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GORENSTEIN QUASI-RESOLVING SUBCATEGORIES

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Abstract. In this paper, we introduce the notion of Gorenstein quasi-resolving subcategories (denoted by $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$) in term of quasi-resolving subcategory $\mathcal{X}$. We define a resolution dimension relative to the Gorenstein quasi-resolving categories $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. In addition, we study the stability of $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and apply the obtained properties to special subcategories and in particular to modules categories. Finally, we use the restricted flat dimension of right B-module M to characterize the finitistic dimension of the endomorphism algebra B of a $\mathcal{GQ}_{\mathcal{X}}$-projective A-module M.

1. Introduction

Throughout this paper, $\mathcal{A}$ denotes an abelian category. By a subcategory of $\mathcal{A}$ we always mean a full subcategory which is closed under isomorphisms. By $\mathcal{P}(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$ we denote the classes of all projective, injective objects of a category $\mathcal{A}$, respectively. For a subcategory $\mathcal{X}$ of $\mathcal{A}$, we denote the $\mathcal{X}\cap\mathcal{P}(\mathcal{A})$ by $\mathcal{P}_{\mathcal{X}}$. Let A be a ring. We denote by A-Mod (resp., A-mod) the category of all left (finitely generated) A-modules and $\mathcal{P}(A)$ (resp., $P(A)$) the category of all projective (resp., finitely generated projective) A-modules.

Auslander and Bridger [1] generalized finitely generated projective modules to finitely generated modules of Gorenstein dimension zero over (commutative) Noetherian rings. Eilenberg and Moore first introduced the viewpoint of relative homological algebra in [5]. Since then the relative homological algebra, especially the Gorenstein homological algebra, got a rapid development. Furthermore, Enochs and Jenda [6] introduced Gorenstein projective modules for arbitrary modules over a general ring, which is a generalization of finitely generated modules of Gorenstein dimension zero.

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Gorenstein projective and injective modules share many nice properties of projective modules and injective modules, respectively, [6, 7] and [10]. In the most results of Gorenstein homological algebra, the condition 'noetherian' is essential. Ding and his coauthors [4, 11] introduced the notion of Gorenstein FP-injective and strongly Gorenstein flat modules .

Let $\mathcal{C}$ be an additive full subcategory of $\mathcal{A}$. Sather-Wagstaff, Sharif and White [12] introduced the Gorenstein category $\mathcal{G}(\mathcal{C})$ as a common generalization of some known modules such as Gorenstein dimension zero [1], Gorenstein projective modules, Gorenstein injective modules, V-Gorenstein projective modules, V-Gorenstein injective modules [8], and so on. The Gorenstein category $\mathcal{G}(\mathcal{C})$ is defined as following.

$$\begin{aligned} \mathcal{G}(\mathcal{C}) \ = \{M \in \mathcal{A} \,|\, & \text{there exists a } \mathrm{Hom}_{\mathcal{A}}(\mathcal{C}, -)\text{-exact and } \mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact} \\ & \text{exact sequence } \cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots \text{ in } \mathcal{A} \\ & \text{with all } C_i, C^i \in \mathcal{C} \text{ such that } M \simeq \mathrm{Im}(C_0 \to C^0)\}. \end{aligned}$$

Set $\mathcal{G}^0(\mathcal{C}) = \mathcal{C}$ and $\mathcal{G}^1(\mathcal{C}) = \mathcal{G}(\mathcal{C})$, and inductively set $\mathcal{G}^{n+1}(\mathcal{C}) = \mathcal{G}(\mathcal{G}^n(\mathcal{C}))$. They proved that when $\mathcal{C}$ is self-orthogonal, $\mathcal{G}(\mathcal{C})$ processes many nice properties. They showed that $\mathcal{G}^{n+1}(\mathcal{C}) = \mathcal{G}(\mathcal{C})$ [12].

Let A be an artin algebra. The famous finitistic dimension conjecture says that there exists a uniform bound of the finite projective dimension of all finitely generated left A-modules of finite projective dimension. In [14], Wei used the restricted flat dimension to compare the finitistic dimension of the algebra A and the endomorphism algebra B of an A-algebra T. In [15], Zhang compare some homological dimension of the algebra A and the endomorphism algebra B of a Gorenstein projective A-module M.

The notion of a quasi-resolving subcategory was introduced in [16] and its nice homological theory was shown. In this paper, we want to associate quasi-resolving subcategories with Gorenstein homological theory. So, we introduce the notion of a Gorenstein quasi-resolving subcategory. We study the homological theory of a Gorenstein quasi-resolving subcategory. In particular, we consider the finitistic dimension of the endomorphism algebras of $\mathcal{GQ}_{\mathcal{X}}$-projective modules.

In Section 2, we introduce the notion of Gorenstein quasi-resolving subcategory (Definition 2). We prove that a Gorenstein quasi-resolving subcategory is also a quasi-resolving subcategory and closed under direct summands (Theorems 2.4 and 2.6). We give the equivalent conditions of an object in a Gorenstein quasi-resolving subcategory (Proposition 2.2). In Section 3, we define a resolution dimension relative to the Gorenstein quasi-resolving subcategory $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and we study $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence to character an object in a Gorenstein quasi-resolving subcategory. In Section 4, we study the stability of the Gorenstein quasi-resolving category $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ (Theorem 4.5). In Section 5, we consider the finitistic dimension of the endmorphism algebra B of a $\mathcal{GQ}_{\mathcal{X}}$-projective module M. We use the restricted flat dimension of M_B

and finitistic $\mathcal{P}_{\mathcal{X}}$-dimension of A to characterize the finitistic dimension of B (Theorem 5.4).

Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. By an $\mathcal{C}$-resolution on an object M of $\mathcal{A}$ we mean an exact sequence

$$\cdots \to C_n \xrightarrow{f_n} \cdots \longrightarrow C_2 \longrightarrow C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \longrightarrow 0$$

with all $C_i \in \mathcal{X}$. Let $res(\mathcal{C}) = \{M \in \mathcal{A} \mid M$ has a $\mathcal{C}$-resolution$\}$. The $\mathrm{Ker}f_{n-1}$ is called an *n-$\mathcal{C}$-syzygy* of M, denoted by $\Omega^n_{\mathcal{C}}(M)$ (we set $\Omega^0_{\mathcal{C}}(M) = M$). An object M of $\mathcal{A}$ is said to have $\mathcal{C}$-resolution dimension $\leq n$, denoted by $\mathcal{C}$-rdim(M) $\leq n$, if there is a $\mathcal{C}$-resolution of the form

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

of M. If n is the least such number, then we set $\mathcal{X}$-rdim(M) $= n$ and if there is no such n, we set $\mathcal{X}$-rdim(M) $= \infty$. We denote by *findim*$_{\mathcal{C}}(A)$ the *finitistic $\mathcal{C}$-dimension* of A, that is,

$$findim_{\mathcal{C}}(A) = \sup\{\mathcal{C}\text{-rdim}(M) \mid M \text{ is an } A\text{-module with } \mathcal{C}\text{-rdim}(M) < \infty\}.$$

Dually, we can define $cores(\mathcal{X}) = \{M \in \mathcal{A} \mid M$ *has an $\mathcal{X}$-coresolution*$\}$ and $\mathcal{X}$-coresolution dimension.

Let $\mathcal{F}$ be a class of $\mathcal{A}$. By $\mathcal{F}$-precover of an object $M \in \mathcal{A}$, we mean a morphism $\varphi : F \longrightarrow M$ where $F \in \mathcal{F}$ such that for any morphism $f : F' \longrightarrow M$ with $F' \in \mathcal{F}$, there is a morphism $g : F \longrightarrow F'$ such that $f = g\varphi$. Dually, one may give the notion of $\mathcal{F}$-preenvelope of an object $M \in \mathcal{A}$.

Let M be a right A-module. Following [3], M_A is said to have little *restricted flat* dimension at most m if for each $i > m$ the functor $\mathrm{Tor}^A_i(M, -)$ vanishes on the category of finitely generated modules with finite flat dimension. The little restricted flat dimension of M_A is denoted by $rfd(M_A)$.

For any object M in $\mathcal{A}$, $\mathrm{Add}M$ denotes the class of direct summands of direct sums of copies of M. For a subcategory $\mathcal{X}$ of $\mathcal{A}$, note that the functor $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$ means $\mathrm{Hom}_{\mathcal{A}}(-, X)$ for any $X \in \mathcal{X}$ and $\mathrm{Ext}^i_{\mathcal{A}}(-, \mathcal{X})$ means $\mathrm{Ext}^i_{\mathcal{A}}(-, X)$ for any $X \in \mathcal{X}$.

