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The Second Reidemeister Moves and Colorings of Virtual Knot Diagrams

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Abstract. Two virtual knot diagrams are said to be *equivalent*, if there is a sequence S of Reidemeister moves and virtual moves relating them. The difference of writhes of the two virtual knot diagrams gives a lower bound for the number of the first Reidemeister moves in S. In previous work, we introduced a polynomial $q_K(t)$ for a virtual knot diagram K which gave a lower bound for the number of the third Reidemeister moves in the sequence S. In this paper we define a new polynomial from a coloring of a virtual knot diagram. Using this polynomial, we give a lower bound for the number of the second Reidemeister moves in S. The polynomial also suggests the design of the sequence S.

1. Introduction

L. H. Kauffman introduced virtual knots, which generalize classical knots, and virtual knot invariants ([7, 8]). Non-classical virtual knots can be represented in a thickened orientable surface of genus greater than 0. Some invariants of classical knots, such as the bracket polynomial, the fundamental group and the quandle, can be naturally extended to virtual knots. Other invariants have be developed for virtual knots. In [5, 6], we gave polynomials related to virtual knot diagrams to study Reidemeister moves. The polynomials provide us with information about the third Reidemeister moves in the deformation of a virtual knot diagram. In this paper we introduce a polynomial, obtained from a coloring of a virtual knot

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diagram, and use it to give a lower bound for the number of second Reidemeister moves in a Reidermeister sequence between two virtual knot diagrams. We also give examples in which we fully determine the minimal number of Reidemeister moves by using the new polynomial, the q-polynomial and the writhe of a virtual knot diagram.

In this paper all knot diagrams are assumed to be oriented. We define the *sign* of a crossing in a knot diagram as shown in Figure 1. We denote the sign of a crossing c by $s(c)$. The *writhe* $w(K)$ of a knot diagram K is defined to be the sum of signs of all crossings in K. In [3], Hayashi introduced th cowrithe of a knot diagram K by considering the Gauss diagram of K, and showed that the cowrithe gives a lower bound for the number of second Reidemeister moves and third Reidemeister moves in a sequence of Reidemeister moves deforming a knot diagram to another one.

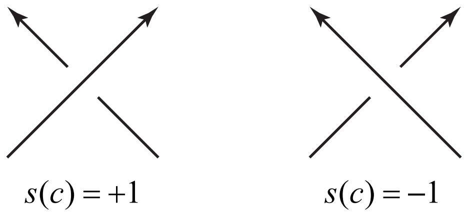


Figure 1: The sign of a crossing.

In [5], we introduced a polynomial $q_K(t)$ of a virtual knot diagram K by considering degrees of mixed pairs of crossings and gave a lower bound for the number of second Reidemeister moves and third Reidemeister moves in a deformation of two equivalent virtual knot diagrams.

In [6], we introduced a bridge diagram and parity polynomials of a virtual knot diagram. If we calculate the parity polynomials of two equivalent virtual knot diagrams, the result in the paper gives a lower bound for the number of third Reidemeister moves needed to deform a diagram to the other one.

All Vassiliev invariants of degree < 2 for classical knots are constants. But there are many nontrivial Vassiliev invariants of degree 1 for virtual knots. Henrich ([4]) introduced Vassiliev invariants of degree 1 for virtual knots and showed that there are infinitely many independent Vassiliev invariants of virtual knots of degree 1. In particular for a crossing c in a virtual knot diagram K, the intersection index $i(c)$ was defined and a polynomial invariant $p_t(K)$ in t of the form

$$p_t(K) = \sum_c s(c)(t^{i(c)} - 1)$$

was introduced, where the sum runs over all crossings c in K.

