The Relation Between Units and Nilpotents

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Abstract. We discuss the relation between units and nilpotents of a ring, concentrating on the transitivity of units on nilpotents under regular group actions. We first prove that for a ring R, if $U(R)$ is right transitive on $N(R)$, then Köthe's conjecture holds for R, where $U(R)$ and $N(R)$ are the group of all units and the set of all nilpotents in R, respectively. A ring is called *right UN-transitive* if it satisfies this transitivity, as a generalization, a ring is called *unilpotent-IFP* if $aU(R) \subseteq N(R)$ for all $a \in N(R)$. We study the structures of right UN-transitive and unilpotent-IFP rings in relation to radicals, NI rings, unit-IFP rings, matrix rings and polynomial rings.

1. Preliminaries

All rings considered in this article are associative with identity unless otherwise stated. Let R be a ring. The group of all units and the set of all idempotents in R are written by $U(R)$ and $I(R)$, respectively. A nilpotent element of R is said

to be a *nilpotent* for simplicity. The Wedderburn radical (i.e., sum of all nilpotent ideals), the upper nilradical (i.e., the sum of all nil ideals), the lower nilradical (i.e., the intersection of all prime ideals), and the set of all nilpotents in R are denoted by $N_0(R), N^*(R), N_*(R)$, and $N(R)$, respectively. Write $N(R)' = N(R)\backslash\{0\}$. It is well-known that $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$. We denote by $\mathbb{Z}_n$ the ring of integers modulo n. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0.

A ring R is usually called *reduced* if $N(R) = 0$. Due to Bell [2], a ring R is called *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. Both commutative rings and reduced ring are IFP clearly. There are many non-reduced commutative rings (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is usually called *abelian* if every idempotent is central. IFP rings are abelian by a simple computation. A ring R is called *NI* [14] if $N^*(R) = N(R)$. It is clear that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. It is easily checked that IFP rings are NI but not conversely.

Following [11], a ring R is said to be *unit-IFP* if $ab = 0$ for $a, b \in R$ implies $aU(R)b = 0$. Unit-IFP rings are abelian by [11, Lemma 1.2]. IFP rings are clearly unit-IFP. The rings below show that the classes of unit-IFP rings and NI rings do not imply each other.

Example 1.1. (1) Let $K = \mathbb{Z}_2$ and $A = K\langle a, b \rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Identify a, b with their images in R for simplicity. Note that the ring coproduct $R = R_1 *_K R_2$ with $R_1 = K[a]$ and $R_2 = \frac{K[b]}{K[b]b^2K[b]}$, where $K[a]$ (resp., $K[b]$) is the polynomial ring with an indeterminate a (resp., b) over K. Then R is unit-IFP but not NI by [11, Example 1.1] and [1, Example 4.8].

(2) Consider $T_n(R)$ over an NI ring R for $n \geq 2$. Then $T_n(R)$ is NI by [8, Proposition 4.1] but not unit-IFP by [11, Lemma 1.2] since $T_n(R)$ is not abelian.

2. Transitivity and Unilpotent-IFP Rings

In this section we introduce two kinds of ring properties through which we study the relation between units and nilpotents. The first is related to the transitivity of units on nilpotents under regular group actions and the second is related to the property of inserting units into nilpotent products of elements.

Recall first the following definitions. Let R be a ring and suppose that there exist two (left and right) regular actions of $U(R)$ on $N(R)$. The *orbit* of $a \in N(R)$ is $o_r(a) = \{au \mid u \in U(R)\} = aU(R)$ (resp., $o_l(a) = \{ua \mid u \in U(R)\} = U(R)a$) under the right (resp., left) regular action of $U(R)$ on $N(R)$. We write $o(a)$ when $o_l(a) = o_r(a)$.

For a ring R, $U(R)$ shall be called *right* (resp., *left*) *transitive on $N(R)$* provided

that $N(R) = 0$ or else, if there exists $a \in N(R)'$ such that $o_r(a) = N(R)'$ (resp., $o_l(a) = N(R)'$) under the right (resp., left) regular action. If $U(R)$ is both right and left transitive on $N(R)$, then $U(R)$ is said to be *transitive* on $N(R)$. Observe that $U(R)$ in a reduced ring R is transitive on $N(R)$ by definition. However there exists a unit-IFP ring that does not satisfy the transitivity as in the part (2) of the following remark.

Remark 2.1. (1) Let R be a non-reduced ring. We claim that if $U(R)$ is right (resp., left) transitive on $N(R)$ then $o_r(q) = N(R)'$ (resp., $o_l(q) = N(R)'$) for any $q \in N(R)'$. Let $U(R)$ be right transitive on $N(R)$. Then $o_r(a) = aU(R) = N(R)'$ for some $a \in N(R)'$. For any $q \in N(R)'$, there exists $u \in U(R)$ such that $au = q$. Then $a = qu^{-1}$, hence it implies that $o_r(a) = qu^{-1}U(R) = o_r(q)$. Thus $o_r(q) = N(R)'$ for any $q \in N(R)'$. The argument for the left case is done by symmetry.

(2) Consider the unit-IFP ring R in Example 1.1(1). Then $U(R) = \{k_0 + k_1 b + bfb \mid k_0 \in K \backslash \{0\}, k_1 \in K$ and $f \in R\}$ by [13, Theorem 1.2], and $N(R) = \{kb + bfb \mid k \in K$ and $f \in R\}$ with $N(R)^2 = 0$ by [13, Theorem 1.3]. So $o_r(kb + bfb) = \{k_0kb + b(k_0f)b \mid k_0 \in K \backslash \{0\}\} = o_l(kb + bfb) \subsetneqq N(R)'$ for any $0 \neq kb + bfb \in N(R)'$. In fact, if $k = 0$ then $b \notin o_r(kb + bfb) = \{b(k_0f)b\}$; if $bfb = 0$ then $bab \notin o_r(kb + bfb) = \{k_0kb\}$; and if $k \neq 0$, $bfb \neq 0$ then $b, bab \notin o_r(kb + bfb) = \{k_0kb + b(k_0f)b \mid k_0k \neq 0$ and $b(k_0f)b \neq 0)\}$. Thus $U(R)$ is neither right nor left transitive on $N(R)$.