2. Gorenstein quasi-resolving subcategories

In this section, we give the notion of Gorenstein quasi-resolving subcategories. Some examples of Gorenstein quasi-resolving subcategories are given. Let us recall several well-known definitions of subcategories. Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$.

Give an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ in $\mathcal{A}$.

(1) $\mathcal{X}$ is said to be closed under extensions if $L, N \in \mathcal{X}$, then so is M.
(2) $\mathcal{X}$ is said to be closed under kernels of epimorphisms if $M, N \in \mathcal{X}$, then so is L.
(3) $\mathcal{X}$ is said to be closed under cokernels of monomorphisms if $L, M \in \mathcal{X}$, then so is N.

(4) $\mathcal{X}$ is said to be epic-admissible exact if it is closed under kernels of epimorphisms and extensions.
(5) $\mathcal{X}$ is said to be mono-admissible exact if it is closed under cokernels of monomorphisms and extensions.

We recall the definition of quasi-resolving subcategories in [16].

Definition 1. Let $\mathcal{X}$ be a subcategory of an abelian category $\mathcal{A}$ and $\mathcal{P}_{\mathcal{X}} = \mathcal{X} \cap \mathcal{P}(\mathcal{A})$. A subcategory $\mathcal{X}$ of $\mathcal{A}$ is called quasi-resolving if $\mathcal{X}$ is epic-admissible exact and $\mathcal{X} \subseteq res(\mathcal{P}_{\mathcal{X}})$.

SETUP: We assume that $\mathcal{X}$ is a quasi-resolving subcategory throughout this article.

Definition 2. A complete $\mathcal{P}_{\mathcal{X}}$-resolution of M is an exact sequence in $\mathcal{A}$

$$\mathbf{P}^\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \tag{2.1}$$

with all P_i and P^i in $\mathcal{P}_{\mathcal{X}}$, which is $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{X})$ exact, such that $M \simeq \operatorname{Im}(P_0 \to P^0)$.

An object $M \in \mathcal{A}$ is called $\mathcal{GQ}_{\mathcal{X}}$-projective if there is a complete $P_{\mathcal{X}}$-resolution as in (2.1) with $M \simeq \operatorname{Im}(P_0 \to P^0)$. The subcategory $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ of all $\mathcal{GQ}_{\mathcal{X}}$-projective objects in $\mathcal{A}$ is called Gorenstein quasi-resolving.

We give some examples of $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

Example 2.1. (1) If $\mathcal{X} = \mathcal{P}(\mathcal{A})$, then $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is just the subcategory of $\mathcal{GP}(\mathcal{A})$ consisting of all Gorenstein projective objects in $\mathcal{A}$.

(2) If $\mathcal{X} = \operatorname{Add}P$ for some $P \in \mathcal{P}(\mathcal{A})$, then $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is just the subcategory of $G_P(\mathcal{A})$ consisting of all P-Gorenstein projective objects in $\mathcal{A}$ [16].

(3) Let $P \in P(A)$. Denote

$$\mathcal{SF}(P) = \{X \in A\text{-mod} \mid X \oplus P^m \simeq P^n \text{ for some } m, n \in \mathbb{Z}^+\}.$$

Then $\mathcal{SF}(P)$ is a quasi-resolving subcategory of A-mod. Take $\mathcal{X} = \mathcal{SF}(P)$. We obtain $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is just the subcategory of $G_p(A)$ consisting of all P-Gorenstein projective modules in A-mod [16].

We give a characterization of an object in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

Proposition 2.2. *The following statements are equivalent for an object $M \in \mathcal{A}$.*

(1) *M is in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*
(2) *M satisfies the following three conditions*:
 (i) $\operatorname{Ext}_{\mathcal{A}}^{i\geq 1}(M, \mathcal{X}) = 0$.
 (ii) *$M \in res(\mathcal{P}_{\mathcal{X}})$.*
 (iii) *There is an exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ in $\mathcal{A}$ with all $P^i \in \mathcal{P}_{\mathcal{X}}$ such that $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{X})$ preserves the exactness of this sequence.*

(3) *There is an exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$ in $\mathcal{A}$ with $P \in \mathcal{P}_{\mathcal{X}}$, $M \in res(\mathcal{P}_{\mathcal{X}})$ and $L \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*

Proof. By Definition 2, it is easy to see (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2) Applying the functor $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$ to $0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$, we have that $\mathrm{Ext}_{\mathcal{A}}^{i\geq 1}(M,\mathcal{X}) = 0$ since $\mathrm{Ext}_{\mathcal{A}}^{i\geq 1}(L,\mathcal{X}) = 0$. It follows that the exact sequence stays exact under the functor $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$. Since $L \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, there is an exact sequence $0 \longrightarrow L \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ which is exact under the functor $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$. So, we get the desired $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$. $\square$

Corollary 2.3. *$M \in \mathcal{P}_{\mathcal{X}}$ if and only if $M \in \mathcal{X} \cap \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*

Proof. The necessity is clear.

Let $M \in \mathcal{X} \cap \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then by definition, there is an exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$$

with $P \in \mathcal{P}_{\mathcal{X}}$ and $L \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. By Proposition 2.2, we have $\mathrm{Ext}_{\mathcal{A}}^{i\geq 1}(L,M) = 0$ since $M \in \mathcal{X}$. Hence the exact sequence splits. Thus M is a direct summand of P. Then $M \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{X} = \mathcal{P}_{\mathcal{X}}$. $\square$

The following theorem is important for this section.

Theorem 2.4. *$\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is a quasi-resolving subcategory of $\mathcal{A}$.*

Proof. Since $\mathcal{P}_{\mathcal{X}} \subseteq \mathcal{GQR}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{P}(\mathcal{A})$ and $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A}) \subseteq res(\mathcal{P}_{\mathcal{X}})$, we have $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A}) \subseteq res(\mathcal{P}_{\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})})$.

We need to prove that if $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is exact with $M_3 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, then $M_1 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ if and only if $M_2 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Assume $M_1 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. By Proposition 2.2, we obtain $\mathrm{Ext}_{\mathcal{A}}^{i\geq 1}(M_2,\mathcal{X}) = 0$. We get two $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequences

$$0 \longrightarrow M_1 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

and

$$0 \longrightarrow M_3 \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \cdots$$

with all P_i, $Q_i \in \mathcal{P}_{\mathcal{X}}$. Take $K_1 \simeq \mathrm{Im}(P_0 \longrightarrow P_1)$ and $L_1 \simeq \mathrm{Im}(Q_0 \longrightarrow Q_1)$. We obtain the following commutative diagram with exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \xrightarrow{\binom{1}{0}} & P_0 \oplus Q_0 & \xrightarrow{(0\ 1)} & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & N_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $\mathcal{X}$ is closed under extensions, then $P_0 \oplus Q_0 \in \mathcal{P}_{\mathcal{X}}$. By the snake lemma, we obtain the middle column is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Continuing the process, we have an exact sequence such as $0 \longrightarrow M_2 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$ in $\mathcal{A}$ which is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact, where all $A^i \in \mathcal{P}_{\mathcal{X}}$. Since $M_1, M_3 \in res(\mathcal{P}_{\mathcal{X}})$, we have $M_2 \in res(\mathcal{P}_{\mathcal{X}})$ by the horseshoe lemma. By Proposition 2.2, we have $M_2 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

On the other hand, assume $M_2 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then we have an exact sequence $0 \longrightarrow M_2 \longrightarrow P \longrightarrow M_0 \longrightarrow 0$ with $P \in \mathcal{P}_{\mathcal{X}}$ and $M_0 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_0 & = & M_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is closed under extensions, we obtain $N \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ from the third column. By [16, Theorem 1.5], we have that $res(\mathcal{P}_{\mathcal{X}})$ is quasi-resolving. Then by the exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ we have $M_1 \in res(\mathcal{P}_{\mathcal{X}})$, since $M_2, M_3 \in res(\mathcal{P}_{\mathcal{X}})$. By Proposition 2.2(3), we have $M_1 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. □

Remark 2.5. (1) We can remove the condition $M \in res(\mathcal{P}_{\mathcal{X}})$ in Proposition 2.2(3) since $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is closed under kernels of epimorphisms.

(2) $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is closed under finite direct sums since $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is closed under extensions.

The following result is crucial for this paper.

