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# Coefficient Estimates for a Subclass of Bi-univalent Functions Associated with Symmetric q-derivative Operator by Means of the Gegenbauer Polynomials

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Abstract. In the present paper, a subclass of analytic and bi-univalent functions is defined using a symmetric $q-$derivative operator by means of Gegenbauer polynomials. Coefficients bounds for functions belonging to this subclass are obtained. Furthermore, the Fekete-Szegö problem for this subclass is solved. A number of known or new results are shown to follow upon specializing the parameters involved in our main results.

## 1. Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of all analytic functions $f$ defined in the open unit disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form

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$$f(\xi)=\xi+\sum_{n=2}^{\infty}a_n\xi^n, \quad (\xi\in\mathbb{U}). \tag{1.1}$$

Further, let $\mathcal{S}$ denote the class of all functions $f\in\mathcal{A}$ which are univalent in $\mathbb{U}$.

Let the functions $f$ and $g$ be analytic in $\mathbb{U}$. We say that the function $f$ is subordinate to $g$, written as $f\prec g$, if there exists a Schwarz function $w$, which is analytic in $\mathbb{U}$ with

$$w(0)=0 \text{ and } |w(\xi)|<1 \qquad (\xi\in\mathbb{U})$$

such that

$$f(\xi)=g(w(\xi)).$$

If the function $g$ is univalent in $\mathbb{U}$, then the following equivalence holds

$$f(\xi)\prec g(\xi) \quad \text{if and only if} \quad f(0)=g(0)$$

and

$$f(\mathbb{U})\subset g(\mathbb{U}).$$

It is well known that every function $f\in\mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(\xi))=\xi \qquad (\xi\in\mathbb{U})$$

and

$$f^{-1}(f(w))=w \qquad (|w|<r_0(f);\ r_0(f)\geq\frac{1}{4})$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function is said to be bi-univalent in $\mathbb{U}$ if both $f(\xi)$ and $f^{-1}(\xi)$ are univalent in $\mathbb{U}$.

Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Example of functions in the class $\Sigma$ are

$$\frac{\xi}{1-\xi}, \qquad \log\frac{1}{1-\xi}, \qquad \log\sqrt{\frac{1+\xi}{1-\xi}}.$$

However, the familiar Koebe function is not a member of $\Sigma$. Other common examples of functions in $\mathbb{U}$ such as

$$\frac{2\xi-\xi^2}{2} \text{ and } \frac{\xi}{1-\xi^2}$$

are also not members of $\Sigma$.

Lewin [19] investigated the bi-univalent function class $\Sigma$ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [9] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [22], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = 4/3$.

The coefficient estimate problem for each of the Taylor–Maclaurin coefficients $|a_n|$ $(n \geq 3; n \in \mathbb{N})$ is presumably still an open problem.

Similar to the familiar subclasses $\mathcal{S}^*(\varsigma)$ and $\mathcal{K}(\varsigma)$ of starlike and convex function of order $\varsigma(0 \leq \varsigma < 1)$, respectively, Brannan and Taha [10] (see also [29]) introduced certain subclasses of the bi-univalent function class $\Sigma$, $\mathcal{S}^*_\Sigma(\varsigma)$ and $\mathcal{K}_\Sigma(\varsigma)$ of bi-starlike functions and of bi-convex functions of order $\varsigma(0 < \varsigma \leq 1)$, respectively. For each of the function classes $\mathcal{S}^*_\Sigma(\varsigma)$ and $\mathcal{K}_\Sigma(\varsigma)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ . For some intriguing examples of functions and characterization of the class $\Sigma$, see [1, 13, 21, 27].

Orthogonal polynomials have been studied extensively as early as they were discovered by Legendre in 1784 [18]. In mathematical treatment of model problems, orthogonal polynomials arise often to find solutions of ordinary differential equations under certain conditions imposed by the model.