Cheng used a similar idea to define a polynomial invariant of virtual knots. Based on Manturov's parity axioms ([10]), Cheng ([1]) introduced the odd writhe polynomial $f_K(t)$ of a virtual knot K. A virtual knot diagram can be represented as a Gauss diagram with chords. Cheng used odd chords and a coloring of a Gauss diagram to define the polynomial. Cheng and Gao ([2]) generalized the odd writhe polynomial $f_K(t)$ and defined the writhe polynomial $W_K(t)$. Using a Cheng coloring of a flat diagram obtained from a virtual knot diagram, Kauffman introduced the affine index polynomial of a virtual knot and proved that it is an invariant of virtual knots ([9]). The affine index polynomial $P_K(t)$ is given in the form

$$P_K(t) = \sum_c s(c)(t^{w_K(c)} - 1),$$

where $w_K(c)$ is the weight of a crossing c and the sum runs over all crossings c in K. $W_K(t)$ and $P_K(t)$ contain similar information on K since $W_K(t) = (P_K(t) + Q_K)t$, where Q_K is a numerical invariant ([2]).

The *Gauss diagram* of a knot diagram K is an oriented circle with chords corresponding to crossings in K. The two endpoints of a chord correspond to the preimages of the crossing in K. A chord corresponding to a crossing c is oriented from the preimage of the over crossing point of c to the preimage of the under crossing point of c. A chord is assumed to be equipped with the sign of the crossing corresponding to the chord. See Figure 2, where c_i's denote crossings in K and their corresponding chords in the Gauss diagram of K. Refer to [7] for more details.

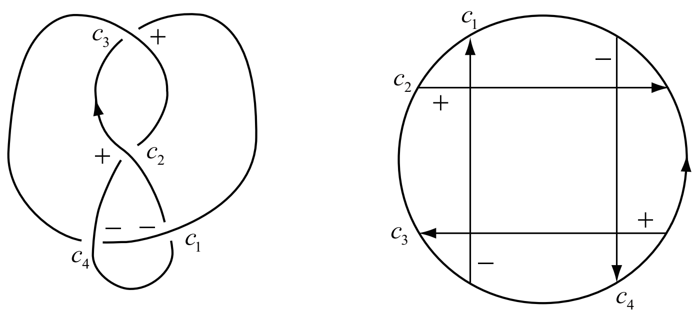


Figure 2:

Let K be a knot diagram with crossings x_1 and x_2. If the two chords corresponding to x_1 and x_2 can be described as shown in Figure 3, then the pair (x_1, x_2) is called a *mixed pair of crossings*. For example the knot in Figure 2 has four mixed pairs $(c_1, c_3), (c_2, c_1), (c_3, c_4)$ and (c_4, c_2) of crossings.

The *cowrithe* of K is defined to be the sum of $s(x_1)s(x_2)$ for all mixed pairs (x_1, x_2) of crossings in K. Let K_1 and K_2 be equivalent knot diagrams and S be a sequence of Reidemeister moves deforming K_1 to K_2. The writhe is invariant

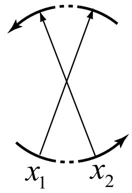


Figure 3:

under second Reidemeister move and third Reidemeister move of diagrams. Since the writhes of two knot diagrams related by a first Reidemeister move differ by 1, we get a lower bound $|w(K_1) - w(K_2)|$ for the number of the first Reidemeister moves in S. Hayashi ([3]) showed that the difference of the cowrithes of K_1 and K_2 is a lower bound for the number of second Reidemeister moves and third Reidemeister moves in S.

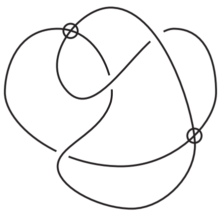


Figure 4: A virtual knot diagram.

A *virtual knot diagram* is a knot diagram allowed to have some virtual crossings which are denoted by encircled 4-valent vertices. See Figure 4 for a virtual knot diagram with two virtual crossings and three crossings. Two knot diagrams are said to be *equivalent* if there is a sequence of Reidemeister moves as shown in Figure 5 between them.

The moves of diagrams shown in Figure 6 are called *virtual moves*. Two virtual knot diagrams K_1 and K_2 are said to be *virtually isotopic* or *equivalent* if there is a sequence of Reidemeister moves and virtual moves from K_1 to K_2. A *virtual knot* is defined to be the equivalence class of a virtual knot diagram under virtual isotopy. From now on, all virtual knots are assumed to be oriented.