Theorem 2.6. *$\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is closed under direct summands.*

Proof. Take $M \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. There is a complete $\mathcal{P}_{\mathcal{X}}$-resolution of M as follows

$$\mathbf{P}^\bullet : \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} \cdots$$

with all P_i and P^i in $\mathcal{P}_{\mathcal{X}}$, which is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact, such that $M \simeq \mathrm{Im}(P_0 \to P^0)$. Assume that $M = Y \oplus W$. By Proposition 2.2, we have $\mathrm{Ext}_{\mathcal{A}}^{i \geq 1}(M, \mathcal{X}) = 0$. Then $\mathrm{Ext}_{\mathcal{A}}^{i \geq 1}(W, \mathcal{X}) = 0 = \mathrm{Ext}_{\mathcal{A}}^{i \geq 1}(Y, \mathcal{X})$.

There are exact sequences

$$0 \longrightarrow Y \xrightarrow{\binom{1}{0}} M \xrightarrow{(0\ 1)} W \longrightarrow 0$$

and

$$0 \longrightarrow W \xrightarrow{\binom{0}{1}} M \xrightarrow{(1\ 0)} Y \longrightarrow 0.$$

Let $f_0 = i\pi$ with $\pi : P_0 \longrightarrow M$ and $i : M \longrightarrow P^0$. Then

$$0 \longrightarrow W_1 \longrightarrow P_0 \xrightarrow{(0\ 1)\pi} W \longrightarrow 0$$

is a $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact exact sequence. Similarly,

$$0 \longrightarrow Y_1 \longrightarrow P_0 \xrightarrow{(1\ 0)\pi} Y \longrightarrow 0$$

is a $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact exact sequence.

Put $M_i = \mathrm{Im} f_i$ for any $i \geq 0$ and $M_0 = M$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_1 & = & M_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & P_0 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & M & \xrightarrow{(0\ 1)} & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the diagram, we obtain $L \simeq W_1$. Since the second and third rows and second row are $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact, we have the first column is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Now we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2 & \longrightarrow & W_2 & \longrightarrow & Y_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus P_0 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & W_1 & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $P_1 \oplus P_0 \in \mathcal{P}_{\mathcal{X}}$. Since the second and third rows and the first and third columns are $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$ exact, we have the first row and the second column are $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Similarly, for Y we can obtain an exact sequence $0 \longrightarrow Y_2 \longrightarrow P_1 \oplus P_0 \longrightarrow Y_1 \longrightarrow 0$ with $P_1 \oplus P_0 \in \mathcal{P}_{\mathcal{X}}$, which is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. We get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_3 & \longrightarrow & W_3 & \longrightarrow & Y_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_2 & \longrightarrow & P_2 \oplus P_1 \oplus P_0 & \longrightarrow & P_1 \oplus P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2 & \longrightarrow & W_2 & \longrightarrow & Y_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We get the exact sequence $0 \longrightarrow W_3 \longrightarrow P_2 \oplus P_1 \oplus P_0 \longrightarrow W_2 \longrightarrow 0$ with $P_2 \oplus P_1 \oplus P_0 \in \mathcal{P}_{\mathcal{X}}$, which is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Similarly, we can get the exact sequence

$$0 \longrightarrow Y_3 \longrightarrow P_2 \oplus P_1 \oplus P_0 \longrightarrow Y_2 \longrightarrow 0$$

with $P_2 \oplus P_1 \oplus P_0 \in \mathcal{P}_{\mathcal{X}}$, which is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact. Repeating the process, we get a $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact exact sequence

$$\cdots \longrightarrow \oplus_{i=n}^{0} P_i \cdots \longrightarrow \oplus_{i=2}^{0} P_i \longrightarrow \oplus_{i=1}^{0} P_i \longrightarrow P_0 \longrightarrow W \longrightarrow 0$$

with $\oplus_{i=n}^{0} P_i \in \mathcal{P}_{\mathcal{X}}$. Similarly, we have the $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact exact sequence

$$\cdots \longrightarrow \oplus_{i=n}^{0} P_i \cdots \longrightarrow \oplus_{i=2}^{0} P_i \longrightarrow \oplus_{i=1}^{0} P_i \longrightarrow P_0 \longrightarrow Y \longrightarrow 0$$

with $\oplus_{i=n}^{0} P_i \in \mathcal{P}_{\mathcal{X}}$. So we have $W,\ Y \in res(\mathcal{P}_{\mathcal{X}})$.

Put $M^i = \mathrm{Im} f^i$ for any $i \geq 0$ and $M^0 = M$. Then

$$0 \longrightarrow W \xrightarrow{i\binom{0}{1}} P^0 \longrightarrow W^1 \longrightarrow 0$$

is a $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact exact sequence. Similarly,

$$0 \longrightarrow Y \xrightarrow{i\binom{1}{0}} P^0 \longrightarrow Y^1 \longrightarrow 0$$

is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequence. Consider the push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \xrightarrow{\binom{1}{0}} & M & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & P^0 & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M^1 & = & M^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the diagram, we have $K \simeq W^1$. Since the middle column and the first and second rows in this push-out diagram is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact, so is the third column. We have the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & W^1 & \longrightarrow & M^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^0 & \longrightarrow & P^0 \oplus P^1 & \longrightarrow & P^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^1 & \longrightarrow & W^2 & \longrightarrow & M^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have an exact sequence $0 \longrightarrow W^1 \longrightarrow P^0 \oplus P^1 \longrightarrow W^2 \longrightarrow 0$ with $P^0 \oplus P^1 \in \mathcal{P}_{\mathcal{X}}$. From this diagram, we have the second column and the third rows are $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact since the first and second row and the first column are $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact. Dual to the proof of $M, Y \in res(\mathcal{P}_{\mathcal{X}})$, we have two $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequences

$$0 \longrightarrow W \longrightarrow P^0 \longrightarrow \oplus_{i=0}^{1} P^i \longrightarrow \cdots \longrightarrow \oplus_{i=0}^{n} P^i \longrightarrow \cdots$$

and

$$0 \longrightarrow Y \longrightarrow P^0 \longrightarrow \oplus_{i=0}^{1} P^i \longrightarrow \cdots \longrightarrow \oplus_{i=0}^{n} P^i \longrightarrow \cdots$$

with $\oplus_{i=0}^{n} P^i \in \mathcal{P}_{\mathcal{X}}$. It follows that $M, Y \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. $\square$

Lemma 2.7 ([16])**.** *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$ and let*

$$0 \longrightarrow K_n^{'} \longrightarrow X_{n-1}^{'} \longrightarrow \cdots \longrightarrow X_1^{'} \longrightarrow X_0^{'} \longrightarrow M \longrightarrow 0 \tag{2.2}$$

and

$$0 \longrightarrow K_n^{''} \longrightarrow X_{n-1}^{''} \longrightarrow \cdots \longrightarrow X_1^{''} \longrightarrow X_0^{''} \longrightarrow M \longrightarrow 0 \tag{2.3}$$

be two exact sequences on $\mathcal{A}$ with all $X_i^{'}, X_i^{''} \in \mathcal{X}$ for $0 \leq i,\ j \leq n-1$. If $M \in res(\mathcal{X})$, then $K_n^{'} \in \mathcal{X}$ if and only if $K_n^{''} \in \mathcal{X}$.

Corollary 2.8. *Let*

$$0 \longrightarrow K_n^{'} \longrightarrow X_{n-1}^{'} \longrightarrow \cdots \longrightarrow X_1^{'} \longrightarrow X_0^{'} \longrightarrow M \longrightarrow 0 \tag{2.4}$$

and

$$0 \longrightarrow K_n^{''} \longrightarrow X_{n-1}^{''} \longrightarrow \cdots \longrightarrow X_1^{''} \longrightarrow X_0^{''} \longrightarrow M \longrightarrow 0 \tag{2.5}$$

be two exact sequences on $\mathcal{A}$ with $X_i^{'}, X_i^{''} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ for $0 \leq i,\ j \leq n-1$. If $M \in res(\mathcal{GQR}_{\mathcal{X}}(\mathcal{A}))$, then $K_n^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ if and only if $K_n^{''} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

3. Gorenstein quasi-resolving resolution dimension

In this section, we discuss the $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-resolution dimensions.

Definition 3. An object M of $\mathcal{A}$ is said to have $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-resolution dimension $\leq n$, denoted by $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) $\leq n$, if there is a $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-resolution of the form

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

of M. If n is the least such number, then we set $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) $= n$ and if there is no such n, we set $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) $= \infty$.

The following result plays a crucial role in this section.