The importance of the orthogonal polynomials for the contemporary mathematics, as well as for wide range of their applications in the physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in the mathematical statistics. Their applications in the quantum mechanics, scattering theory, automatic control, signal analysis and axially symmetric potential theory are also known [7, 11].

A special case of orthogonal polynomials are Gegenbauer polynomials. They are representatively related with typically real functions $T_R$ as discovered in [17], where the integral representation of typically real functions and generating function of Gegenbauer polynomials are using common algebraic expressions. Undoubtedly, this led to several useful inequalities appear from Gegenbauer polynomials realm.

Typically real functions play an important role in the geometric function theory because of the relation $T_R = \overline{co}S_R$ and its role of estimating coefficient bounds, where $S_R$ denotes the class of univalent functions in the unit disk with real coefficients, and $\overline{co}S_R$ denotes the closed convex hull of $S_R$.

Very recently, Amourah et al. [4] considered the Gegenbauer polynomials $H_\alpha(x, \xi)$, which are given by

$$H_\alpha(x,\xi) = \frac{1}{\left(1 - 2x\xi + \xi^2\right)^\alpha}, \tag{1.3}$$

where $x \in [-1, 1]$ and $\xi \in \mathbb{U}$. For fixed $x$ the function $H_\alpha$ is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$H_\alpha(x,\xi)=\sum_{n=0}^{\infty}C_n^\alpha(x)\xi^n, \tag{1.4}$$

where $C_n^\alpha(x)$ is Gegenbauer polynomial of degree $n$.

Obviously, $H_\alpha$ generates nothing when $\alpha=0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$H_0(x,\xi)=1-\log\left(1-2x\xi+\xi^2\right)=\sum_{n=0}^{\infty}C_n^0(x)\xi^n \tag{1.5}$$

for $\alpha=0$. Moreover, it is worth to mention that a normalization of $\alpha$ to be greater than $-1/2$ is desirable [11, 25]. Gegenbauer polynomials can also be defined by the following recurrence relations

$$C_n^\alpha(x)=\frac{1}{n}\left[2x\left(n+\alpha-1\right)C_{n-1}^\alpha(x)-\left(n+2\alpha-2\right)C_{n-1}^\alpha(x)\right], \tag{1.6}$$

with the initial values

$$C_0^\alpha(x)=1,\ C_1^\alpha(x)=2\alpha x \text{ and } C_2^\alpha(x)=2\alpha\left(1+\alpha\right)x^2-\alpha. \tag{1.7}$$

Special cases of Gegenbauer polynomials $C_n^\alpha(x)$ is Chebyshev Polynomials, when $\alpha=1$, and if $\alpha=\frac{1}{2}$, we get the Legendre Polynomials.

The theory of $q$-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, $q$-difference and $q$-integral equations, as well as geometric function theory of complex analysis. The application of $q$-calculus was initiated by Jackson [15]. Recently, many researchers studied $q$-calculus such as Srivastava et al. [28], Muhammad and Darus [20], Kanas and Răducanu [16], Aldweby and Darus [2] (see also, [24, 26, 28]) and also the reference cited therein.

For the convenience, we provide some basic definitions and concept details of $q-$calculus which are used in this paper. We shall follow the notation and terminology in [14].

**Definition 1.1.** ([15]) For $0<q<1$ the Jackson's $q$-derivative of a function $f\in\mathcal{A}$ is, by definition, given as follows

$$\mathbb{D}_q f(\xi)=\begin{cases}\dfrac{f(\xi)-f(q\xi)}{(1-q)\xi} & for \quad \xi\neq 0,\\ f'(0) & for \quad \xi=0,\end{cases} \tag{1.8}$$

From (1.8), we have

$$\mathbb{D}_q f(\xi)=1+\sum_{n=2}^{\infty}[n]_q a_n\xi^{n-1} \tag{1.9}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \; n \in \mathbb{N} = \{1, 2, ...\}, \tag{1.10}$$

is sometimes called *the basic number n*. If $q \to 1-, [n]_q \to n$.