We shall call a ring R *right* (resp., *left*) *UN-transitive* if $U(R)$ is right (resp., left) transitive on $N(R)$, and R is called *UN-transitive* if $U(R)$ is transitive on $N(R)$. We first have the following by Remark 2.1(1).

Lemma 2.2. *A non-reduced ring R is right (resp., left) UN-transitive if and only if $o_r(a) = N(R)'$ (resp., $o_l(a) = N(R)'$) for any $a \in N(R)'$.*

Given a ring R and $k \geq 1$, write $Nil_k(R) = \{a \in R \mid a^k = 0\}$. Note $N(R) = \cup_{i=1}^{\infty} Nil_i(R)$. Note that Köthe's conjecture (i.e., the sum of two nil left ideals is nil) holds for a given ring R when $N(R)$ is additively closed.

Proposition 2.3. *Let R be a right or left UN-transitive ring. Then we have the following assertions.*

(1) $N(R)^2 = 0$.

(2) $N(R) = Nil_2(R)$ and $N(R)$ is a subring of R.

(3) *Köthe's conjecture holds for R.*

Proof. (1) If $N(R) = 0$, then we are done. Assume $N(R) \neq 0$. Let $q \in N(R)'$, say $q^n = 0$ with $n \geq 2$. Then $N(R)' = o_r(q)$ by Lemma 2.2, whence $q^{n-1} = qu$ for some $u \in U(R)$. Multiplying by q on the left, we get $0 = q^2u$ and so $q^2 = 0$. This concludes $N(R) = Nil_2(R)$.

Further, we claim $N(R)^2 = 0$. Let $p, q \in N(R)'$. Then $p^2 = 0 = q^2$ as above. Since $o_r(p) = N(R)' = o_r(q)$ by Lemma 2.2, $pu_1 = q$ and $qu_2 = p$ for some $u_1, u_2 \in U(R)$. So $qpu_1 = q^2 = 0$ and $pqu_2 = p^2 = 0$, whence $pq = 0 = qp$.

(2) $N(R) = Nil_2(R)$ by (1). Let $p, q \in N(R)$. Then $(p+q)^2 = p^2 + pq + qp + q^2 =$

0 by (1), so that $p + q \in N(R)$. Consequently, $N(R)$ is a subring of R.

(3) This is evident from (2).

The proof for the left UN-transitive case is similar.   □

Each converse of Proposition 2.3 needs not hold by Remark 2.1(2). The rings below shall provide the motivation for the main argument in this article.

Example 2.4. (1) Let D be a division ring and $R = T_2(D)$. Then R is NI with $N(R) = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ and $U(R) = \{(a_{ij}) \in R \mid a_{11}, a_{22} \in U(D)\}$, noting $U(D) = D \backslash \{0\}$. So, for any $M = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in N(R)'$ (i.e., $a \neq 0$), we have

$$o_l(M) = \begin{pmatrix} 0 & U(D)a \\ 0 & 0 \end{pmatrix} = U(D)e_{12} \text{ and } o_r(M) = \begin{pmatrix} 0 & aU(D) \\ 0 & 0 \end{pmatrix} = U(D)e_{12}.$$

Thus $o_l(M) = o_r(M) = N(R)'$ and so R is UN-transitive.

(2) We follow the construction in [6, Example 1.2(2)] which applies [15, Definition 1.3]. Let A be a commutative ring with an endomorphism σ and M be an A-module. For $A \oplus M$, the addition and multiplication are given by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 \sigma(r_2))$. Then this construction also forms a ring.

For a field K, let $K(x)$ be the quotient field of the polynomial ring $K[x]$ and σ be the non-surjective monomorphism of $K(x)$ defined by $\sigma\left(\frac{f(x)}{g(x)}\right) = \frac{f(x^2)}{g(x^2)}$. Let $R = K(x) \oplus K(x)$ with the preceding multiplication. Note that $U(R) = U(K(x)) \oplus K(x) = K(x)' \oplus K(x)$, where $K(x)' = K(x) \backslash \{0\}$. Then R is isomorphic to the subring

$$\left\{ \begin{pmatrix} h(x) & k(x) \\ 0 & \sigma(h(x)) \end{pmatrix} \mid h(x), k(x) \in K(x) \right\} \text{ of } T_2(K(x)),$$

$$\text{via } (h(x), k(x)) \mapsto \begin{pmatrix} h(x) & k(x) \\ 0 & \sigma(h(x)) \end{pmatrix},$$

by the argument in [6, Example 1.2(2)]. Since $T_2(K(x))$ is clearly NI, R is NI by [8, Proposition 2.4]. Note that $N(R) = \{0\} \oplus K(x) = N^*(R)$ and $R/N^*(R) \cong K(x)$.

For any $a = (0, f(x)) \in N(R)'$, we have

$$o_l(a) = (K(x)' \oplus K(x))(0, f(x)) = \{0\} \oplus K(x)'f(x) = \{0\} \oplus K(x)'(= N(R)')$$

and $o_r(a) = (0, f(x))(K(x)' \oplus K(x)) = \{0\} \oplus f(x)\sigma(K(x)')$. Here if $f(x) \in \sigma(K(x)')$ then $f(x)\sigma(K(x)') = \sigma(K(x)')$; and if $f(x) \notin \sigma(K(x)')$ then $1 \notin f(x)\sigma(K(x)')$. Thus $f(x)\sigma(K(x)') \subsetneq K(x)'$, and $o_r(a) \subsetneq o_l(a)$ follows. These also imply that R is left UN-transitive but not right UN-transitive.