Proposition 3.1. *Let $0 \longrightarrow A \longrightarrow G_0 \xrightarrow{f} G_1 \longrightarrow M \longrightarrow 0$ be an exact sequence in $\mathcal{A}$ with $G_0,\ G_1 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then we have the following exact sequences*

$$0 \longrightarrow A \longrightarrow P \longrightarrow G \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow A \longrightarrow H \longrightarrow Q \longrightarrow M \longrightarrow 0$$

with $P, Q \in \mathcal{P}_{\mathcal{X}}$ and $G, H \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

Proof. Since $G_0 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, there is an exact sequence $0 \longrightarrow G_0 \longrightarrow P \longrightarrow G_2 \longrightarrow 0$ in $\mathcal{A}$ with $P \in \mathcal{P}_{\mathcal{X}}$ and $G_2 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then we have the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & \mathrm{Im}f & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & B & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_2 & = & G_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Im}f & \longrightarrow & G_1 & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & G_2 & = & G_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the middle column of the second diagram, we obtain $G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. So we have the desired exact sequence $0 \longrightarrow A \longrightarrow P \longrightarrow G \longrightarrow M \longrightarrow 0$

Dually, taking pullback, one gets the second desired exact sequence. □

The following theorem is one of main results in this section.

Theorem 3.2. *Let $n \geq 1$ and $0 \longrightarrow A \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ be an exact sequence in $\mathcal{A}$ with all $G_i \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then there are exact sequences*

(1) $0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ *and* $0 \longrightarrow H \longrightarrow B \longrightarrow A \longrightarrow 0$ *with all $P_i \in \mathcal{P}_{\mathcal{X}}$ and $H \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*

(2) $0 \longrightarrow A \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$ *and* $0 \longrightarrow M \longrightarrow N \longrightarrow G \longrightarrow 0$ *with all $Q_i \in \mathcal{P}_{\mathcal{X}}$ and $G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*

Proof. We proceed by induction on n. When $n = 1$, we have an exact sequence $0 \longrightarrow A \longrightarrow G_0 \longrightarrow M \longrightarrow 0$. Since $G_0 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, we have an exact sequence $0 \longrightarrow H \longrightarrow P \longrightarrow G_0 \longrightarrow 0$ with $P \in \mathcal{P}_{\mathcal{X}}$ and $H \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Consider the pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Now suppose that $n \geq 2$ and we have the exact sequence

$$0 \longrightarrow A \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

Put $K = \mathrm{Im}(G_2 \longrightarrow G_1)$. Then we have an exact sequence

$$0 \longrightarrow K \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

By Proposition 3.1, we have an exact sequence

$$0 \longrightarrow K \longrightarrow G_1' \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Then we get an exact sequence

$$0 \longrightarrow A \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_2 \longrightarrow G_1' \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Put $N = \mathrm{Im}(G_1' \longrightarrow P_0)$. By the assumption, we have exact sequences

$$0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow H \longrightarrow B \longrightarrow A \longrightarrow 0$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$. Then we have the desired exact sequence

$$0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

(2) The proof is dual to that of (1), so we omit it. □

Another main result in this section is the following.

Theorem 3.3. *Let M be an object in $\mathcal{A}$ with $\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) < \infty$. Then the following are equivalent for any $n \geq 0$.*

(1) *$\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) \leq n$.*

(2) *There is an exact sequence*

$$0 \longrightarrow G_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$ and $G_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

(3) *For each non-negative integer* t *with* $0 \leq t \leq n$, *there is an exact sequence*

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_t \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_t \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ *and* $X_i \in \mathcal{P}_{\mathcal{X}}$ *for* $i \neq t$.

(4) $\mathrm{Ext}^{n+i}_{\mathcal{A}}(M, X) = 0$ *for any object* $X \in \mathcal{X}$ *and* $i \geq 1$.

(5) $\mathrm{Ext}^{n+1}_{\mathcal{A}}(M, Y) = 0$ *for any* Y *in* $\mathcal{A}$ *with* $\mathcal{X}$*-rdim*$(Y) < \infty$.

(6) $\mathrm{Ext}^{n+i}_{\mathcal{A}}(M, Y) = 0$ *for any* Y *in* $\mathcal{A}$ *with* $\mathcal{X}$*-rdim*$(Y) < \infty$ *and* $i \geq 1$.

(7) $\mathrm{Ext}^{n+i}_{\mathcal{A}}(M, P) = 0$ *for any object* $P \in \mathcal{P}_{\mathcal{X}}$ *and* $i \geq 1$.

(8) $\mathrm{Ext}^{n+1}_{\mathcal{A}}(M, Z) = 0$ *for any* Z *in* $\mathcal{A}$ *with* $\mathcal{P}_{\mathcal{X}}$*-rdim*$(Z) < \infty$.

(9) $\mathrm{Ext}^{n+i}_{\mathcal{A}}(M, Z) = 0$ *for any* Z *in* $\mathcal{A}$ *with* $\mathcal{P}_{\mathcal{X}}$*-rdim*$(Z) < \infty$ *and* $i \geq 1$.

Proof. (1) ⇒ (2) By Theorem 3.2, take $A = G_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. It is easy to see $B \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

(2) ⇒ (1) and (3) ⇒ (1) are trivial.

(1) ⇒ (3) We proceed by induction on n. Suppose $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) ≤ 1. Then there is an exact sequence $0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ in $\mathcal{A}$ with $G_0, G_1 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. By Proposition 3.1, we get the exact sequence $0 \longrightarrow P_1 \longrightarrow G_0^{'} \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow G_1^{'} \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with $P_0, P_1 \in \mathcal{P}_{\mathcal{X}}$ and $G_0^{'}$, $G_1^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

Now suppose that $n \geq 2$. Then there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with all $G_i \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Let $K = \mathrm{Im}(G_2 \longrightarrow G_1)$. Applying Proposition 3.1 to the exact sequence $0 \longrightarrow K \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$, we get an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_2 \longrightarrow G_1^{'} \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_0 \in \mathcal{P}_{\mathcal{X}}$ and $G_1^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Put $N = \mathrm{Im}(G_1^{'} \longrightarrow P_0)$. Then $\mathcal{GQR}_{\mathcal{X}}$-rdim$(N) \leq n-1$. By the induction hypothesis, there is an exact sequence

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_t \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_t \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $X_i \in \mathcal{P}_{\mathcal{X}}$ for $i \neq t$ and $1 \leq t \leq n$.

Now we need only to prove (3) for $t = 0$. Put $B = \mathrm{Im}(G_1 \longrightarrow G_0)$. By the induction hypothesis, we get an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow G^{'} \longrightarrow B \longrightarrow 0$$

with $G^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $P_i \in \mathcal{P}_{\mathcal{X}}$ for any $2 \leq i \leq n$. Set $A = \mathrm{Im}(P_2 \longrightarrow G_1^{'})$. Applying Proposition 3.1 to the exact sequence $0 \longrightarrow A \longrightarrow G_1^{'} \longrightarrow G_0 \longrightarrow M \longrightarrow 0$, we get the desired exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow G_0^{'} \longrightarrow M \longrightarrow 0$$

with $G_0^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and all $P_i \in \mathcal{P}_{\mathcal{X}}$.

(6) ⇒ (5) and (6) ⇒ (4) are trivial.

It is easy to get (1) ⇒ (4) and (4) ⇒ (6) by Proposition 2.2 and dimension shifting.

(5) $\Rightarrow$ (1) and (8) $\Rightarrow$(1) Let $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) $\leq m$. By (3), there is an exact sequence

$$0 \longrightarrow X_m \longrightarrow X_{m-1} \cdots \longrightarrow X_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with $G_0 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $X_i \in \mathcal{P}_{\mathcal{X}} \subseteq \mathcal{X}$ for $1 \leq i \leq m$.

We claim that $m \leq n$. Otherwise, let $m > n$. Put $M_i = \mathrm{Im}(X_i \longrightarrow X_{i-1})$ with $X_0 = G_0$ for any $1 \leq i \leq m$. Note that $\mathcal{X}$-rdim$(M_{n+1}) \leq \mathcal{P}_{\mathcal{X}}$-rdim$(M_{n+1}) \leq m-n-1$. So $\mathrm{Ext}^1_{\mathcal{A}}(M_n, M_{n+1}) \simeq \mathrm{Ext}^{n+1}_{\mathcal{A}}(M, M_{n+1}) = 0$. Hence the exact sequence $0 \longrightarrow M_{n+1} \longrightarrow X_n \longrightarrow M_n \longrightarrow 0$ splits. Then M_n is a direct summand of $X_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. By Theorem 2.6, $M_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ which is a contradiction.