For a function $h(\xi) = \xi^n$, we obtain

$$\mathbb{D}_q h(\xi) = \mathbb{D}_q \xi^n = \frac{1-q^n}{1-q}\xi^{n-1} = [n]_q \xi^{n-1},$$

and

$$\lim_{q\to 1-} \mathbb{D}_q h(\xi) = \lim_{q\to 1-} \left([n]_q \xi^{n-1}\right) = n\xi^{n-1} = h'(\xi),$$

where $h'$ is the ordinary derivative.

**Definition 1.2.** ([8]) The symmetric $q-$derivative $\widetilde{\mathbb{D}}_q f$ of a function $f$ given by (1.1) is defined as follows:

$$\left(\widetilde{\mathbb{D}}_q f\right)(\xi) = \begin{cases} \frac{f(q\xi)-f(q^{-1}\xi)}{(q-q^{-1})\xi} & \xi \neq 0 \\ f'(0) & \xi = 0 \end{cases}. \tag{1.11}$$

From (1.11), we deduce that $\widetilde{\mathbb{D}}_q \xi^n = \widetilde{[n]}_q \xi^{n-1}$, and a power series of $\widetilde{\mathbb{D}}_q f$ is

$$\left(\widetilde{\mathbb{D}}_q f\right)(\xi) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n \xi^{n-1},$$

when $f$ has the form (1.1) and the symbol $\widetilde{[n]}_q$ denotes the number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Clearly, we have the following relations

$$\widetilde{\mathbb{D}}_q \left(f(\xi) + g(\xi)\right) = \left(\widetilde{\mathbb{D}}_q f\right)(\xi) + \left(\widetilde{\mathbb{D}}_q g\right)(\xi),$$

$$\begin{aligned}\widetilde{\mathbb{D}}_q \left(f(\xi) g(\xi)\right) &= g\left(q^{-1}\xi\right)\left(\widetilde{\mathbb{D}}_q f\right)(\xi) + f\left(q\xi\right)\left(\widetilde{\mathbb{D}}_q g\right)(\xi) \\ &= g\left(q\xi\right)\left(\widetilde{\mathbb{D}}_q f\right)(\xi) + f\left(q^{-1}\xi\right)\left(\widetilde{\mathbb{D}}_q g\right)(\xi),\end{aligned}$$

and

$$\left(\widetilde{\mathbb{D}}_q f\right)(\xi) = \mathbb{D}_q f(q^{-1}\xi).$$

From (1.2) and (1.11), we also deduce that

$$\left(\widetilde{\mathbb{D}}_q g\right)(w) = \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})\, w}$$
$$= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q \left(2a_2^2 - a_3\right) w^2 - \widetilde{[4]}_q \left(5a_2^3 - -5a_2 a_3 + a_4\right) w^3 + \cdots . \tag{1.12}$$

Recently, many researchers have been exploring bi-univalent functions associated with orthogonal polynomials, few to mention (see,[31], [30]). For Gegenbauer polynomial, as far as we know, there is little work associated with bi-univalent functions in the literatures. Inspired by the works of Amourah et al. [4], we introduce the following new subclasses of bi-univalent functions, as follows:

**Definition 1.3.** Let $\alpha$ is a nonzero real constant. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\widetilde{\mathfrak{B}}_{\Sigma}^{q}(x, \alpha)$ if the following subordinations are satisfied:

$$\left(\widetilde{\mathbb{D}}_q f(\xi)\right) \prec H_\alpha(x, \xi) \tag{1.13}$$

and

$$\left(\widetilde{\mathbb{D}}_q g(w)\right) \prec H_\alpha(x, w), \tag{1.14}$$

where $x \in (\frac{1}{2}, 1]$, the function $g(w) = f^{-1}(w)$ is defined by (1.2) and $H_\alpha$ is the generating function of the Gegenbauer polynomial given by (1.3).

