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FINITE p-GROUPS WHOSE NORMAL CLOSURES OF NON-NORMAL SUBGROUPS HAVE TWO ORDERS

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Abstract. We describe the structure of finite p-groups in which all normal closures of non-normal subgroups have two orders for $p > 2$.

1. Introduction

A finite group is called a Dedekind group if all its subgroups are normal. Such groups were classified by Dedekind in 1897. Many scholars studied finite groups which are "near" Dedekind groups in different ways. For example, Passman [4] classified the finite p-groups having a gap and the finite p-groups all of whose non-normal subgroups are cyclic. As one special case of the second classification, he classified the finite p-groups all of whose non-normal subgroups are of order p. Along the same line, Zhang et al. [7] classified finite groups whose non-normal subgroups are of order p or pq, where p, q are primes. Zhang et al. [8, 9] classified the finite p-groups whose non-normal subgroups have orders at most p^3. An [1] classified the finite p-groups whose non-normal subgroups have orders p^m and p^{m+1}.

In [5], Qu and the author investigated finite groups which are "near" Dedekind groups from the point of view of normal closure of non-normal subgroups. For a finite group G, they introduced the following notation:

$$\Delta(G) = \{|H^G| \mid H \text{ is not normal in } G\}.$$

They also called a group G with $|\Delta(G)| = k$ is a Δ_k-group. A finite group G is a Dedekind group if and only if $\Delta(G) = \emptyset$. Those groups G classified by Passman in [4] are exactly those p-groups with $\Delta(G) = \{p^2\}$. Finite p-groups classified by Zhang et al. in [7] are exactly those p-groups with $\Delta(G) \subseteq \{p^2, p^3\}$. Finite p-groups G classified by Zhang et al. in [8, 9] are exactly those p-groups with $\Delta(G) \subseteq \{p^2, p^3, p^4\}$. Finite p-groups G classified by An in [1] are exactly those p-groups with $\Delta(G) \subseteq \{p^{m+1}, p^{m+2}\}$. Obviously, An, Passman

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and Zhang classified finite p-groups in which $\Delta(G)$ has both a specified size and is contained in a specific set of numbers.

Notice that a finite group is a Δ_k-group for some a fixed positive integer k. Hence determining all Δ_k-groups is hopeless. However, for small k, the structure of Δ_k-groups can be determined. Qu and the author in this paper in [5] determined the structure of Δ_1-groups, where the value of the element of $\Delta(G)$ is not restricted. In this paper, we continue the work in [5]. For an odd prime p, we study the structure of Δ_2-p-groups, where a Δ_2-p-group G means G is both a finite p-group and a Δ_2-group. The main result in this paper is:

Theorem 1.1. *Assume G is a Δ_2-p-group and p is an odd prime.*

(1) $d(G) \leq 5$ *and* $|G'| \leq p^5$.

(2) *If* $|G'| = p$, *then G is one of the groups in Theorem 2.4.*

(3) *If* $|G'| \geq p^2$, *then* $p^4 \leq |G| \leq p^{10}$ *except for G being metacyclic and*

$$G \cong \langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle, \text{ where } n \geq m \geq 2.$$

The notation and terminology are standard, see [2]. For a nilpotent group G, we use $c(G)$ to denote the nilpotent class of G, and use C_n and C_p^n to denote the cyclic group of order n and the elementary abelian group of order p^n, respectively.

We use $\mathrm{M}_p(n, m)$, where $n \geq 2$, to denote the p-group

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle,$$

and $\mathrm{M}_p(n, m, 1)$, with $n \geq m$, and if $p = 2$, then $m + n \geq 3$, to denote the p-group

$$\langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle.$$

If H and K are subgroups of G with $G = HK$ and $[H, K] = 1$, we call G a *central product* of H and K, denoted by $G = H * K$. Clearly, $H \cap K \leq Z(G)$. In this paper, when we refer to a central product we always assume $H \cap K \neq 1$. Let G be a finite p-group. For any positive integer s, we define

$$\Omega_s(G) = \langle a \in G \mid a^{p^s} = 1 \rangle \text{ and } \mho_s(G) = \langle a^{p^s} \mid a \in G \rangle.$$

In this paper we always assume p is an odd prime.