(3) Let $R = K(x) \oplus K(x)$ be the ring and σ be the non-surjective monomorphism of $K[x]$, as in (2). Give R the multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 +$

$m_1 r_2$) which is defined by [15, Definition 1.3]. Then this construction also forms a ring, and R is isomorphic to the subring

$$\left\{ \begin{pmatrix} h(x) & 0 \\ k(x) & \sigma(h(x)) \end{pmatrix} \mid h(x), k(x) \in K(x) \right\} \text{ of } T_2'(K(x)),$$

$$\text{via } (h(x), k(x)) \mapsto \begin{pmatrix} h(x) & 0 \\ k(x) & \sigma(h(x)) \end{pmatrix},$$

by the argument in [6, Example 1.2(1)], where $T_n'(R)$ denotes the n by n lower triangular matrix ring over R. Then, by a similar argument to (2), R is an NI ring with $U(R) = K(x)' \oplus K(x)$ and $N(R) = \{0\} \oplus K(x) = N^*(R)$.

Let $a = (0, f(x))$ be arbitrary in $N(R)'$. Then we can show that $o_l(a) \subsetneq o_r(a)$ by the symmetric computation to (2), and moreover R is right UN-transitive but not left UN-transitive on $N(R)$.

Next we consider a generalized condition of one-sided UN-transitivity by considering "$\subseteq$", in place of "$=$".

Proposition 2.5. (1) *For a ring R, the following conditions are equivalent:*

 (i) $ab \in N(R)$ *for* $a, b \in R$ *implies* $aU(R)b \subseteq N(R)$;

 (ii) $a \in N(R)$ *implies* $ras \in N(R)$ *for all* $r, s \in U(R)$;

 (iii) $o_l(a) \subseteq N(R)$ *for all* $a \in N(R)$;

 (iv) $o_r(a) \subseteq N(R)$ *for all* $a \in N(R)$;

 (v) *If* $a_1 \cdots a_n \in N(R)$ *for* $a_1, \ldots, a_n \in R$ *and* $n \geq 2$, *then for all* $u_1, \ldots, u_{n+1} \in U(R)$, $u_1 a_1 u_2 a_2 \cdots u_n a_n u_{n+1} \in N(R)$.

(2) *Let R be a ring which satisfies any of the preceding conditions. Then $u + a \in U(R)$ for all $u \in U(R)$ and $a \in N(R)$.*

Proof. (1) (i) $\Rightarrow$ (ii): Assume that (i) holds. Let $a \in N(R)$. Then we have the following implications: $a = 1a \in N(R) \Rightarrow 1ra \in N(R)$ for all $r \in U(R) \Rightarrow ra = (ra)1 \in N(R) \Rightarrow (ra)s1 \in N(R)$ for all $s \in U(R) \Rightarrow ras \in N(R)$.

(v) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious. The proof of (i) $\Rightarrow$ (v) is done by using the condition (i), iteratively.

(iv) $\Rightarrow$ (i): Assume that (iv) holds. Let $ab \in N(R)$ for $a, b \in R$. Then $ba \in N(R)$ and so $o_r(ba) = baU(R) \subseteq N(R)$ by assumption. Thus $aU(R)b \subseteq N(R)$, showing that the condition (i) is satisfied.

(2) Note that $1 + n \in U(R)$ for all $n \in N(R)$. Let $u \in U(R)$ and $a \in N(R)$. Then $u + a = u(1 + u^{-1}a)$ and we have $u^{-1}(u + a) = 1 + u^{-1}a$. Since $u^{-1}a \in N(R)$ by (1), $1 + u^{-1}a \in U(R)$. Thus $u + a = u(1 + u^{-1}a) \in U(R)$. $\square$

Based on the facts above, we consider the following as a generalization of not only one-sided UN-transitive rings, but unit-IFP rings and NI rings.

A ring shall be called *unilpotent-IFP* if it satisfies any of the conditions in Proposition 2.5(1). In this case, we will usually use the condition (i).

Every NI ring R is unilpotent-IFP since $R/N^*(R)$ is reduced, and every unit-IFP ring is also unilpotent-IFP by [11, Lemma 1.2], but each converse needs not hold by Example 1.1. Moreover it is clear that the class of unilpotent-IFP rings contains right (left) UN-transitive rings, but not conversely by Example 1.1 or Example 2.4(2, 3).

The non-unilpotent-IFP rings below provide useful manner to argue about the unilpotent-IFPness of given rings.

Example 2.6. (1) Consider $Mat_n(A)$ over any ring A for $n \geq 2$, and take $\alpha = e_{21}, \gamma = e_{22}$ and $\beta = e_{21} + e_{12} + e_{33} + \cdots + e_{nn}$ in $Mat_n(A)$. Then $\alpha\gamma = 0$ but $\alpha\beta\gamma = e_{22} \in I(Mat_n(A))$, noting $\beta \in U(Mat_n(A))$. Hence $\alpha U(Mat_n(A))\gamma \not\subseteq N(Mat_n(A))$.

(2) Consider the rings constructed in [10, Example 2(2)]. Let $K = \mathbb{Z}_2$, the field of integers modulo 2, $A = K[a]$, and $B = K[b]$, where $K[a]$ and $K[b]$ be the polynomial rings with indeterminates a and b over K, respectively. Let J be the ideal of B generated by b^2. Set $C = A \oplus A$ and $D = B/J$. Identify b with its image for simplicity. Let $R = C *_K D$ which stands for the ring coproduct of C and D over K. Then $1 + b$ is a unit and $(1, 0)b(0, 1)$ is a nilpotent of R, in fact, $[b(1, 0)b(0, 1)]^2 = 0$. But

$$[(1 + b)(1, 0)b(0, 1)]^{2^n} = (1, 0)b(0, 1)[b(1, 0)b(0, 1)]^{2^n - 1} + [b(1, 0)b(0, 1)]^{2^n},$$

which is not a nilpotent, for all n. Hence $[b(1, 0)b(0, 1)]U(R)[b(1, 0)b(0, 1)] \not\subseteq N(R)$.