As we can for a classical knot diagram, we can define a virtual knot from its Gauss diagram by disregarding virtual crossings. See Figure 7, where c_i's denote crossings in K and their corresponding chords in the Gauss diagram of K. We naturally extend the sign, writhe and cowrithe of a knot diagram to a virtual knot diagram. For a virtual knot diagram K, $C(K)$ and $M(K)$ will denote the set of crossings and the set of all mixed pairs of crossings in K respectively.

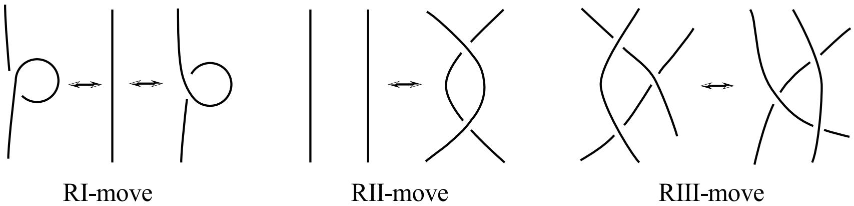


Figure 5: Reidemeister moves.

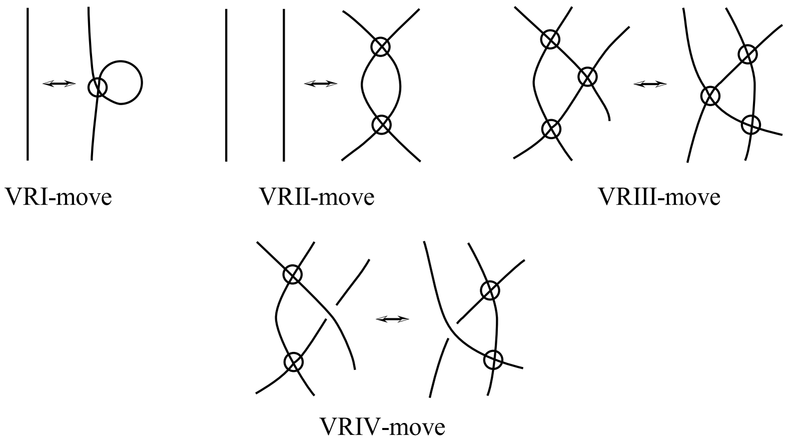


Figure 6: Virtual moves.

For two virtually isotopic knot diagrams K_1 and K_2, we denote the minimal numbers of the first Reidemeister moves, second Reidemeister moves, and third Reidemeister moves needed to deform K_1 to K_2 by $n_1(K_1, K_2)$, $n_2(K_1, K_2)$ and $n_3(K_1, K_2)$ respectively. We also denote the minimal number of all Reidemeister moves needed to deform K_1 to K_2 by $n(K_1, K_2)$. For a virtual knot diagram K, we will color the diagram with integers and assign a grade $g(c)$ for each crossing c in K by using the coloring. $h_K(t)$ will be defined in the form

$$h_K(t) = \sum_{c \in C(K)} s(c)(t^{g(c)} - 1).$$

In Section we introduce a coloring of a virtual knot diagram K and define a polynomial $h_K(t)$. Then we show that the polynomial $h_K(t)$ is invariant under the first and third Reidemeister moves. If two virtual knot diagrams K_1 and K_2 differ by a single second Reidemeister move then we will show that $h_{K_1}(t) - h_{K_2}(t) =$

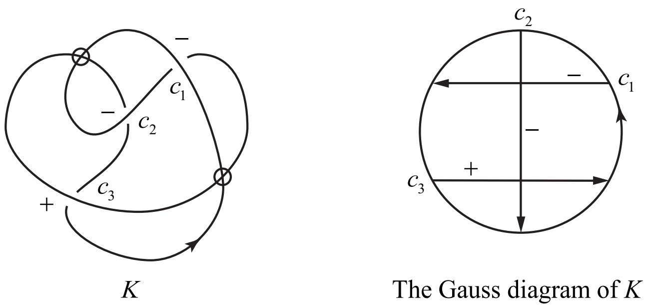


Figure 7: Virtual moves.

$(t^i - 1) - (t^{-i} - 1)$ for some integer i. From this result we get a lower bound for the number of second Reidemeister moves needed to deform K_1 to K_2 when they are equivalent. We give examples showing that our result is distinguished from Hayashi's result in [3] and our previous result in [5].