It is easy to see the equivalences of (6), (7), (8) and (9). This completes the proof. □

From Theorem 3.3, we have the following consequence.

Corollary 3.4. *Let M be an object in $\mathcal{A}$ with $\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) < \infty$. Then*

$$\begin{aligned}
&\mathcal{GQR}_{\mathcal{X}}\text{-}rdim(M)\\
&= \sup\{n \geq 0 \,|\, \mathrm{Ext}^n_{\mathcal{A}}(M,Y) \neq 0 \textit{ for some object } Y \textit{ in } \mathcal{X}\}\\
&= \sup\{n \geq 0 \,|\, \mathrm{Ext}^n_{\mathcal{A}}(M,Y) \neq 0 \textit{ for some object } Y \textit{ in } \mathcal{A} \textit{ with } \mathcal{X}\text{-}rdim(Y) < \infty\}\\
&= \sup\{n \geq 0 \,|\, \mathrm{Ext}^n_{\mathcal{A}}(M,P) \neq 0 \textit{ for some object } P \textit{ in } \mathcal{P}_{\mathcal{X}}\}\\
&= \sup\{n \geq 0 \,|\, \mathrm{Ext}^n_{\mathcal{A}}(M,Z) \neq 0 \textit{ for some object } Z \textit{ in } \mathcal{A} \textit{ with } \mathcal{P}_{\mathcal{X}}\text{-}rdim(Z) < \infty\}.
\end{aligned}$$

Corollary 3.5. *By an argument similar to the proof of the equivalence of* (1) *and* (7) *in Theorem 3.3, we have that if $\mathcal{P}_{\mathcal{X}}$-rdim$M < \infty$, then*

$$\mathcal{P}_{\mathcal{X}}\text{-}rdim(M) = \sup\{n \geq 0 \,|\, \mathrm{Ext}^n_{\mathcal{A}}(M,P) \neq 0 \textit{ for some object } P \textit{ in } \mathcal{P}_{\mathcal{X}}\}.$$

So let $M \in \mathcal{A}$, if $\mathcal{P}_{\mathcal{X}}$-rdim$M < \infty$, then $\mathcal{P}_{\mathcal{X}}$-rdim$(M) = \mathcal{GQR}_{\mathcal{X}}$-rdim$(M)$.

As an immediate consequence of Theorems 3.3 and 3.2, we get the following corollaries.

Corollary 3.6. *Let M be an object in $\mathcal{A}$ such that $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) $\leq n$. Then*

(1) *There is an exact sequence $0 \longrightarrow N \longrightarrow G \longrightarrow M \longrightarrow 0$ in $\mathcal{A}$ with $G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $\mathcal{P}_{\mathcal{X}}$-rdim$(N) \leq n-1$.*

(2) *There is an exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow H \longrightarrow 0$ in $\mathcal{A}$ with $H \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $\mathcal{P}_{\mathcal{X}}$-rdim$(N) \leq n-1$.*

Proof. (1) follows by Theorem 3.3(3).

(2) follows by Theorem 3.2(2) and Theorem 3.3(3). □

Corollary 3.7. *Let $0 \to X \to G \to Y \to 0$ be an exact sequence in $\mathcal{A}$. Assume that G is in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and X has a finite $\mathcal{GQR}_{\mathcal{X}}$-resolution.*

(1) *If $Y \notin \mathcal{GQR}_{\mathcal{X}}$, then $\mathcal{GQR}_{\mathcal{X}}$-rdim$(Y) = \mathcal{GQR}_{\mathcal{X}}$-rdim$(X) + 1$.*

(2) *If $Y \in \mathcal{GQR}_{\mathcal{X}}$, so is X.*

Proof. Since X has a finite $\mathcal{GQR}_{\mathcal{X}}$-resolution, so does Y. Applying $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$ to the given exact sequence, we obtain that $\mathrm{Ext}^i_{\mathcal{A}}(X,-)|_{\mathcal{X}} \simeq \mathrm{Ext}^{i+1}_{\mathcal{A}}(Y,-)|_{\mathcal{X}}$ for any $i \geq 1$. Then (1) and (2) follows by Theorem 3.3. □

By the corollary, we have the following result.

Proposition 3.8. *Let* $0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ *be an exact sequence with* $G_1, G_0 \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. *If* $\mathrm{Ext}^1_{\mathcal{A}}(M,P) = 0$ *for any* $P \in \mathcal{P}_{\mathcal{X}}$, *then* $M \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.

Proof. By Corollary 3.6, there is an exact sequence $0 \longrightarrow P \longrightarrow G \longrightarrow M \longrightarrow 0$ with $P \in \mathcal{P}_{\mathcal{X}}$ and $G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then the exact sequence splits. By Theorem 2.6, we obtain $M \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. □

The following result shows that for any object in $\mathcal{A}$ with a finite $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-resolution dimension, we can construct a $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-precover.

Theorem 3.9. *Let M be an object in $\mathcal{A}$ with $\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) \leq n$ if and only if M admits a $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-precover of $G \longrightarrow M$ such that if $0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$ is exact, then $\mathcal{P}_{\mathcal{X}}$-rdim$(L) \leq n-1$.*

Proof. The sufficiency is obvious. We only need to show its necessity.

Since $\mathcal{GQR}_{\mathcal{X}}$-rdim(M) $\leq n$, there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $P_i \in \mathcal{P}_{\mathcal{X}}$ and $G_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then there is a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequence $0 \longrightarrow G_n \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_{n-1} \longrightarrow K_n \longrightarrow 0$ with all $Q_i \in \mathcal{P}_{\mathcal{X}}$ and $K_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. So we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & G_n & \longrightarrow & Q_0 & \longrightarrow & \cdots & \longrightarrow & Q_{n-1} & \longrightarrow & K_n & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

This diagram gives a chain map between complexes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Q_0 & \longrightarrow & \cdots & \longrightarrow & Q_{n-1} & \longrightarrow & K_n & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Its mapping cone is exact, that is, we have an exact sequence

$$0 \longrightarrow Q_0 \longrightarrow Q_1 \oplus P_{n-1} \longrightarrow \cdots \longrightarrow Q_{n-1} \oplus P_1 \longrightarrow K_n \oplus P_0 \longrightarrow M \longrightarrow 0.$$

Let $K_n \oplus P_0 = G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $L = \mathrm{Im}(Q_{n-1} \oplus P_1 \longrightarrow G)$. Then we have an exact sequence $0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$. Note that $\mathcal{P}_{\mathcal{X}}$-rdim$(L) \leq n-1$. We obtain that $\mathrm{Ext}^{i\geq 1}_{\mathcal{A}}(G', L) = 0$ by Proposition 2.2 and the dimension shifting for any $G^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Applying $\mathrm{Hom}_{\mathcal{A}}(G^{'}, -)$ to $0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$, we obtain that $G \longrightarrow M$ is a $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-precover. □

We give some applications of Theorem 3.9.

Corollary 3.10. *Let M be a left A-module with a finite Gorenstein projective dimension n. Then M admits a surjective Gorenstein projective precover $f : G \longrightarrow M$ where* Ker$f$ *satisfies projective dimension $\leq n-1$.*

Corollary 3.11. *Let M be a left A-module with a finite P-Gorenstein projective dimension n for some $P \in \mathcal{P}(A)$. Then M admits a surjective P-Gorenstein projective precover $f : G \longrightarrow M$ where* Ker$f$ *satisfies* AddP-*rdim*(Ker$f$) $\leq n-1$.

It is known that the pure injective plays an important role in the homological algebra, and the relative version also have been investigated by many authors ([2, 9, 13], etc). Inspired by this, we give the following definition.

Definition 4. An exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is called $\mathcal{GQR}_{\mathcal{X}}$-pure exact if for any $X \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, the induced sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, L) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, M) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, N) \longrightarrow 0$ is exact.

An object X of $\mathcal{A}$ is called $\mathcal{GQR}_{\mathcal{X}}$-pure projective (resp., $\mathcal{GQR}_{\mathcal{X}}$-pure injective) if $\mathrm{Hom}_{\mathcal{A}}(X, -)$ (resp., $\mathrm{Hom}_{\mathcal{A}}(-, X)$) leaves any $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence exact.

Definition 5. If an exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is a $\mathcal{GQR}_{\mathcal{X}}$-pure exact, then f is called a $\mathcal{GQR}_{\mathcal{X}}$-pure injection and g is called a $\mathcal{GQR}_{\mathcal{X}}$-pure surjection.

The following proposition shows that the necessity of studying $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence.