Based on the structures of rings in Example 2.4, a ring R shall be called *left unilpotent-duo* (resp., *right unilpotent-duo*) if $o_r(a) \subseteq o_l(a)$, i.e., $aU(R) \subseteq U(R)a$ (resp., $o_l(a) \subseteq o_r(a)$, i.e., $U(R)a \subseteq aU(R)$)) for all $a \in N(R)$. A ring is said to be *unilpotent-duo* if it is both left and right unilpotent-duo. Reduced rings are clearly unilpotent-duo. This unilpotent-duo property is not left-right symmetric as we see in Example 2.4(2, 3). In fact the ring R in Example 2.4(2) is left unilpotent-duo but not right unilpotent-duo; while, the ring R in Example 2.4(3) is right unilpotent-duo but not left unilpotent-duo.

Theorem 2.7. (1) *Every right or left unilpotent-duo ring is unilpotent-IFP.*

(2) *Let R be a ring with an involution $*$. Then*

(i) *R is left unilpotent-duo if and only if it is right unilpotent-duo.*

(ii) *R is left UN-transitive if and only if it is right UN-transitive. Especially, if R is non-reduced right UN-transitive, there exists $a \in N(R)'$ such that*

$$o_l(a) = o_l(a^*) = o_r(a) = o_r(a^*) = N(R)'.$$

(3) *Let R be a non-reduced ring. If R is UN-transitive with $o(a) = N(R)'$ for some $a \in N(R)'$, then R is unilpotent-duo.*

Proof. (1) Let R be a right unilpotent-duo ring and suppose $a \in N(R)$ for $a \in R$. Then $a^n = 0$ for some $n \geq 1$. Let $u \in U(R)$. Since R is right unilpotent-duo, $ua = au_1$ for some $u_1 \in U(R)$ and $u_1ua = au_2$ for some $u_2 \in U(R)$. Continuing in

this manner, there exists $u_n \in U(R)$ such that $u_{n-1}ua = au_n$ where $u_{n-1} \in U(R)$ for $n \geq 2$. Then

$$(ua)^n = ua(ua)^{n-1} = a(u_1ua)(ua)^{n-2} = a^2(u_2ua)(ua)^{n-3} = \cdots$$
$$= a^{n-1}(u_{n-1}ua) = a^n u_n = 0,$$

and hence $ua \in N(R)$. Thus R is unilpotent-IFP. The proof for the left unilpotent-duo ring is similar.

2-(i) Let R be left unilpotent-duo and suppose that $a \in N(R)$ and $u \in U(R)$. Then $a^* \in N(R)$ and $u^* \in U(R)$. Since R is left unilpotent-duo, $a^*u^* = va^*$ for some $v \in U(R)$. This yields

$$ua = ((ua)^*)^* = (a^*u^*)^* = (va^*)^* = av^* \in aU(R),$$

implying that R is right unilpotent-duo. The converse is shown similarly.

2-(ii) First, if $N(R) = 0$ then we are done. We assume $N(R) \neq 0$ and let R be left UN-transitive. Then $o_l(a) = U(R)a = N(R)'$ for some $a \in N(R)'$. Let $b \in N(R)'$ with $b = ua$ for some $u \in U(R)$. Then $a^*u^* = b^* \in N(R)$. Since $N(R)' = U(R)a$, $b^* = a^*u^* = va$ for some $v \in U(R)$, and so

$$b = (b^*)^* = (va)^* = a^*v^* \in a^*U(R).$$

This implies $N(R)' = a^*U(R) = o_r(a^*)$, noting $a^*U(R) \subseteq N(R)'$ by Proposition 2.5. Therefore R is right UN-transitive. The converse is similar to above.

Next, suppose that R is non-reduced UN-transitive. Then, by the preceding argument, there exists $a \in N(R)'$ such that

$$U(R)a = o_l(a) = N(R)' = o_r(a^*) = a^*U(R).$$

From this, we obtain $a^* = ua$ and $a = a^*v$ for some $u, v \in U(R)$; hence $a = u^{-1}a^*$ and $a^* = av^{-1}$. Thus we also have

$$N(R)' = o_l(a) = U(R)a = U(R)u^{-1}a^* = U(R)a^* = o_l(a^*)$$

and

$$N(R)' = o_r(a^*) = a^*U(R) = av^{-1}U(R) = aU(R) = o_r(a).$$

Therefore $o_l(a^*) = o_l(a) = o_r(a^*) = o_r(a) = N(R)'$.

(3) Suppose that R is UN-transitive with $o(a) = N(R)'$ for some $a \in N(R)'$. Consider $U(R)b$ with $b \in N(R)'$ and let $c = ub \in U(R)b$. Since $N(R)' = o(a) = U(R)a = aU(R)$, $b = ga = ag_1$ for some $g, g_1 \in U(R)$. So we have

$$c = ub = uga = ag_2 = ag_1g_1^{-1}g_2 = bg_1^{-1}g_2 \in bU(R),$$

where $(ug)a = ag_2$ for some $g_2 \in U(R)$ because $N(R)' = o(a)$. Thus R is right unilpotent-duo. The proof of left unilpotent-duo case is similar. $\qquad\square$

The converse of Theorem 2.7(1) does not hold in general by Example 3.8(1) to follow.

Let K be a commutative ring and G be any group. Recall the standard involution $*$ on the group ring KG in [3], i.e., $(\sum a_i g_i)^* = \sum a_i g_i^{-1}$ for all $a_i \in K$ and $g_i \in G$. Thus we obtain the following as a corollary of Theorem 2.7.

Corollary 2.8. *Let K be a commutative ring and G be any group. Then we have the following results.*

(1) The group ring KG is left unilpotent-duo if and only if it is right unilpotent-duo.

(2) The group ring KG is left UN-transitive if and only if it is right UN-transitive.

One may ask whether if R is a commutative ring then R is UN-transitive. However the answer is negative by the following.

Example 2.9. Consider the infinite direct product $R = \prod_{i=1}^{\infty} \mathbb{Z}_{2^i}$, and the subring S of R generated by the direct sum $\oplus_{i=1}^{\infty} \mathbb{Z}_{2^i}$ and the identity of R.