We note that $\lim_{q \to 1-} \widetilde{\mathfrak{B}}_{\Sigma}^{q}(x, \alpha) = \mathfrak{B}_{\Sigma}(x, \alpha)$,where the class $\mathfrak{B}_{\Sigma}(x, \alpha)$ defined as follows:

**Definition 1.4.** Let $\alpha$ is a nonzero real constant. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{B}_{\Sigma}(x, \alpha)$ if the following subordinations are satisfied:

$$f'(\xi) \prec H_\alpha(x, \xi) \tag{1.15}$$

and

$$g'(w) \prec H_\alpha(x, w)$$

where $x \in (\frac{1}{2}, 1]$, the function $g(w) = f^{-1}(w)$ is defined by (1.2) and $H_\alpha$ is the generating function of the Gegenbauer polynomial given by (1.3).

**Remark 1.5.** We note that the subclasses $\widetilde{\mathfrak{B}}^q_\Sigma(x,1) = \widetilde{H}^q_\Sigma(x)$ and $\mathfrak{B}_\Sigma(x,1) = H(x)$, were introduced and studied by Altinkaya and Yalçin [3].

The following result will be required for proving our results.

**Lemma 1.6.** ([23]) *Let $\mathcal{P}$ be the class of Caratheodory function with positive real part consisting of all analytic functions $p : \mathbb{U} \to \mathbb{C}$ satisfying $p(0) = 1$ and* $\mathrm{Re}(p(\xi)) > 0$.*If the function $p \in \mathcal{P}$ is defined by*

$$p(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots ,$$

*then*

$$|p_n| \le 2, \ \ n \in \mathbb{N}. \tag{1.16}$$

In this paper, we use the Gegenbauer polynomial expansions to provide estimates for the initial coefficients of the subclass of bi-univalent functions $\widetilde{\mathfrak{B}}^q_\Sigma(x,\alpha)$ defined by symmetric $q$-derivative operator. We also solve Fekete-Szegö problem for functions in this class.

Unless otherwise mentioned, we assume in the reminder of this paper that, $0 < q < 1$, $x \in (\frac{1}{2}, 1]$ and $\alpha$ is a nonzero real constant.

## 2. Coefficient Bounds of the Class $\widetilde{\mathfrak{B}}^q_\Sigma(x,\alpha)$

This section is devoted to find initial coefficient bounds of the class $\widetilde{\mathfrak{B}}^q_\Sigma(x,\alpha)$ of bi–univalent functions.

**Theorem 2.1.** *Let $f \in \Sigma$ given by (1.1) belongs to the class $\widetilde{\mathfrak{B}}^q_\Sigma(x,\alpha)$. Then*

$$|a_2| \le \frac{2\,|\alpha|\,x\sqrt{2\,|\alpha|\,x}}{\sqrt{\left|\left(4\widetilde{[3]}_q\alpha - 2\alpha\widetilde{[2]}_q^2(1+\alpha)\right)x^2 + 2\alpha\widetilde{[2]}_q^2 x + \alpha\widetilde{[2]}_q^2\right|}},$$

*and*

$$|a_3| \le \frac{4\alpha^2x^2}{\widetilde{[2]}_q^2} + \frac{2\,|\alpha|\,x}{\widetilde{[3]}_q}.$$

*Proof.* Let $f \in \widetilde{\mathfrak{B}}^q_\Sigma(x,\alpha)$. From Definition 1.3, for some analytic functions $\psi$, $v$ such that $\psi(0) = v(0) = 0$ and $|\psi(\xi)| < 1$, $|v(w)| < 1$ for all $\xi, w \in \mathbb{U}$, then we can write

$$\left(\widetilde{\mathbb{D}}_q f\right)(\xi) = H_\alpha(x, w(\xi)) \tag{2.1}$$

and

$$\left(\widetilde{\mathbb{D}}_q g\right)(w) = H_\alpha(x, v(w)), \tag{2.2}$$

Next, define the functions $p, q \in P$ by

$$p(\xi) = \frac{1+\psi(\xi)}{1-\psi(\xi)} = 1 + c_1\xi + c_2\xi^2 + \cdots$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + \cdots .$$