2. A Polynomial of a Virtual Knot Diagram

For a given virtual knot diagram K, we get its projection $F(K)$ by replacing all crossings with flat crossings. A preimage of an edge in the 4-valent graph $F(K)$ is called an *arc*. If K has n crossings and $n > 0$ then it has $2n$ arcs. For example, the virtual knot diagram with three crossings c_1, c_2, and c_3 in Figure 8 has six arcs. We will label each arc of K with an integer. We choose an arc, label it 0, and

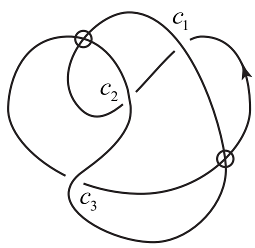


Figure 8:

proceed to walk along the orientation of K. If we go over a positive crossing or go under a negative crossing we label increase out lablel by 1 and apply it to the next arc. Conversely, if we go under a positive crossing or go over a negative crossing we decrease our label 1. See Figure 9 for a labeling of arcs.

See Figure 10 for a coloring of a virtual knot diagram. In Figure 9 and Figure

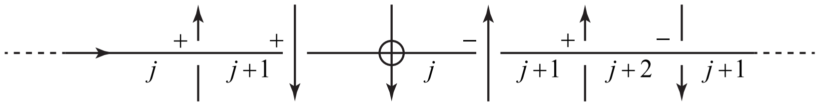


Figure 9:

10, the sign of each crossing is denoted by + or −. We can check, using induction on the number of crossings, that any virtual knot diagram can be colored.

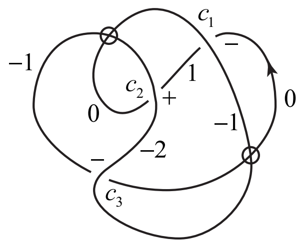


Figure 10:

Assume that K is colored as previously mentioned. We will assign an integer for each crossing. Let c be a positive crossing or a negative crossing in K and p, q, r, s be the labels of the arcs around c as shown in Figure 11. Then the *grade* $g(c)$ of c is defined to be the integer $p - s$, which is independent of colorings of K.

Figure 11:

We define $h_K(t)$ by the equation

$$h_K(t) = \sum_{c \in C(K)} s(c)(t^{g(c)} - 1).$$

For example if K is the virtual knot diagram with crossings c_1, c_2 and c_3 as shown in Figure 10, then $g(c_1) = 1$, $g(c_2) = -2$, and $g(c_3) = 1$. Then we can calculate

$h_K(t)$ as following

$$\begin{aligned} h_K(t) &= \sum_{c\in C(K)} s(c)(t^{g(c)}-1) \\ &= -(t-1)+(t^{-2}-1)-(t-1) \\ &= (t^{-2}-1)-2(t-1). \end{aligned}$$

For a classical knot diagram K, $h_K(t)=0$ since each crossing in K has grade 0.

If K_1 and K_2 are virtual knot diagrams related by a first Reidemeister move and colored as shown in Figure 12, then for the crossing c in K_1, $g(c)=0$ and

$$h_{K_1}(t)-h_{K_2}(t)=s(c)(t^{g(c)}-1)=0.$$

Similarly for the other types of the first Reidemeister move we get similar result. Then we have the following

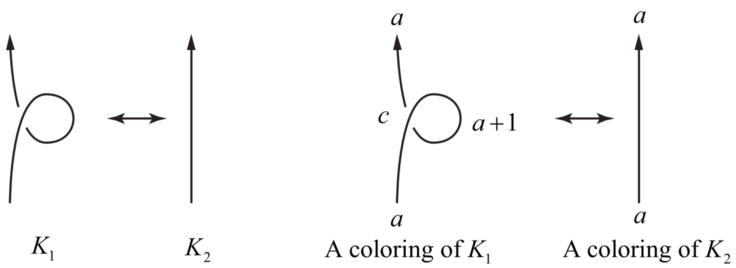


Figure 12:

Lemma 2.1. *If K_1 and K_2 are virtual knot diagrams related by a first Reidemeister move then*

$$h_{K_1}(t)=h_{K_2}(t).$$

There are two types of second Reidemeister moves considering the orientations of the two strands involved with second Reidemeister moves as shown in Figure 13 and Figure 14. The two crossings c_1 and c_2 appearing in second Reidemeister move in Figure 13 and in Figure 14 have different signs and grades in general. We will calculate the difference between $h_{K_1}(t)$ and $h_{K_2}(t)$ when K_1 and K_2 are related by a second Reidemeister move.