Let $(a_i) \in N(R)'$ such that $o_r((a_i)) = (a_i)U(R) = N(R)'$. Then $(a_i)^k = 0$ for some $k \geq 2$, and $(b(i)_j) \in (a_i)U(R)$ for all $i \geq 1$, where $b(i)_i = 2$ and $b(i)_j = 0$ for $i \neq j$. From $(a_i)U(R) = N(R)'$, we obtain that $[(a_i)(u_i)]^k = (a_i)^k(u_i)^k = 0$ for all $(u_i) \in U(R)$; hence $(c_i)^k = 0$ for all $(c_i) \in N(R)$. However $(b(i)_j)^k \neq 0$ for all $i \geq k+1$, and so $(b(i)_j) \notin N(R)'$ for all $i \geq k+1$. This induces a contradiction because $(b(i)_j)^i = 0$. Thus there cannot exist $(a_i) \in N(R)'$ such that $o_r((a_i)) = N(R)'$. That is, R is not UN-transitive.

Let $(a_i) \in N(S)'$ such that $o_r((a_i)) = (a_i)U(S) = N(S)'$. Then there exists $h \geq 2$ such that $a_i = 0$ for all $i \geq h$. So, letting $(d_j) \in S$ such that $d_h = 2$ and $d_j = 0$ for all $j \neq h$, $(d_j) \notin (a_i)U(S)$ and $(d_j) \notin N(S)'$ follows. This induces a contradiction because $(d_j)^h = 0$. Thus there cannot exist $(a_i) \in N(S)'$ such that $o_r((a_i)) = N(S)'$. That is, S is not UN-transitive.

Recall that unilpotent-IFP rings need not be NI by Example 1.1(1). We see conditions under which unilpotent-IFP rings may be NI. Note that for a ring R, if R is one-sided UN-transitive, then R is unilpotent-IFP.

Theorem 2.10. *Let R be a non-reduced ring and suppose that there exists $a \in N(R)'$ such that $aR \subseteq N(R)$. If R is right or left UN-transitive, then R is an NI ring such that $N_0(R) = N_*(R) = N^*(R) = N(R) = aR = RaR$ and $N^*(R)^2 = 0$.*

Proof. By hypothesis, $aR \subseteq N(R)$ with $a \in N(R)'$. Suppose that R is right UN-transitive. Then $o_r(a) = N(R)'$ by Lemma 2.2, whence we have

$$N(R)' = aU(R) \subseteq aR \subseteq N(R),$$

from which we infer that $aR = N(R) = aU(R) \cup \{0\}$.

Now consider RaR. Since aR is nil, we see that $ras \in N(R)$ for all $r, s \in R$. By Proposition 2.3, $N(R)$ is a trivial subring of R (i.e., $N(R)^2 = 0$) and so RaR is also nil, entailing $RaR = N(R) = N^*(R)$. Furthermore, $aR = bR$ for any $b \in N(R)'$

since $o_r(a) = N(R)' = o_r(b)$ by Lemma 2.2. Consequently we now have

$$RaR = N(R) = aR = bR = RbR = N^*(R) \text{ for any } b \in R,$$

concluding that R is NI. Moreover since $N(R)^2 = 0$ by Proposition 2.3(1), we see that $N_0(R) = N_*(R) = N^*(R) = N(R)$ and $N^*(R)^2 = 0$.

The proof of the left transitive case can be done by symmetry. □

Regarding Theorem 2.10, it is evident that if R is an NI ring then R is a unilpotent-IFP ring such that $aR \subseteq N(R)$ for all $a \in N(R)$. But Example 2.9 illuminates that there exists an NI ring which is neither right nor left UN-transitive.

Polynomial rings over NI rings need not be NI by Smoktunowicz [16]. But if given a ring R satisfies the condition of Theorem 2.10, then $R[x]$ is NI as we see in the following.

Corollary 2.11. *Let R be a non-reduced ring and suppose that there exists $a \in N(R)'$ such that $aR \subseteq N(R)$. If R is left or right UN-transitive, then $R[x]$ is an NI ring such that $N(R[x]) = N^*(R[x]) = R[x]aR[x]$.*

Proof. By hypothesis and Theorem 2.10, we first obtain that $N^*(R)^2 = 0$ and $N_0(R) = N_*(R) = N^*(R) = N(R) = aR = RaR$. Then $R[x]$ is an NI ring such that $N(R[x]) = N^*(R[x]) = N^*(R)[x]$, by help of the proof of [8, Proposition 4.4]. This yields

$$N(R[x]) = N^*(R[x]) = N^*(R)[x] = (aR)[x] = (RaR)[x] = R[x]aR[x].$$

□

Remark 2.12. Let R be a ring and let $a \in N(R)'$. Then $a^k = 0$ for some $k \geq 2$. Assume $N^*(R) = aR = RaR$. Then we have

$$N^*(R)^k = (aR)^k = (a\underline{R})(aR)(aR)^{k-2} = (a\underline{a}R)(aR)^{k-2} = \cdots$$
$$= (a^{k-1}\underline{R})(aR) = a^{k-1}\underline{aR} = 0.$$

Next argue about the actual form of elements in $N^*(R)^k$. Let $b_i \in N^*(R)$ for $i = 1, 2, \ldots, k$. Then $b_i = ac_i$ for some $c_i \in R$. So we have

$$b_1 b_2 \cdots b_k = (ac_1)(ac_2) \cdots (ac_k) = (a\underline{c_1})(ac_2)(ac_3) \cdots (ac_k) = a\underline{ad_1}c_2(ac_3) \cdots (ac_k)$$
$$= a^2 \underline{d_1 c_2}(ac_3) \cdots (ac_k) = a^2 \underline{ad_2}c_3(ac_4) \cdots (ac_k) = a^3 \underline{d_2 c_3}(ac_4) \cdots (ac_k)$$
$$= \cdots = a^{k-1}\underline{d_{k-2}c_{k-1}}(ac_k) = a^{k-1}\underline{ad_{k-1}}c_k = a^k d_{k-1}c_k = 0,$$

where $c_1 a = ad_1$, $d_1 c_2 a = ad_2, \ldots, d_{k-2}c_{k-1}a = ad_{k-1}$ with $d_j \in R$ for $j = 1, 2, \ldots, k-1$.