In the following, one can derive

$$\psi(\xi) = \frac{p(\xi)-1}{p(\xi)+1} = \frac{1}{2}c_1\xi + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right)\xi^2 + \cdots \tag{2.3}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}d_1 w + \frac{1}{2}\left(d_2 - \frac{1}{2}d_1^2\right)w^2 + \cdots . \tag{2.4}$$

From the equalities (2.1), (2.2), (2.3) and (2.4), we obtain that

$$\left(\widetilde{\mathbb{D}}_q f\right)(\xi) = 1 + \frac{1}{2}C_1^\alpha(x)c_1\xi + \left[\frac{1}{4}C_2^\alpha(x)c_1^2 + \frac{1}{2}C_1^\alpha(x)\left(c_2 - \frac{1}{2}c_1^2\right)\right]\xi^2 + \cdots \tag{2.5}$$

and

$$\left(\widetilde{\mathbb{D}}_q g\right)(w) = 1 + \frac{1}{2}C_1^\alpha(x)d_1 w + \left[\frac{1}{4}C_2^\alpha(x)d_1^2 + \frac{1}{2}C_1^\alpha(x)\left(d_2 - \frac{1}{2}d_1^2\right)\right]w^2 + \cdots . \tag{2.6}$$

Thus, upon comparing the corresponding coefficients in (2.5) and (2.6), we have

$$\widetilde{[2]}_q a_2 = \frac{1}{2}C_1^\alpha(x)c_1, \tag{2.7}$$

$$\widetilde{[3]}_q a_3 = \frac{1}{2}C_1^\alpha(x)\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}C_2^\alpha(x)c_1^2, \tag{2.8}$$

$$-\widetilde{[2]}_q a_2 = \frac{1}{2}C_1^\alpha(x)d_1, \tag{2.9}$$

and

$$\widetilde{[3]}_q\left(2a_2^2 - a_3\right) = \frac{1}{2}C_1^\alpha(x)\left(d_2 - \frac{1}{2}d_1^2\right) + \frac{1}{4}C_2^\alpha(x)d_1^2. \tag{2.10}$$

It follows from (2.7) and (2.9) that

$$c_1 = -d_1 \tag{2.11}$$

and

$$2\widetilde{[2]}_q^2 a_2^2 = \frac{1}{4}\left[C_1^\alpha(x)\right]^2\left(c_1^2+d_1^2\right). \tag{2.12}$$

If we add (2.8) and (2.10), we get

$$2\widetilde{[3]}_q a_2^2 = \frac{1}{2}C_1^\alpha(x)\left(c_2+d_2\right)+\frac{1}{4}\left(C_2^\alpha(x)-C_1^\alpha(x)\right)\left(c_1^2+d_1^2\right). \tag{2.13}$$

Substituting the value of $\left(c_1^2+d_1^2\right)$ from (2.12) in the right hand side of (2.13), we deduce that

$$2\left[\widetilde{[3]}_q-\widetilde{[2]}_q^2\frac{C_2^\alpha(x)-C_1^\alpha(x)}{\left[C_1^\alpha(x)\right]^2}\right]a_2^2 = \frac{1}{2}C_1^\alpha(x)\left(c_2+d_2\right). \tag{2.14}$$

Using (2.6), (1.16) and (2.14), we find that

$$|a_2| \le \frac{2\left|\alpha\right|x\sqrt{2\left|\alpha\right|x}}{\sqrt{\left|\left(4\widetilde{[3]}_q\alpha-2\alpha\widetilde{[2]}_q^2(1+\alpha)\right)x^2+2\alpha\widetilde{[2]}_q^2x+\alpha\widetilde{[2]}_q^2\right|}}.$$

