{Aut}(\Omega_m) = \left\{(z_1, z_2) \mapsto \left(e^{i\theta_1}\frac{(1-|a|^2)^{\frac{1}{2m}} z_1}{(1-\overline{a}z_2)^{\frac{1}{m}}}, e^{i\theta_2}\frac{z_2 - a}{1-\overline{a}z_2}\right) : \theta_1, \theta_2 \in \mathbb{R}, a \in \Delta\right\}$$

and $W := \{(0,w) : |w| = 1\} \subset \partial\Omega_m$ is the set of weakly pseudoconvex boundary points of Ω_m for $m > 1$. Note that $\partial\Omega_m \setminus W$ is the set of strictly pseudoconvex boundary points.

From now on, we will find the pseudohermitian curvature of the exhaustion

$$\Omega_{m,\epsilon} := \{(z_1, z_2) \in \mathbb{C}^2 : \phi(z_1, z_2) = \epsilon\}$$

with $\epsilon \ll 1$. In particular, we will observe the behavior of the pseudohermitian curvature when $(0, z_2) \to (0, w)$ with $|w| = 1$, i.e. weakly pseudoconvex points of

Ω_m. At $z=(0,z_2)$, we have the following:

$$K(z,z)=\frac{1}{\pi^2}\frac{m+1}{m}\frac{1}{(1-|z_2|^2)^{2+\frac{1}{m}}},$$

$$\phi=-\left(\frac{1}{\pi^2}\frac{m+1}{m}\right)^{\frac{-m}{2m+1}}(1-|z_2|^2),$$

$$A_1=\frac{\partial A_2}{\partial z_1}=\frac{\partial A_1}{\partial z_2}=\frac{\partial A_2}{\partial \overline{z}_1}=\frac{\partial A_1}{\partial \overline{z}_2}=\frac{\partial A_1}{\partial z_1}=\frac{\partial^2 A_1}{(\partial \overline{z}_1)^2}=0$$

$$A_2=-\phi\frac{\overline{z}_2}{1-|z_2|^2}=\left(\frac{1}{\pi^2}\frac{m+1}{m}\right)^{\frac{-m}{2m+1}}\overline{z}_2,\quad \frac{\partial A_2}{\partial \overline{z}_2}=\left(\frac{1}{\pi^2}\frac{m+1}{m}\right)^{\frac{-m}{2m+1}}.$$

Hence at $z=(0,z_2)$, by straightforward calculation we have

$$\begin{aligned}
&r=\frac{1}{|A_2|^2}\frac{\partial A_2}{\partial \overline{z}_2}=\left(\frac{1}{\pi^2}\frac{m+1}{m}\right)^{\frac{m}{2m+1}}\frac{1}{|z_2|^2},\\
&|B_1|^2=\frac{\partial A_1}{\partial \overline{z}_1}=\frac{\partial^2\phi}{\partial z_1\partial \overline{z}_1}=-\frac{2m}{m+1}\frac{1}{(1-|z_2|^2)^{1/m}}\phi,\quad B_2=0,\\
&A=\frac{1}{iA_2\overline{B_1}}\frac{\partial B_1}{\partial \overline{z}_1},\quad \overline{C_1}=\frac{1}{B_1}\frac{\partial B_1}{\partial \overline{z}_1},\quad \overline{C_2}=\frac{1}{B_1}\frac{\partial B_1}{\partial \overline{z}_2}-\frac{r}{2}\overline{A_2},\\
&R=\frac{-2}{|B_1|^2}\mathrm{Re}\left(\frac{\partial C_1}{\partial \overline{z}_1}\right).
\end{aligned}\tag{3.1}$$