3. Structure of Unilpotent-IFP Rings

In this section we study the structure of unilpotent-IFP rings as well as the relations between unilpotent-IFP rings and related rings, studying the structures of some kinds of unilpotent-IFP rings which are considered ordinarily in (noncommutative) ring theory.

We first note that the class of unilpotent-IFP rings is not closed under homomorphic images, since every ring is a homomorphic image of a free ring (which is reduced and therefore unilpotent-IFP). But we obtain elementary properties for unilpotent-IFP rings as follows. The direct product of rings R_i $(i \in \Lambda)$ is denoted by $\prod_{i \in \Lambda} R_i$.

Proposition 3.1. (1) *If S is a subring of a unilpotent-IFP ring R with the identity of R, then S is unilpotent-IFP.*

(2) *Let I be a nil ideal of a ring R. Then R/I is unilpotent-IFP if and only if so is R.*

(3) *Let $\{R_i \mid i \in \Lambda\}$ be a family of rings and $R = \prod_{i \in \Lambda} R_i$, where Λ is finite. Then R_i is a unilpotent-IFP ring for all $i \in \Lambda$ if and only if R is unilpotent-IFP.*

Proof. (1) Note that $U(S) \subseteq U(R)$ and $N(S) = S \cap N(R)$. Suppose that R is unilpotent-IFP and let $a \in N(S)$ and $u \in U(S)$. Then $au \in N(R) \cap S = N(S)$, and so S is unilpotent-IFP.

(2) Let I be a nil ideal of R. Note that

$$N(R/I) = \{a + I \mid a \in N(R)\} \text{ and } U(R/I) = \{u + I \mid u \in U(R)\}.$$

For $a \in N(R)$ and $u \in U(R)$, $au + I \in N(R/I)$ if and only if $au \in N(R)$. Thus the proof can be shown easily.

(3) Since Λ is finite, $N(R) = \prod_{i \in \Lambda} N(R_i)$. Note $U(R) = \prod_{i \in \Lambda} U(R_i)$. For $a = (a_i)_i \in N(R)$ and $u = (u_i)_i \in U(R)$, $au = (a_i u_i)_i \in N(R)$ if and only if $a_i u_i \in N(R_i)$ for all $i \in \Lambda$ since Λ is finite. Thus the proof can be shown easily. □

We next study some sorts of unilpotent-IFP rings which are able to provide plentiful information to related studies. For a ring R and $n \geq 2$, let $D_n(R)$ be the ring of all matrices in $T_n(R)$ whose diagonal entries are all equal and $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R)$ is isomorphic to $R[x]/x^n R[x]$.

Let R be any ring and $n \geq 2$. Then $Mat_n(R)$ is not unilpotent-IFP by Example 2.6(1), and $T_n(R)$ cannot be unit-IFP by help of [11, Lemma 1.2(2)], but we have the following.

Proposition 3.2. *Let R be a ring and $n \geq 2$. The following conditions are equivalent:*

(1) *R is unilpotent-IFP;*

(2) *$T_n(R)$ is unilpotent-IFP;*

(3) *$D_n(R)$ is unilpotent-IFP;*

(4) $V_n(R)$ *is unilpotent-IFP.*

Proof. It suffices to show (1) $\Rightarrow$ (2) by Proposition 3.1(1). Let R be unilpotent-IFP. Consider the nil ideal $I = \{(a_{ij}) \in T_n(R) \mid a_{ii} = 0$ for all $i\}$ of $T_n(R)$. Then $T_n(R)/I$ is isomorphic to an n-copies of R; hence $T_n(R)/I$ is unilpotent-IFP by Proposition 3.1(3). Thus $T_n(R)$ is unilpotent-IFP by Proposition 3.1(2). □

By the same idea as in the proof of Proposition 3.2, we consider a similar proposition which also provides examples of unilpotent-IFP rings, being concerned with modules.

Proposition 3.3. *Let R, S be rings and $_RM_S$ be an (R, S)-bimodule. Then $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is unilpotent-IFP if and only if R and S are unilpotent-IFP.*

The following is an application of Proposition 2.5(1). Let R be an algebra over a commutative ring S. Following Dorroh [4], the *Dorroh extension* of R by S is the Abelian group $R \times S$ with multiplication given by $(s_1, r_1)(s_2, r_2) = (s_1 s_2, s_1 r_2 + s_2 r_1 + r_1 r_2)$ for $r_i \in R$ and $s_i \in S$.

Proposition 3.4. *Let R be an algebra with identity over a commutative reduced ring S. Then R is unilpotent-IFP if and only if so is the Dorroh extension D of R by S.*

Proof. It suffices to show the necessity by help of Proposition 3.1(1). Clearly s is identified with $s1$ in R for all $s \in S$. Note that $R = \{r + s \mid (r, s) \in D\}$ and $N(D) = (0, N(R))$ since S is a commutative reduced ring.

Suppose that R is unilpotent-IFP. Let $(u, b) \in U(D)$, then $u \in U(S)$. Say $(u, b)^{-1} = (u^{-1}, c)$. Since $(u, b)(u^{-1}, c) = (1, 0) = (u^{-1}, c)(u, b)$, $uc + u^{-1}b + bc = 0$, and $u^{-1}b + uc + cb = 0$. Thus $(u + b)(u^{-1} + c) = 1 = (u^{-1} + c)(u + b)$ in R, and so $u + b \in U(R)$. Now consider $(0, a) \in N(D)$. Then $a(u + b) \in N(R)$ since R is unilpotent-IFP, and it implies that $(0, a)(u, b) = (0, a(u + b)) \in N(D)$. Therefore D is unilpotent-IFP by Proposition 2.5(1). □

In what follows we consider some conditions under which the set of all nilpotent elements in unilpotent-IFP rings forms a subring, which it is compared with Proposition 2.3.