Lemma 2.2. *If K_1 and K_2 are virtual knot diagrams related by a second Reidemeister move then*

$$h_{K_1}(t)-h_{K_2}(t)=(t^{i}-1)-(t^{-i}-1)$$

for some integer i.

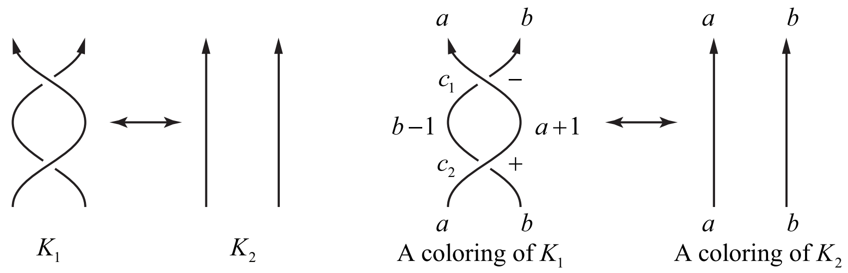


Figure 13:

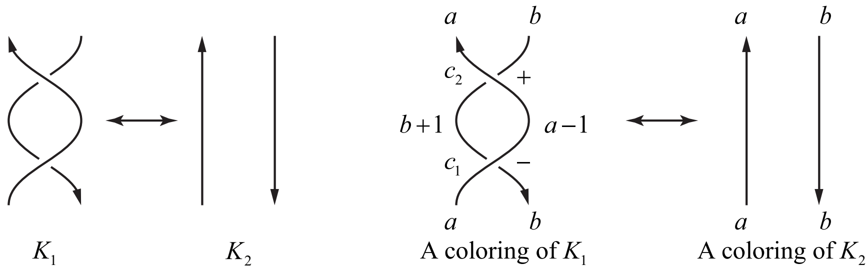


Figure 14:

Proof. Let K_1 and K_2 differ by a second Reidemeister move and be colored as shown in Figure 13. For the two crossings c_1 and c_2 in K_1,

$$g(c_1) = b - a - 1 = -g(c_2).$$

Then

$$\begin{aligned} h_{K_1}(t) - h_{K_2}(t) &= s(c_1)(t^{g(c_1)} - 1) + s(c_2)(t^{g(c_2)} - 1) \\ &= -(t^{b-a-1} - 1) + (t^{-b+a+1} - 1) \\ &= (t^i - 1) - (t^{-i} - 1), \end{aligned}$$

for some integer $i = -b + a + 1$. Similarly, if K_1 and K_2 are related by a second Reidemeister move of type in Figure 14, we can also show that $h_{K_1}(t) - h_{K_2}(t) = (t^i - 1) - (t^{-i} - 1)$ for some integer i. □

We note that $s(c_1) = -s(c_2)$ and $g(c_1) = -g(c_2)$ in the proof of Lemma 2.2. If K_1 is deformed to K_2 by a third Reidemeister move and c_1, c_2, c_3 are three crossings in K_1 involved with the move, then there are corresponding crossings c'_1, c'_2, c'_3 in K_2 such that c_n and c'_n has the same sign and grade for $n = 1, 2, 3$. Using this observation we can show the following