Theorem 3.12. *Let M be an object in $\mathcal{A}$ with a finite $\mathcal{GQR}_{\mathcal{X}}$-resolution dimension. Then the following are equivalent*:

(1) *M is in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*

(2) *For any $\mathcal{GQR}_{\mathcal{X}}$-pure surjection $g : Y \longrightarrow Z$ and any $h : M \longrightarrow Z$, there exists $l : M \longrightarrow Y$ such that $h = gl$.*

(3) *The functor* $\mathrm{Hom}_{\mathcal{A}}(M, -)$ *leaves any $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence exact.*

(4) *Every $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence $0 \longrightarrow L \longrightarrow N \longrightarrow M \longrightarrow 0$ splits.*

Proof. (1) ⇒ (2) Since $g : Y \longrightarrow Z$ is a $\mathcal{GQR}_{\mathcal{X}}$-pure surjection, there is a $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$. Since M is in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, then the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, Z) \longrightarrow 0$$

is exact.

(2) ⇒ (3) and (3) ⇒ (4) are easy.

(4) ⇒ (1) Assume that M has a finite $\mathcal{GQR}_{\mathcal{X}}$-resolution dimension n. By Theorem 3.9, there is an exact sequence $0 \longrightarrow L \longrightarrow G \xrightarrow{h} M \longrightarrow 0$ in $\mathcal{A}$ with

$G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $\mathcal{P}_{\mathcal{X}}$-rdim$(L) \leq n-1$ such that h is a $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$-precover. Then for any $X \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, the induced sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, L) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, G) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, M) \longrightarrow 0$ is exact. Hence $0 \longrightarrow L \longrightarrow G \stackrel{h}{\longrightarrow} M \longrightarrow 0$ is a $\mathcal{GQR}_{\mathcal{X}}$-pure exact sequence. Thus it splits by (4). Then M is a direct summand of G. By Proposition 2.6, M is in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. $\square$

4. Stability of Gorenstein quasi-resolving categories

Proposition 4.1. *Let M be an object in $\mathcal{A}$. Assume that there is an exact sequence in $\mathcal{A}$*

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^n \longrightarrow \cdots \tag{4.1}$$

with all $G^n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then we have

(1) *There is an exact sequence*

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^n \longrightarrow \cdots \tag{4.2}$$

with all $P^n \in \mathcal{P}_{\mathcal{X}}$.

(2) *Let D be an object in $\mathcal{A}$ such that any short exact sequence in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact. If (4.1) is $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact, then so is (4.2).*

Proof. (1) Put $M^1 = \mathrm{Im}(G^0 \longrightarrow G^1)$ and $M^2 = \mathrm{Im}(G^1 \longrightarrow G^2)$. Then we have the following two exact sequences

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow M^1 \longrightarrow 0$$

and

$$0 \longrightarrow M^1 \longrightarrow G^1 \cdots \longrightarrow G^n \longrightarrow \cdots .$$

There is an exact sequence $0 \longrightarrow G^0 \longrightarrow P^0 \longrightarrow G \longrightarrow 0$ with $P^0 \in \mathcal{P}_{\mathcal{X}}$ and $G \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & M^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & P^0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now consider following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M^1 & \longrightarrow & G^1 & \longrightarrow & M^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & G^{'} & \longrightarrow & M^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the second column of this diagram, we obtain $G^{'} \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. We get an exact sequence

$$0 \longrightarrow L \longrightarrow G^{'} \longrightarrow G^2 \longrightarrow \cdots \longrightarrow G^n \longrightarrow \cdots \tag{4.3}$$

with $G^{'}$, $G^i \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ for any $i \geq 2$. Repeating the process to (4.3) and so on, we can get the desired sequence.

(2) Let D be an object in $\mathcal{A}$ such that any short exact sequence in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact. By assumption, (4.1) is $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact. Since the middle column and the first rows are both $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact in the above two diagrams, we obtain the middle rows are both $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact in the above two diagrams. Continuing this process, we can deduce that (4.2) is $\mathrm{Hom}_{\mathcal{A}}(-, D)$-exact. □

The next result is dual to Proposition 4.1.

Proposition 4.2. *Let M be an object in $\mathcal{A}$. Assume that there is an exact sequence in $\mathcal{A}$*

$$\cdots \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0 \tag{4.4}$$

with all $G_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Then we have

(1) *There is an exact sequence*

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{4.5}$$

with all $P_n \in \mathcal{P}_{\mathcal{X}}$.

(2) *Let D be an object in $\mathcal{A}$ such that any short exact sequence in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ is* $\mathrm{Hom}_{\mathcal{A}}(-, D)$ *exact. If* (4.4) *is* $\mathrm{Hom}_{\mathcal{A}}(-, D)$ *exact, then so is* (4.5).

Proof. It is completely dual to the proof Proposition 4.1, so we omit it. □

Combining Propositions 4.1 and 4.2, we have the following equivalent conditions.

Corollary 4.3. *Let M be an object in $\mathcal{A}$. Then M admits a* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-*exact exact sequence* $0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^n \longrightarrow \cdots$ *with all* $G^n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ *if and only if there is a* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-*exact exact sequence* $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^n \longrightarrow \cdots$ *with all* $P^n \in \mathcal{P}_{\mathcal{X}}$.

The next result is dual to Corollary 4.3.

Corollary 4.4. *Let M be an object in $\mathcal{A}$. Then M admits a* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-*exact exact sequence* $\cdots \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ *with all* $G_n \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ *if and only if there is a* $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-*exact exact sequence* $\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ *with all* $P_n \in \mathcal{P}_{\mathcal{X}}$.

Set $[\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^1 = \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, and inductively set the following subcategory of $\mathcal{A}$:

$$\begin{aligned}[\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^{n+1} = \{M \in \mathcal{A} \,|\, & \text{there exists a } \operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})\text{-exact exact sequence} \\ & \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots \text{ in } \mathcal{A} \\ & \text{with all } G_i, G^i \in \mathcal{A} \text{ such that } M \simeq \operatorname{Im}(G_0 \to G^0)\}.\end{aligned}$$

Now, we are ready to prove the main result in the section.

Theorem 4.5. *For any $n \geq 1$, $[\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^n = \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$.*

Proof. It is easy to see that $\mathcal{P}_{\mathcal{X}} \subseteq \mathcal{GQR}_{\mathcal{X}}(\mathcal{A}) \subseteq [\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^2 \subseteq [\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^3 \subseteq \cdots$ is an ascending chain of additive subcategories of $\mathcal{A}$.

Let M be an object in $[\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^2$. Then there is a $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequence $\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ in $\mathcal{A}$ with all $G_i, G^i \in \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$, such that $M \simeq \operatorname{Im}(G_0 \to G^0)$. By Corollaries 4.3 and 4.4, there are two $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact exact sequences

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all $P_n \in \mathcal{P}_{\mathcal{X}}$, and

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^n \longrightarrow \cdots$$

with all $P^n \in \mathcal{P}_{\mathcal{X}}$. So

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^n \longrightarrow \cdots$$

is a complete $\mathcal{P}_{\mathcal{X}}$-resolution of M with $M \simeq \operatorname{Im}(P_0 \longrightarrow P^1)$ and hence M is an object in $\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$ and $[\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^2 \subseteq \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. Thus we have $[\mathcal{GQR}_{\mathcal{X}}(\mathcal{A})]^2 = \mathcal{GQR}_{\mathcal{X}}(\mathcal{A})$. By using induction on n, we get easily the assertion. $\square$

Set the following subcategory of A-Mod.

$$\begin{aligned}[\mathcal{GP}(R)]^2 = \{M \in \operatorname{Mod} A \,|\, & \text{there exists a } \operatorname{Hom}_A(-,\mathcal{P}(A))\text{-exact exact sequence} \\ & \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots \text{ in } \operatorname{Mod} A \\ & \text{with all } G_i, G^i \in \operatorname{Mod} A \text{ such that } M \simeq \operatorname{Im}(G_0 \to G^0)\}.\end{aligned}$$

We give some applications of Theorem 4.5.