Proposition 3.5. *Let R be a unilpotent-IFP ring with $N(R) = Nil_2(R)$. Then we have the following results.*
  (1) $N(R)$ is a subring of R, and $ab = -ba$ for all $a, b \in N(R)$.
  (2) $N(R)$ is a commutative subring of R, when R is of characteristic 2.

Proof. (1) Let $a, b \in N(R) = Nil_2(R)$. Since R is unilpotent-IFP, $(ab)^2 = a^2b + abab = a(1 + b)ab \in N(R)$ from $a(ab) = 0 \in N(R)$ and $1 + b \in U(R)$. Thus $ab \in N(R)$ and $ba \in N(R)$ follows. Since $(ab)^2 = 0$ and $(ba)^2 = 0$, $(a + b)^4 = (ab + ba)^2 = abab + baba = (ab)^2 + (ba)^2 = 0$ and so $a + b \in N(R)$. Hence $N(R)$ forms a subring of R. Moreover, $(a + b)^2 = 0$ implies that $ab + ba = 0$.
  (2) It is an immediate consequence of (1). □

The following elaborates Proposition 3.5.

Example 3.6. (1) We recall the unit-IFP (and so unilpotent-IFP) ring R in Example 1.1. Then $N(R) = Kb + bRb = Nil_2(R)$ as mentioned earlier; hence $N(R)$ is a commutative subring of R.

(2) We recall the ring R as in Example 2.6(2). Then it is obvious $U(C) = \{(1,1)\}$ by [10, Example 2(2)]. So we have that:

$$N(R) = \{(c,0)r(0,d), (0,e)s(f,0), btb, b \mid r,s,t \in R \text{ and } c,d,e,f \in A\}$$

and $U(R) = \{1 + kw \mid k \in K \text{ and } w \in N(R)\}$, entailing $N(R) = Nil_2(R)$. The characteristic of R is 2, but $N(R)$ is not closed under multiplication as can be seen by $(a,0)b(0,a)b \notin N(R)$. So R is not unilpotent-IFP by Proposition 3.5. In fact, $(a,0)(0,a)b = 0$ but $(a,0)b(0,a)b = (a,0)(1+b)(0,a)b \notin N(R)$ (in spite of $1 + b \in U(R)$).

Recall that Köthe's conjecture holds for a given ring R when $N(R)$ is additively closed. So Köthe's conjecture holds for a unilpotent-IFP ring R with $N(R) = Nil_2(R)$ by Proposition 3.5 as well as for a left or right UN-transitive ring by Proposition 2.3.

Theorem 3.7. (1) *A ring R is unilpotent-IFP and satisfies Köthe's conjecture if and only if $R/N^*(R)$ is a unit-IFP ring.*

(2) *A ring R is NI if and only if $N(R)$ is a subring of R such that $ab \in N(R)$ for $a,b \in R$ implies $a(R \backslash N(R))b \subseteq N(R)$.*

Proof. (1) Suppose that R is unilpotent-IFP and satisfies Köthe's conjecture. Then $S = R/N^*(R)$ clearly satisfies Köthe's conjecture, and S is unilpotent-IFP by Proposition 3.1(2). Assume that $ab = 0$ for $a,b \in S$. Then $bxa \in N(S)$ for all $x \in S$. Note $U(S) = \{u + N^*(R) \mid u \in U(R)\}$. Since S unilpotent-IFP, $bxau \in N(S)$ for all $u \in U(S)$ by Proposition 2.5(1). This yields $aubx \in N(S)$, and thus aub generates a nil right ideal in S. But $N^*(S) = 0$ and S satisfies Köthe's conjecture, so S contains no nonzero nil one-sided ideals. Therefore $aub = 0$, proving that S is unit-IFP.

Conversely, suppose that $R/N^*(R)$ is unit-IFP. Then it is unilpotent-IFP and satisfies Köthe's conjecture by definition and [11, Theorem 1.3(1)], respectively. So R obviously satisfies Köthe's conjecture and is unilpotent-IFP by Proposition 3.1(2).

(2) The necessity is obvious. For the converse, let $ab \in N(R)$ for $a,b \in R$. Since $N(R)$ is a subring of R and $ba \in N(R)$, we get $aN(R)b \subseteq N(R)$. Consequently we have $aRb \subseteq N(R)$ by the condition that $ab \in N(R)$ for $a,b \in R$ implies $a(R \backslash N(R))b \subseteq N(R)$. Then R is NI by [12, Corollary 1.4]. □

Regarding Theorem 3.7(1), nilpotents always form a subring in a unilpotent-IFP ring satisfying Köthe's conjecture. Indeed, if R is unilpotent-IFP satisfying Köthe's conjecture, then $R/N^*(R)$ is unit-IFP, so nilpotents there form a subring, which in turn implies that nilpotents of R form a subring of R.

Consider the necessity of Theorem 3.7(2). If a ring R satisfies the condition that $ab \in N(R)$ for $a, b \in R$ implies $a(R \backslash N(R))b \subseteq N(R)$. Then R is unilpotent-IFP. For, if the ring R above is not unilpotent-IFP, then there exist $a, b \in R$ such that $ab \in N(R)$ and $aU(R)b \nsubseteq N(R)$. Then $aub \notin N(R)$ for some $u \in U(R)$, contrary to $aub \in a(R \backslash N(R))b \subseteq N(R)$. So one may ask whether if R is a unilpotent-IFP ring such that $N(R)$ is a subring of R then R is NI. But the answer is negative by the unilpotent-IFP ring R in Example 1.1(1) that is not NI. Note that $N(R)$ is a subring of R by [1, Theorem 4.7 and Corollary 3.3], and that $bba = 0$, $baba \notin N(R)$ and $baba \in b(R \backslash N(R))ba$.