By differentiating the equation (2.5) with $\frac{\partial}{\partial \overline{z}_1}$, we obtain

$$\begin{aligned}
\frac{\partial B_1}{\partial \overline{z}_1}\overline{B_1}+B_1\frac{\partial \overline{B_1}}{\partial \overline{z}_1}&=\frac{\partial}{\partial \overline{z}_1}\left(\frac{\partial A_1}{\partial \overline{z}_1}-r|A_1|^2\right)\\
&=\frac{\partial^2 A_1}{\partial^2 \overline{z}_1}-\frac{\partial r}{\partial \overline{z}_1}|A_1|^2-r\frac{\partial |A_1|^2}{\partial \overline{z}_1}.
\end{aligned}\tag{3.2}$$

Since $A_1=0$ and $\frac{\partial A_1}{\partial z_1}=0$ at $z=(0,z_2)$, we have

$$\frac{\partial B_1}{\partial \overline{z}_1}\overline{B_1}+B_1\frac{\partial \overline{B_1}}{\partial \overline{z}_1}=0,$$

and hence

$$\frac{1}{B_1}\frac{\partial B_1}{\partial \overline{z}_1}=-\frac{1}{\overline{B_1}}\frac{\partial \overline{B_1}}{\partial \overline{z}_1}.$$

By differentiating the equation (3.2) with $\frac{\partial}{\partial z_1}$, we obtain

$$\begin{aligned}
&\frac{\partial^2 B_1}{\partial z_1\partial \overline{z}_1}\overline{B_1}+\left|\frac{\partial B_1}{\partial \overline{z}_1}\right|^2+\left|\frac{\partial B_1}{\partial z_1}\right|^2+B_1\frac{\partial^2 \overline{B_1}}{\partial z_1\partial \overline{z}_1}\\
&=\frac{\partial^3 A_1}{\partial z_1\partial^2 \overline{z}_1}-\frac{\partial^2 r}{\partial z_1\partial \overline{z}_1}|A_1|^2-\frac{\partial r}{\partial \overline{z}_1}\frac{\partial |A_1|^2}{\partial z_1}-\frac{\partial r}{\partial z_1}\frac{\partial |A_1|^2}{\partial \overline{z}_1}-r\frac{\partial^2 |A_1|^2}{\partial z_1\partial \overline{z}_1}.
\end{aligned}$$

Evaluating the above equation at $z=(0,z_2)$ gives

$$2\mathrm{Re}\left(\frac{1}{B_1}\frac{\partial^2 B_1}{\partial z_1\partial\overline{z}_1}\right)+2\left|\frac{1}{B_1}\frac{\partial B_1}{\partial\overline{z}_1}\right|^2=\frac{1}{|B_1|^2}\left(\frac{\partial^3 A_1}{\partial z_1\partial^2\overline{z}_1}-r\left|\frac{\partial A_1}{\partial\overline{z}_1}\right|^2\right). \tag{3.3}$$

Since

$$\begin{aligned}\frac{\partial\overline{C_1}}{\partial z_1}&=\frac{\partial}{\partial z_1}\left(\frac{1}{B_1}\frac{\partial B_1}{\partial\overline{z}_1}-\frac{1}{D}\frac{A_1}{B_1}\left(\frac{\partial B_1}{\partial\overline{z}_1}B_2-\frac{\partial B_2}{\partial\overline{z}_1}B_1\right)-\frac{r}{2}\overline{A_1}\right)\\&=-\frac{1}{B_1^2}\frac{\partial B_1}{\partial z_1}\frac{\partial B_1}{\partial\overline{z}_1}+\frac{1}{B_1}\frac{\partial^2 B_1}{\partial z_1\partial\overline{z}_1}-\frac{r}{2}\frac{\partial\overline{A_1}}{\partial z_1}\\&=\left|\frac{1}{B_1}\frac{\partial B_1}{\partial z_1}\right|^2+\frac{1}{B_1}\frac{\partial^2 B_1}{\partial z_1\partial\overline{z}_1}-\frac{r}{2}\frac{\partial\overline{A_1}}{\partial z_1},\end{aligned}$$