Moreover, if we subtract (2.10) from (2.8), we obtain

$$4\widetilde{[3]}_q\left(a_3-a_2^2\right)=\frac{1}{2}C_1^\alpha(x)\left(c_2-d_2\right)+\frac{1}{4}\left(C_2^\alpha(x)-C_1^\alpha(x)\right)\left(c_1^2-d_1^2\right). \tag{2.15}$$

Then, in view of (1.7) and (2.12), equation (2.15) becomes

$$a_3=\frac{\left[C_1^\alpha(x)\right]^2}{8\widetilde{[2]}_q^2}\left(c_1^2+d_1^2\right)+\frac{C_1^\alpha(x)}{4\widetilde{[3]}_q}\left(c_2-d_2\right).$$

Thus applying (1.7) and (1.16), we have

$$|a_3| \le \frac{4\alpha^2x^2}{\widetilde{[2]}_q^2}+\frac{2\left|\alpha\right|x}{\widetilde{[3]}_q}.$$

□

## 3. Fekete–Szegö Problem for the Function Class $\widetilde{\mathfrak{B}}_\Sigma^q(x,\alpha)$

Fekete-Szegö inequality is one of the famous problem related to coefficients of univalent analytic functions. It was first given by [12], who stated that, if $f \in \Sigma$, then

$$|a_3-\eta a_2^2| \le 1+2e^{-2\eta/(1-\mu)}.$$

This bound is sharp when $\eta$ is real.

In this section, we aim to provide Fekete–Szegö inequalities for functions in the class $\widetilde{\mathfrak{B}}_\Sigma^q(x,\alpha)$. These inequalities are given in the following theorem.

**Theorem 3.1.** *Let $f \in \Sigma$ given by (1.1) belongs to the class $\widetilde{\mathfrak{B}}_{\Sigma}^{q}(x,\alpha)$. Then*

$$\left|a_3 - \eta a_2^2\right| \leq$$

$$\begin{cases} \dfrac{2|\alpha|x}{\widetilde{[3]}_q}, & |\eta-1| \leq \left|\dfrac{\left(4\widetilde{[3]}_q\alpha - 2\alpha\widetilde{[2]}_q^2(1+\alpha)\right)x^2 + 2\alpha\widetilde{[2]}_q^2 x + \alpha\widetilde{[2]}_q^2}{4\alpha^2 x^2 \widetilde{[3]}_q}\right| \\ \dfrac{8|\alpha|^3 x^3 |1-\eta|}{\left|\left(4\widetilde{[3]}_q\alpha - 2\alpha\widetilde{[2]}_q^2(1+\alpha)\right)x^2 + 2\alpha\widetilde{[2]}_q^2 x + \alpha\widetilde{[2]}_q^2\right|}, & |\eta-1| \geq \left|\dfrac{\left(4\widetilde{[3]}_q\alpha - 2\alpha\widetilde{[2]}_q^2(1+\alpha)\right)x^2 + 2\alpha\widetilde{[2]}_q^2 x + \alpha\widetilde{[2]}_q^2}{4\alpha^2 x^2 \widetilde{[3]}_q}\right|, \end{cases}$$

where $\eta \in \mathbb{R}$.

*Proof.* From (2.14) and (2.15)

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)\frac{\left[C_1^{\alpha}(x)\right]^3 (c_2 + d_2)}{4\left[\widetilde{[3]}_q \left[C_1^{\alpha}(x)\right]^2 - \left(C_2^{\alpha}(x) - C_1^{\alpha}(x)\right)\widetilde{[2]}_q^2\right]} \\ &\quad + \frac{C_1^{\alpha}(x)}{4\widetilde{[3]}_q}(c_2 - d_2) \\ &= C_1^{\alpha}(x)\left[\left[h(\eta) + \frac{1}{4\widetilde{[3]}_q}\right]c_2 + \left[h(\eta) - \frac{1}{4\widetilde{[3]}_q}\right]d_2\right], \end{aligned}$$

where

$$h(\eta) = \frac{\left[C_1^{\alpha}(x)\right]^2 (1-\eta)}{4\left[\widetilde{[3]}_q \left[C_1^{\alpha}(x)\right]^2 - \left(C_2^{\alpha}(x) - C_1^{\alpha}(x)\right)\widetilde{[2]}_q^2\right]},$$