Lemma 2.3. *If K_1 and K_2 are virtual knot diagrams related by a third Reidemeister move then*

$$h_{K_1}(t) = h_{K_2}(t).$$

Proof. Let K_1 and K_2 differ by a third Reidemeister move and be colored as shown in Figure 15. For the three crossings c_1, c_2, c_3 in K_1 and the three crossings c_1', c_2', c_3' in K_2, we can see that $s(c_n) = s(c_n')$ and $g(c_n) = g(c_n')$ for $n = 1, 2, 3$. Then

$$\begin{aligned} h_{K_1}(t) - h_{K_2}(t) &= \sum_{n=1}^{3} s(c_n)(t^{g(c_n)} - 1) - \sum_{n=1}^{3} s(c_n')(t^{g(c_n')} - 1) \\ &= 0. \end{aligned}$$

Similarly if K_1 and K_2 are related by other types of third Reidemeister moves, we can show that $h_{K_1}(t) - h_{K_2}(t) = 0$. □

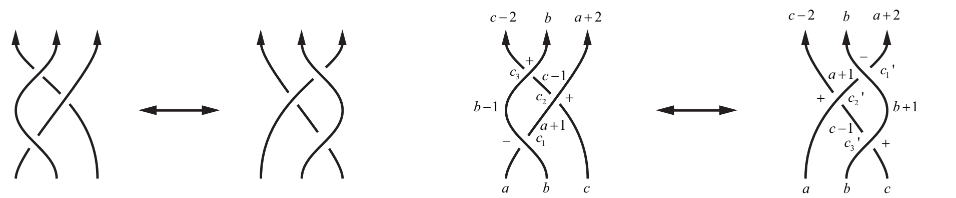


Figure 15:

From the coloring of K_1 in Figure 15, we see that $g(c_1) = a - b + 1$, $g(c_2) = a - c + 2$ and $g(c_3) = b - c + 1$. It is noteworthy that $g(c_2) = g(c_1) + g(c_3)$. If three crossings in a knot diagram do not satisfy such an equation, then we may not apply third Reidemeister move at any deformation of the knot diagram. This will help us design a sequence of Reidemeister moves to deform a virtual knot diagram to another one.

By combining Lemma 2.1, Lemma 2.2 and Lemma 2.3, we get a lower bound for the number of second Reidemeister moves deforming a virtual knot diagram to another one as following.

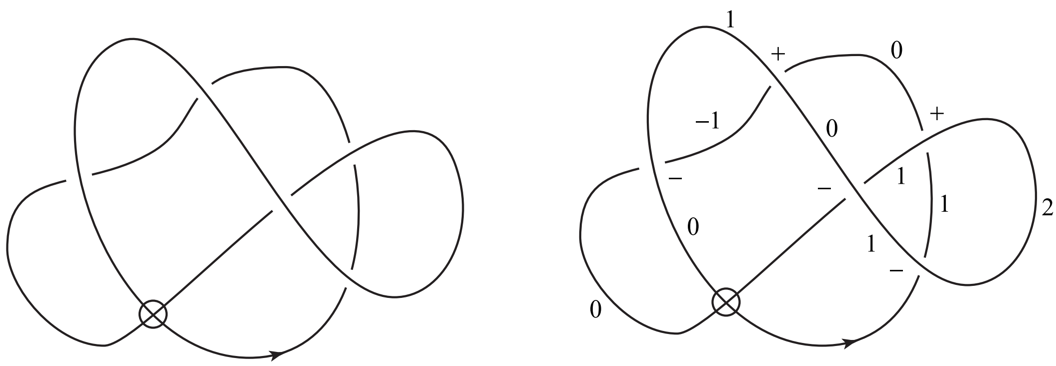
Theorem 2.4. *If K_1 and K_2 are equivalent virtual knot diagrams then we get the following inequality for the number $n_2(K_1, K_2)$ of second Reidemeister moves in a sequence of Reidemeister moves deforming K_1 to K_2.*

$$n_2(K_1, K_2) \geq \frac{1}{2} \sum_{i \in \mathbb{Z}} |a_i|,$$

where $h_{K_1}(t) - h_{K_2}(t) = \sum_{i \in \mathbb{Z}} a_i(t^i - 1)$ and $a_0 = 0$.

The difference of cowrithes of two equivalent knot diagrams gives a lower bound for the number of second Reidemeister moves and third Reidemeister moves needed to deform one to the other ([3]). The result can be naturally extended to virtual knot diagrams, but cowrithe is invariant under second Reidemeister move of the type in Figure 14. We will denote the trivial knot diagram by O.