Corollary 4.6. *If A is a ring, then $[\mathcal{GP}(A)]^n = \mathcal{GP}(A)$ for any $n \geq 1$.*

Proof. Take $\mathcal{A} = A$-Mod and $\mathcal{X} = \mathcal{P}(A)$. It is easy to obtain it. □

Corollary 4.7. *If A is a ring, then $[G_P(A)]^n = G_P(A)$ for some $P \in \mathcal{P}(A)$ and any $n \geq 1$.*

Proof. Take $\mathcal{A} = A$-Mod and $\mathcal{X} = \mathrm{Add}(P)$. It is easy to obtain it. □

Corollary 4.8. *The following are equivalent for an A-module M.*

(1) *M is Gorenstein projective.*

(2) *There is a $\mathrm{Hom}_A(-, \mathcal{P}(A))$-exact exact sequence*

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in A-Mod with all $G_i, G^i \in \mathcal{GP}(A)$ such that $M \simeq \mathrm{Im}(G_0 \to G^0)$.

(3) *There is a $\mathrm{Hom}_A(-, \mathcal{X})$-exact exact sequence*

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in A-Mod with all $G_i, G^i \in \mathcal{GP}(A)$ such that $M \simeq \mathrm{Im}(G_0 \to G^0)$ whenever $\mathcal{X}$ is a subcategory in A-Mod with $\mathcal{P}(A) \subseteq \mathcal{X} \subseteq \mathcal{GP}(A)$.

(4) *There is a $\mathrm{Hom}_A(-, \mathcal{GP}(A))$-exact exact sequence*

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in A-Mod with all $G_i, G^i \in \mathcal{GP}(A)$ such that $M \simeq \mathrm{Im}(G_0 \to G^0)$.

Proof. (1) $\Leftrightarrow$ (2) holds by the definition of Gorenstein projective modules and Corollary 4.6.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are trivial.

(1) $\Rightarrow$ (4) Since M is Gorenstein projective, there is a $\mathrm{Hom}_A(-, \mathcal{GP}(A))$-exact exact sequence

$$\cdots \to 0 \to M \to M \to 0 \to \cdots .$$

□

Remark 4.9. We call a subcategory $\mathcal{Y}$ of $\mathcal{A}$ quasi-coresolving if $\mathcal{Y}$ is monic-admissible exact and $\mathcal{Y} \subseteq cores(\mathcal{I}_\mathcal{Y})$ and $\mathcal{I}_\mathcal{Y} = \mathcal{Y} \cap \mathcal{I}(\mathcal{A})$. Then a dual notion of a Gorenstein quasi-resolving subcategory can be given as follows.

A complete $\mathcal{I}_\mathcal{X}$-resolution of M is an exact sequence in $\mathcal{A}$

$$(*) \qquad \mathbf{I}^\bullet : \cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with all I_i and I^i in $\mathcal{I}_\mathcal{X}$, which is $\mathrm{Hom}_\mathcal{A}(\mathcal{X}, -)$, such that $M \simeq \mathrm{Im}(I_0 \to I^0)$.

An object $M \in \mathcal{A}$ is called $\mathcal{GQ}_\mathcal{Y}$-injective if there is a complete $I_\mathcal{X}$-resolution as in $(*)$ with $M \simeq \mathrm{Im}(I_0 \to I^0)$. The subcategory $\mathcal{GQC}_\mathcal{Y}(\mathcal{A})$ of all $\mathcal{GQ}_\mathcal{Y}$-injective objects in $\mathcal{A}$ is called Gorenstein quasi-coresolving. We point out that all of the above results also hold true by using completely dual arguments.

5. Finitistic dimension and endomorphism algebra of a $\mathcal{GQ}_{\mathcal{X}}$-projective module

In this section, let A be an artin algebra and let $\mathcal{A}$ be the category of A-modules. All modules are always finitely generated left modules unless otherwise specified.

Definition 6. An algebra A is called $\mathcal{GQR}_{\mathcal{X}}$-finite if $\mathcal{GQR}_{\mathcal{X}}(A)$ has only finitely many isomorphism classes of indecomposable objects.

Clearly, A is $\mathcal{GQR}_{\mathcal{X}}$-finite if and only if there is an A-module E such that $\mathcal{GQR}_{\mathcal{X}}(A) = add_A E$.

When we take $\mathcal{C} = \mathcal{P}_{\mathcal{X}}$, we get the *finitistic $\mathcal{P}_{\mathcal{X}}$-dimension* of A

$$\text{findim}_{\mathcal{P}_{\mathcal{X}}}(A) = \sup\{\mathcal{P}_{\mathcal{X}}\text{-rdim}(M) \mid M \text{ is a finitely generated left } A\text{-module with } \mathcal{P}_{\mathcal{X}}\text{-rdim}(M) < \infty\}.$$

Similarly, we get the *finitistic $\mathcal{GQR}_{\mathcal{X}}$-dimension* of A

$$\text{findim}_{\mathcal{GQR}_{\mathcal{X}}}(A) = \sup\{\mathcal{GQR}_{\mathcal{X}}\text{-rdim}(M) \mid M \text{ is a finitely generated left } A\text{-module with } \mathcal{GQR}_{\mathcal{X}}\text{-rdim}(M) < \infty\}.$$

Lemma 5.1. *Let A be an Artin algebra. Then* $\text{findim}_{\mathcal{GQR}_{\mathcal{X}}}(A) = \text{findim}_{\mathcal{P}_{\mathcal{X}}}(A)$.

Proof. Let M be a left A-module. By Corollary 3.5, if $\mathcal{P}_{\mathcal{X}}$-rdim$(M) < \infty$, then $\mathcal{P}_{\mathcal{X}}$-rdim$(M) = \mathcal{GQR}_{\mathcal{X}}$-rdim$(M)$. It follows that

$$\text{findim}_{\mathcal{P}_{\mathcal{X}}}(A) \leq \text{finding}_{\mathcal{GQR}_{\mathcal{X}}}(A).$$

Let $0 < \mathcal{GQR}_{\mathcal{X}}$-rdim$(M) < \infty$. Then by Corollary 3.6, there is a module K with $\mathcal{P}_{\mathcal{X}}$-rdim$(K)$= $\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) - 1$, and hence $\text{finding}_{\mathcal{GQR}_{\mathcal{X}}}(A) \leq \text{findim}_{\mathcal{P}_{\mathcal{X}}}(A) + 1$.

We may therefore assume that $0 < \text{findim}_{\mathcal{GQR}_{\mathcal{X}}}(A) = m < \infty$. Take a module M with $\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) = m$. We wish to find a module L with $\mathcal{P}_{\mathcal{X}}$-rdim$(L) = m$. By Corollary 3.6, there is an exact sequence $0 \to K \to G \to M \to 0$ with $G \in \mathcal{GQR}_{\mathcal{X}}(A)$ and $\mathcal{P}_{\mathcal{X}}$-rdim$(K) = m - 1$. By Corollary 3.5, $\mathcal{P}_{\mathcal{X}}$-rdim$(K) = m - 1 = \mathcal{GQR}_{\mathcal{X}}$-rdim$(K)$. Since $G \in \mathcal{GQR}_{\mathcal{X}}(A)$, there are a module $P \in \mathcal{P}_{\mathcal{X}}$ and a monomorphism $G \to P$. Since $K \subseteq P$, we can consider the quotient $L = P/K$. Note that $M \simeq G/K$ is a submodule of L, and thus we get a short exact sequence $0 \to M \to L \to L/M \to 0$.

If L is $\mathcal{GQ}_{\mathcal{X}}$-projective, then Corollary 3.7 will imply $\mathcal{GQR}_{\mathcal{X}}$-rdim$(L/M) = m + 1$, since $\mathcal{GQR}_{\mathcal{X}}$-rdim$(M) = m$. But this contracts with the fact $\text{findim}_{\mathcal{GQR}_{\mathcal{X}}}(A) = m$. Hence L is not $\mathcal{GQ}_{\mathcal{X}}$-projective. Therefore, the short exact sequence $0 \to K \to P \to L \to 0$ shows that

$$\mathcal{GQR}_{\mathcal{X}}\text{-rdim}(L) = \mathcal{GQR}_{\mathcal{X}}\text{-rdim}(K) + 1 = m$$

and L has a finite $\mathcal{P}_{\mathcal{X}}$-resolution of finite m. The former shows that $\mathcal{P}_{\mathcal{X}}$-rdim$(L) \geq \mathcal{GQR}_{\mathcal{X}}$-rdim$(L) = m$ and the latter shows that $\mathcal{P}_{\mathcal{X}}$-rdim$(L) \leq m$. Hence $\mathcal{P}_{\mathcal{X}}$-rdim$(L) = m$. So $\text{finding}_{\mathcal{GQR}_{\mathcal{X}}}(A) \leq \text{findim}_{\mathcal{P}_{\mathcal{X}}}(A)$. $\square$