The following elaborates upon the relations among the concepts above.

Example 3.8. (1) There exists a unilpotent-IFP ring that is neither right nor left unilpotent-duo. Let R_1 be the ring R in Example 2.4(2) that is left unilpotent-duo but not right unilpotent-duo; and let R_2 be the ring R in Example 2.4(3) that is right unilpotent-duo but not left unilpotent-duo. Set $R = R_1 \times R_2$. Then R is unilpotent-IFP by Theorem 2.7(1) and Proposition 3.1(3). Note $U(R) = U(R_1) \times U(R_2)$ and $N(R) = N(R_1) \times N(R_2)$. So R is neither right nor left unilpotent-duo.

(2) There exists an IFP ring that is neither right nor left unilpotent-duo. We use the ring in [7, Example 2]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$; and let $B = \{f \in A \mid$ the constant term of f is zero$\}$. Let I be the ideal of A generated by

$$a_0 b_0, a_1 b_2 + a_2 b_1, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, a_2 b_2, a_0 r b_0, a_2 r b_2,$$

$$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), \text{ and } r_1 r_2 r_3 r_4,$$

where $r, r_1, r_2, r_3, r_4 \in B$; and set $R = A/I$. Then R is IFP by the argument in [7, Example 2]. Identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Note $U(R) = \{1 + g \mid g \in B\} = 1 + B$ since $B^4 = 0$. Consider $o_r(c)$ and $o_l(c)$. Then $o_r(c) \nsubseteq o_l(c)$ and $o_r(c) \nsupseteq o_l(c)$ because $c(1 + a_0) \notin o_l(c)$ and $(1 + a_0)c \notin o_r(c)$ for $(1 + a_0) \in U(R)$. For assuming $c(1 + a_0) = (1 + g_1)c$ and $(1 + a_0)c = c(1 + g_2)$ for some $g_i \in B$, we get $0 \neq ca_0 = g_1 c$ and $0 \neq a_0 c = cg_2$ (i.e., $ca_0 - g_1 c, a_0 c - cg_2 \in I$), contrary to the construction of I. Therefore R is neither right nor left unilpotent-duo.

The following diagram shows all implications among the concepts above.
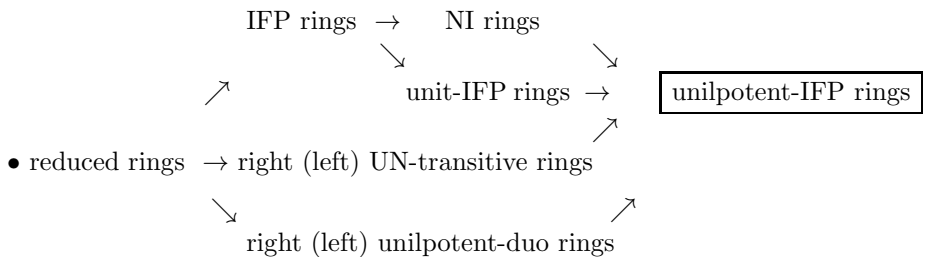
In what follows we consider a condition under which the ring properties mentioned above coincide. Following [5], a ring R is said to be *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$.

Proposition 3.9. *For a von Neumann regular ring R, the following conditions are equivalent:*

(1) R *is reduced;* (2) R *is IFP;* (3) R *is unit-IFP;* (4) R *is abelian;* (5) R *is right(left) UN-transitive;* (6) R *is unilpotent-IFP;* (7) R *is right or left unilpotent-duo.*

Proof. The implications $(1) \Rightarrow (2)$, $(1) \Rightarrow (7)$, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (6)$ are obvious. $(3) \Rightarrow (4)$, $(1) \Leftrightarrow (4)$ and $(7) \Rightarrow (6)$ are shown by [11, Lemma 1.2(2)], [5, Theorem 3.2] and Theorem 2.7(1), respectively.

The implications $(1) \Rightarrow (5)$ and $(5) \Rightarrow (6)$ are obvious.

$(6) \Rightarrow (4)$: Let R be unilpotent-IFP. Assume on the contrary that there exist $e^2 = e, r \in R$ such that $er(1 - e) \neq 0$. Say $a = er(1 - e)$. Then $a^2 = 0$ and so $1 - a \in U(R)$. Since R is von Neumann regular, $a = aba$ for some $b \in R$. Then $bab(1 - ab) = 0$. Since R is unilpotent-IFP, $bab(1 - a)(1 - ab) \in N(R)$ because $1 - a \in U(R)$. But

$$bab(1 - a)(1 - ab) = (bab - ba)(1 - ab) = -ba + ba^2b = -ba \notin N(R),$$

contrary to $bab(1 - a)(1 - ab) \in N(R)$. Thus R is abelian. □

Recall that a ring R is said to be *directly finite* (or *Dedekind finite*) if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is well-known that abelian rings are directly finite. NI rings are directly finite by [8, Proposition 2.7(1)]. We also obtain this result as a corollary of the following.

Proposition 3.10. *Every unilpotent-IFP ring is directly finite.*

Proof. Let R be a unilpotent-IFP ring and assume on the contrary that R is not directly finite. Then $ab = 1$ and $ba \neq 1$ for some $a, b \in R$. In what follows we refer to the argument for one-sided inverses in [9, page 1]. Consider $x = 1 - ba$ and $y = b - b^2a = b(1 - ba)$. Then $xy = 0$. Since $z = (1 - ba)a \in N(R)$, $1 + z \in U(R)$. Then $x(1 + z)y = xzy \in N(R)$ because R is unilpotent-IFP. But

$$xzy = (1 - ba)((1 - ba)a)(b(1 - ba)) = 1 - ba \notin N(R),$$

contrary to $xzy \in N(R)$. Thus R is directly finite. □

The converse of Proposition 3.10 needs not hold as can be seen by $Mat_n(D)$, over a division ring D for $n \geq 2$, which is Artinian (hence directly finite) but not unilpotent-IFP by Example 2.6(1).

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