Example 2.5. Let K_1 be a virtual knot diagram and colored as shown in Figure 16. Since $w(K_1) = 1$, $n(K_1, O) \geq 1$. Since $h_{K_1}(t) = -2(t^{-1} - 1) + 2(t - 1)$ by applying Theorem 2.4 we see that $n_2(K_1, O) \geq 2$. In fact we can deform K_1 to the trivial knot diagram by using second Reidemeister move twice and the first Reidemeister move once and $n(K_1, O) = 3$. Note that the cowrithe of K_1 is 1 and we may not determine $n_2(K_1, O)$ with cowrithe of K_1 and with a sequence of Reidemeister moves.



K_1 A coloring of K_1

Figure 16:

If two given virtual knot diagrams seem to be equivalent, we can try to deform one to the other by using Reidemeister moves and virtual moves. By using grades of crossings we may get hints to check whether the two virtual knot diagrams are equivalent or not. If the grade of a crossing is 0 then it may be cancelled by a first Reidemeister move. If two crossings c_1 and c_2 in a virtual knot diagram have different signs with $g(c_1) + g(c_2) = 0$ then c_1 and c_2 may be cancelled by a second Reidemeister move at some step of a sequence of Reidemeister moves. If a triple (c_1, c_2, c_3) of crossings in a virtual knot diagram satisfy $g(c_1) = g(c_2) + g(c_3)$, we may apply third Reidemeister move at some step of a sequence of Reidemeister moves. We can also try to use the following result to find the minimal number of third Reidemeister moves needed in a deformation.

In [5], we defined a polynomial $q_K(t)$ of a virtual knot diagram K. For a chord c of a Gauss diagram, let r_+ and r_- be the numbers of positive chords and negative

$C(K_2)$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
Sign	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$
Grade	-1	-2	1	2	-1	1	0	0

Table 1:

chords from left to right respectively and let l_+ and l_- be the numbers of positive chords and negative chords from right to left respectively. We define the *degree* $\deg(c)$ of a crossing c by the equation

$$\deg(c) = r_+ - r_- - l_+ + l_-.$$

For $c_1, c_2 \in C(K)$ we also define $\deg(c_1, c_2)$ by the formula

$$\deg(c_1, c_2) = \deg(c_1) - \deg(c_2).$$

Theorem 2.6. ([5]) *Let K_1 and K_2 be two equivalent virtual knot diagrams and S be a sequence of Reidemeister moves deforming K_1 to K_2. Let $g_K(t) = q_K(t) - q_K(1)$ and $g_{K_1}(t) - g_{K_2}(t) = \sum_{i=m}^{l} a_i(t^i - 1)$. Then we get inequalities*

$$n_3(S) \geq \frac{\sum_{i=m}^{l} |a_i|}{3} \text{ and } n_3(S) \geq |\sum_{i=m}^{l} a_i|,$$

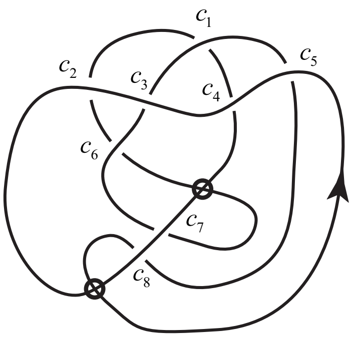
where $n_3(S)$ is the number of third Reidemeister moves in S.

Example 2.7. Let K_2 be a virtual knot diagram and $c_1, \cdots, c_8$ be crossings in K_2 as shown in Figure 17. We can easily calculate the sign and the grade of c_i for each $i = 1, \cdots, 8$ as arranged in Table 1.

Then $h_{K_2}(t) = (t^{-2} - 1) + 2(t^{-1} - 1) - 2(t - 1) - (t^2 - 1)$. By applying Theorem 2.4 we see that $n_2(K_2, O) \geq 3$. Since $w(K_2) = 2$, we see that $n_1(K_2, O) \geq 2$.