Then, in view of (1.7) and (1.16), we conclude that

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2|\alpha|x}{\widetilde{[3]}_q} & 0 \leq |h(\eta)| \leq \frac{1}{4\widetilde{[3]}_q}, \\ 8\,|\alpha|\,x\,|h(\eta)| & |h(\eta)| \geq \frac{1}{4\widetilde{[3]}_q}. \end{cases}$$

Which completes the proof of Theorem 3.1. □

## 4. Corollaries and Consequences

In this section, we apply our main results in order to deduce each of the following new corollaries and consequences.

**Corollary 4.1.** *Let $f \in \Sigma$ given by (1.1) belongs to the class $\widetilde{\mathfrak{B}}_{\Sigma}^{q}(x,\frac{1}{2})$. Then*

$$|a_2| \leq \frac{x\sqrt{2x}}{\sqrt{\left|\left(2\widetilde{[3]}_q - \frac{3}{2}\widetilde{[2]}_q^2\right)x^2 + \widetilde{[2]}_q^2 x + \frac{1}{2}\widetilde{[2]}_q^2\right|}},$$

$$|a_3| \leq \frac{x^2}{\widetilde{[2]}_q^2} + \frac{x}{\widetilde{[3]}_q},$$

*and* $\left|a_3 - \eta a_2^2\right| \leq$

$$\begin{cases} \frac{x}{\widetilde{[3]}_q}, & |\eta - 1| \leq \left| \frac{\left(2\widetilde{[3]}_q - \frac{3}{2}\widetilde{[2]}_q^2\right)x^2 + \widetilde{[2]}_q^2 x + \frac{1}{2}\widetilde{[2]}_q^2}{x^2 \widetilde{[3]}_q} \right| \\ \frac{x^3|1-\eta|}{\left|\left(2\widetilde{[3]}_q - \frac{3}{2}\widetilde{[2]}_q^2\right)x^2 + \widetilde{[2]}_q^2 x + \frac{1}{2}\widetilde{[2]}_q^2\right|}, & |\eta - 1| \geq \left| \frac{\left(2\widetilde{[3]}_q - \frac{3}{2}\widetilde{[2]}_q^2\right)x^2 + \widetilde{[2]}_q^2 x + \frac{1}{2}\widetilde{[2]}_q^2}{x^2 \widetilde{[3]}_q} \right|, \end{cases}$$

where $\eta \in \mathbb{R}$.

**Corollary 4.2.** *Let $f \in \Sigma$ given by (1.1) belongs to the class $\mathfrak{B}_\Sigma(x, \alpha)$. Then*

$$|a_2| \leq \frac{|\alpha|\, x\sqrt{2x}}{\sqrt{|(1-2\alpha)\,x^2 + 2x + 1|}},$$

$$|a_3| \leq \alpha^2 x^2 + \frac{2\,|\alpha|\,x}{3},$$

$$\text{and} \quad \left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2|\alpha|x}{3}, & |\eta - 1| \leq \left|\frac{(1-2\alpha)x^2+2x+1}{3\alpha x^2}\right| \\ \frac{2|\alpha|^2 x^3|1-\eta|}{|(1-2\alpha)x^2+2x+1|}, & |\eta - 1| \geq \left|\frac{(1-2\alpha)x^2+2x+1}{3\alpha x^2}\right|, \end{cases}$$

where $\eta \in \mathbb{R}$.

**Concluding Remark.** By taking $\alpha = 1$, one can deduce the above results for various subclasses of $\Sigma$ studied by Altinkaya and Yalçin [3].

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