Lemma 5.2. *Let M be a $\mathcal{GQ}_{\mathcal{X}}$-projective module and $B = \mathrm{End}_A M$. Then for any $X \in B$-mod, $\Omega_B^2(X) = \mathrm{Hom}_A(M, \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N))$ for some $N \in A$-mod.*

Proof. Consider the exact sequence

$$0 \to \Omega_B^2(X) \to P_1 \to P_0 \to X \to 0$$

with P_i projective B-mod, $i = 1, 2$. Applying the functor $M \otimes_B -$, we have an exact sequence

$$(*) \qquad 0 \to Y \to M \otimes_B P_1 \to M \otimes_B P_0 \to M \otimes_B X \to 0$$

with $Y \in A$-mod. Now applying the functor $\mathrm{Hom}_A(M, -)$, we have an induced exact sequence

$$0 \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, M \otimes_B P_1) \to \mathrm{Hom}_A(M, M \otimes_B P_0).$$

Then we have the commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \Omega_B^2(X) & \longrightarrow & P_1 & \longrightarrow & P_0 \\
& \downarrow \phi & & \downarrow \sigma_{P_1} & & \downarrow \sigma_{P_0} \\
0 \longrightarrow & \mathrm{Hom}_A(M, Y) & \longrightarrow & \mathrm{Hom}_A(M, M \otimes_B P_1) & \longrightarrow & \mathrm{Hom}_A(M, M \otimes_B P_0)
\end{array}$$

Since the canonical homomorphisms σ_{P_0} and σ_{P_1} are isomorphisms, ϕ is an isomorphism.

Now consider the exact sequence $(*)$, since $M \otimes_B P_i \in \mathrm{add}_A M$ and $\mathcal{GQR}_{\mathcal{X}}(A)$ is closed under direct sum and direct summands by Theorem 2.6, $M \otimes_B P_i$ is $\mathcal{GQ}_{\mathcal{X}}$-projective. By Theorem 3.2, there exist exact sequences $0 \to Y \to Q_1 \to Q_0 \to N \to 0$ and $0 \to M \otimes_B X \to N \to G \to 0$ in A-mod with $Q_i \in \mathcal{P}_{\mathcal{X}}$ and $G \in \mathcal{GQR}_{\mathcal{X}}(A)$, $i = 0, 1$. Therefore, $Y \simeq \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)$. □

Remark 5.3. In the proof of the above lemma, if $Y \simeq \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)$, then $\mathcal{GQR}_{\mathcal{X}}$-rdim$({}_A M \otimes_B X) = \mathcal{GQR}_{\mathcal{X}}$-rdim$({}_A N)$.

Theorem 5.4. *Let M be a $\mathcal{GQ}_{\mathcal{X}}$-projective A-module and $B = \mathrm{End}_A M$. If A is $\mathcal{GQR}_{\mathcal{X}}$-finite with $\mathcal{GQR}_{\mathcal{X}}(A) = \mathrm{add}_A E$, then we have*

(1) *If* $\mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A = 0$ *or* 1, *then*

$$\mathrm{findim} B \leq 2 + \mathrm{rfd}(M_B) + \mathrm{pd}_B \mathrm{Hom}_A(M, E).$$

(2) *If* $\mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A \geq 2$, *then*

$$\mathrm{findim} B \leq \mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A + \mathrm{rfd}(M_B) + \mathrm{pd}_B \mathrm{Hom}_A(M, E).$$

Proof. If $\mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A$ or $\mathrm{rfd}(M_B)$ is infinite, then we have nothing to say. So we assume that $\mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A = r < \infty$ and $rfd(M_B) = t < \infty$.

Let ${}_B Y \in B$-mod with $\mathrm{pd}({}_B Y) < \infty$. Denote by Y_i the i-th syzygy of Y for each i. Since $rfd(M_B) = t$, we have $\mathrm{Tor}^B_{i+t}(M, Y) = \mathrm{Tor}^B_i(M, Y_t) = 0$ for $i \geq 1$.

Let $0 \to P_m \to \cdots \to P_1 \xrightarrow{f} P_0 \to Y_t \to 0$ be a projective resolution of Y_t in B-mod. Applying the functor $M \otimes_B -$ to the above exact sequence, since $\mathrm{Tor}_i^B(M, Y_t) = 0$ for $i \geq 1$, we obtain the following exact sequence

$$(**) \qquad 0 \to M \otimes_B P_m \to \cdots \to M \otimes_B P_1 \xrightarrow{1 \otimes f} M \otimes_B P_0 \to M \otimes_B Y_t \to 0.$$

Now consider the exact sequence $(**)$, since $M \otimes_B P_i \in \mathrm{add}_A M$ and $\mathcal{GQR}_{\mathcal{X}}(A)$ is closed under direct sums and direct summands, we obtain that $M \otimes_B P_i$ is still $\mathcal{GQ}_{\mathcal{X}}$-projective and thus $\mathcal{GQR}_{\mathcal{X}}\text{-rdim}(M \otimes_B Y_t) < \infty$, $i = 0, 1, \ldots, m$. By Lemma 5.1, we get $\mathcal{GQR}_{\mathcal{X}}\text{-rdim}(M \otimes_B Y_t) \leq r$.

By Lemma 5.2, $\Omega_B^2(Y_t) = \mathrm{Hom}_A(M, \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N))$ for some $N \in A$-mod. In fact, $\Omega^2_{\mathcal{P}_{\mathcal{X}}}(N) \simeq \mathrm{Ker}(1 \otimes f)$ has a finite $\mathcal{GQR}_{\mathcal{X}}$-resolution. By Remark 5.3, we have $\mathcal{GQR}_{\mathcal{X}}\text{-rdim}({}_AM \otimes_B Y_t) = \mathcal{GQR}_{\mathcal{X}}\text{-rdim}({}_AN)$.

(1) If $r = \mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A < 2$, then $\mathcal{GQR}_{\mathcal{X}}\text{-rdim}({}_AM \otimes_B Y_t) = \mathcal{GQR}_{\mathcal{X}}\text{-rdim}({}_AN)$ < 2. Hence $\Omega^2_{\mathcal{P}_{\mathcal{X}}}(N) \in \mathcal{GQR}_{\mathcal{X}}(A)$. We obtain that

$$\begin{aligned} \mathrm{pd}_B(Y) &\leq \mathrm{pd}_B(\Omega_B^2(Y_t)) + t + 2 = \mathrm{pd}_B \mathrm{Hom}_A(M, \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)) + t + 2 \\ &\leq 2 + \mathrm{rfd}(M_B) + \mathrm{pd}_B \mathrm{Hom}_A(M, E). \end{aligned}$$

(2) If $r = \mathrm{findim}_{\mathcal{P}_{\mathcal{X}}} A \geq 2$, then $\mathcal{GQR}_{\mathcal{X}}\text{-rdim}({}_AN) = \mathcal{GQR}_{\mathcal{X}}\text{-rdim}(M \otimes_B Y_t) \leq r$. Hence, $\mathcal{GQR}_{\mathcal{X}}\text{-rdim}(\Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)) \leq r - 2$ by Corollary 3.7. By Theorem 3.2, we obtain the exact sequence

$$(***) \qquad 0 \to P_{r-2} \to \cdots \to P_1 \to G_0 \to \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N) \to 0$$

in A-mod where $G_0 \in \mathcal{GQR}_{\mathcal{X}}(A)$ and $P_i \in \mathcal{P}_{\mathcal{X}}$ for $i = 1, \ldots, r-2$. Moreover by applying the functor $\mathrm{Hom}_A(M, -)$ to $(***)$, we obtain the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(M, P_{r-2}) \to \cdots \to \mathrm{Hom}_A(M, P_1) \\ &\to \mathrm{Hom}_A(M, G_0) \to \mathrm{Hom}_A(M, \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)) \to 0. \end{aligned}$$

Note that $\mathrm{pd}_B \mathrm{Hom}_A(M, P_i) \leq \mathrm{pd}_B \mathrm{Hom}_A(M, E)$, so

$$\mathrm{pd}_B \mathrm{Hom}_A(M, \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)) \leq \mathrm{pd}_B \mathrm{Hom}_A(M, E) + r - 2.$$

Thus we have

$$\begin{aligned} \mathrm{pd}_B(Y) &\leq \mathrm{pd}_B(\Omega_B^2(Y_t)) + t + 2 = \mathrm{pd}_B \mathrm{Hom}_A(M, \Omega^2_{\mathcal{P}_{\mathcal{X}}}(N)) + t + 2 \\ &\leq r + t + \mathrm{pd}_B \mathrm{Hom}_A(M, E). \end{aligned}$$

□

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