by Proposition 2.1 and equations (3.3), (3.1) we obtain

$$\begin{aligned}R&=\frac{-2}{|B_1|^2}\left(\left|\frac{1}{B_1}\frac{\partial B_1}{\partial\overline{z}_1}\right|^2+\mathrm{Re}\left(\frac{1}{B_1}\frac{\partial^2 B_1}{\partial z_1\partial\overline{z}_1}\right)-\frac{r}{2}\frac{\partial\overline{A_1}}{\partial z_1}\right)\\&=\frac{-1}{|B_1|^2}\left(\frac{1}{|B_1|^2}\left(\frac{\partial^3 A_1}{\partial z_1\partial^2\overline{z}_1}-r\left|\frac{\partial A_1}{\partial\overline{z}_1}\right|^2\right)-r\frac{\partial\overline{A_1}}{\partial z_1}\right)\\&=\frac{-1}{|B_1|^4}\frac{\partial^3 A_1}{\partial z_1\partial^2\overline{z}_1}+2r\frac{1}{|B_1|^2}\frac{\partial A_1}{\partial\overline{z}_1}\\&=\frac{-1}{|B_1|^4}\frac{\partial^4\phi}{\partial^2 z_1\partial^2\overline{z}_1}+2r,\end{aligned}\tag{3.4}$$

$$\begin{aligned}\frac{\partial\phi}{\partial z_1}&=\phi\frac{\partial}{\partial z_1}\log(-\phi)=-p\phi\frac{\partial}{\partial z_1}\log K\\&=-p\phi\frac{\partial}{\partial z_1}\log\Big((m+1)\left(1-|z_2|^2\right)^{1/m}-(1-m)|z_1|^2\Big)\\&\quad+3p\phi\frac{\partial}{\partial z_1}\log\Big(\left(1-|z_2|^2\right)^{1/m}-|z_1|^2\Big)\\&=-p\phi\overline{z}_1\left(\frac{-(1-m)}{(m+1)(1-|z_2|^2)^{\frac{1}{m}}-(1-m)|z_1|^2}+\frac{3}{(1-|z_2|^2)^{\frac{1}{m}}-|z_1|^2}\right)\end{aligned}$$

with $p=\frac{m}{2m+1}$, and

$$\begin{aligned}\frac{\partial^2\phi}{\partial^2 z_1}&=p^2\overline{z}_1^2\phi\left(\frac{-(1-m)}{(m+1)(1-|z_2|^2)^{\frac{1}{m}}-(1-m)|z_1|^2}+\frac{3}{(1-|z_2|^2)^{\frac{1}{m}}-|z_1|^2}\right)^2\\&\quad-p\phi\overline{z}_1^2\left(\frac{-(1-m)^2}{\left((m+1)(1-|z_2|^2)^{\frac{1}{m}}-(1-m)|z_1|^2\right)^2}+\frac{3}{\left((1-|z_2|^2)^{\frac{1}{m}}-|z_1|^2\right)^2}\right).\end{aligned}$$

Let

$$M := \frac{-(1-m)}{(m+1)(1-|z_2|^2)^{\frac{1}{m}} - (1-m)|z_1|^2} + \frac{3}{(1-|z_2|^2)^{\frac{1}{m}} - |z_1|^2}$$

and

$$N := \frac{-(1-m)^2}{\Big((m+1)(1-|z_2|^2)^{\frac{1}{m}} - (1-m)|z_1|^2\Big)^2} + \frac{3}{\Big((1-|z_2|^2)^{\frac{1}{m}} - |z_1|^2\Big)^2}.$$

Then

$$\frac{\partial^2 \phi}{\partial^2 z_1} = p^2 \overline{z}_1^2 M^2 \phi - p\phi \overline{z}_1^2 N.$$

Hence at $z = (0, z_2)$, we have

$$\frac{\partial^4 \phi}{\partial^2 z_1 \partial^2 \overline{z}_1} = 2p\phi(pM^2 - N)$$

and by (3.4)

$$R = -\frac{(3m+1)(m-1)}{m(2m+1)} \frac{1}{\phi} + 2r. \tag{3.5}$$

As a result we obtain our main theorem.

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