From the table we may guess that c_7 and c_8 can be cancelled by first Reidemeister moves. A crossing in the set $\{c_1, c_5\}$ can be cancelled with a crossing in the set $\{c_3, c_6\}$ by a second Reidemeister move. Also the crossings c_2 and c_4 may be cancelled by a second Reidemeister move. There are also triples (c_i, c_j, c_k) of crossings such that $g(c_i) = g(c_j) + g(c_k)$ and we may try to apply a third Reidemeister move for the three crossings. Recall that $s(c_i) = s(c_i')$ and $g(c_i) = g(c_i')$ for $i = 1, 2, 3$ in a third Reidemeister move as shown in Figure 15.

Since $q_{K_2}(t) - q_{K_2}(1) = (t^{-1} - 1) + (t - 1)$, we see that $n_3(K_2, O) \geq 1$. In fact we can show that $n_1(K_2, O) = 2$, $n_2(K_2, O) = 3$, $n_3(K_2, O) = 1$ and $n(K_1, K_2) = 6$ as illustrated in Figure 18. In the figure, c_i is transformed to c_i', which has the same sign and grade with c_i for $i = 3, 4$, by a third Reidemeister move. The pairs (c_2, c_4'),



K_2

Figure 17:

(c_3', c_5) and (c_1, c_6) are cancelled by second Reidemeister moves in the sequence of Reidemeister moves and virtual moves. Also note that $g(c_3) = g(c_1) + g(c_4)$ and we can apply a third Reidemeister move for a local part of the diagram containing the three crossings c_1, c_3 and c_4.

We have studied lower bounds for the minimal numbers of Reidemeister moves for two equivalent virtual knot diagrams.

Question. Find upper bounds for the minimal numbers of Reidemeister moves for two equivalent virtual knot diagrams.

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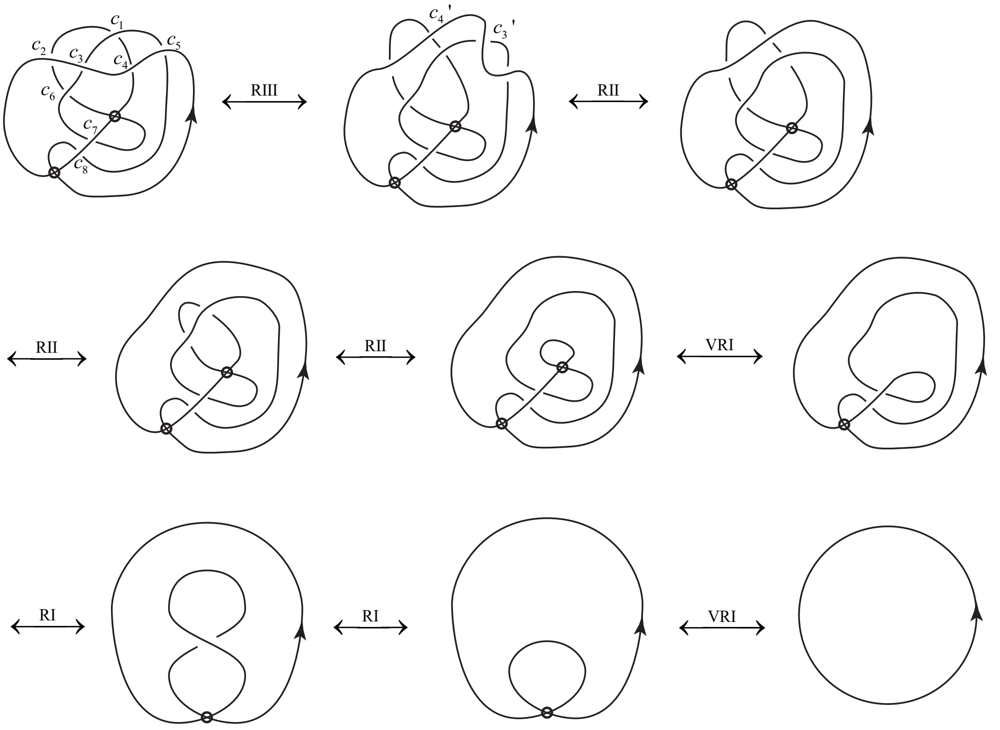

Figure 18:

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