2. Proving $d(G) \leq 5$ for a Δ_2-p-group G

Lemma 2.1 ([5, Theorem 3.6])**.** *Let G be a p-group and $p > 2$. Then G is a Δ_1-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(1) $\mathrm{M}_p(n, m), n \geq m$;

(2) $\mathrm{M}_p(1, 1, 1) * \mathrm{C}_{p^n}$;

(3) $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = 1, [a, c] = b^p, [b, c] = a^p b^{wp} \rangle$, *where* $1 \leq w \leq \frac{p-1}{2}$ *and* $1 + \frac{w^2}{4}$ *is a quadratic non-residue modulo* p;

(4) $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = 1, [a, c] = b^{vp}, [b, c] = a^p b^{wp} \rangle$, *where v is a quadratic non-residue modulo p, $0 \leq w \leq \frac{p-1}{2}$ and $v + \frac{w^2}{4}$ is a quadratic non-residue modulo p.*

Lemma 2.2. *Let G be a finite group and $N \trianglelefteq G$.*

(1) *If $|\Delta(G)| = 2$, then $|\Delta(G/N)| \leq 2$;*
(2) *If $\Delta(G/N) = \{p^s, p^t\}$ and $|\Delta(G)| = 2$, then $\Delta(G) = \{p^{s+l}, p^{t+l}\}$, where $|N| = p^l$.*

Proof. (1) Let $\Delta(G) = \{p^k, p^l\}$ and $|N| = p^s$. Take any non-normal subgroup H/N of G/N. Then H is non-normal in G. Hence $|H^G| = p^k$ or p^l. Since $(H/N)^{G/N} = H^G/N$, $|(H/N)^{G/N}| = p^{k-s}$ or p^{l-s}. It follows that $\Delta(G/N) \subseteq \{p^{k-s}, p^{l-s}\}$, and $|\Delta(G/N)| \leq 2$.

(2) By hypothesis, there exist non-normal subgroups H, K of G such that $|(H/N)^G| = p^s$ and $|(K/N)^G| = p^t$. It is clear that $(H/N)^G = H^G/N$ and $(K/N)^G = K^G/N$. Let $|N| = p^l$. It follows that $|H^G| = p^{s+l}$ and $|K^G| = p^{t+l}$. Since $|\Delta(G)| = 2$, $\Delta(G) = \{p^{s+l}, p^{t+l}\}$. $\square$

Lemma 2.3 ([1, Theorems 13-15])**.** *Assume p is an odd prime and G is a finite p-group with $|G'| = p$. Then the orders of non-normal subgroups of G are p^k and p^{k+1} if and only if G is one of the following non-isomorphic groups*:

(1) $\mathrm{M}_p(n, n+1)$, $n \geq 2$;
(2) $\mathrm{M}_p(2, 1, 1) * \mathrm{C}_{p^n}$, $n \geq 1$;
(3) $\mathrm{M}_p(2, 2, 1) * \mathrm{C}_{p^n}$, $n \geq 1$;
(4) $\mathrm{M}_p(n, m) \times \mathrm{C}_p$, $n \geq m$;
(5) $\mathrm{M}_p(1, 1, 1) * \mathrm{M}_p(m, 1)$, $m \geq 2$;
(6) $(\mathrm{M}_p(1, 1, 1) * \mathrm{C}_{p^n}) \times \mathrm{C}_p$, $n \geq 1$;
(7) $(\mathrm{M}_p(1, 1, 1) * \mathrm{M}_p(1, 1, 1)) * \mathrm{C}_{p^n}$, $n \geq 1$.

Theorem 2.4. *Assume G is a finite p-group with $|G'| = p$. Then G is a Δ_2-group if and only if G is isomorphic to one of the groups listed in Lemma 2.3.*

Proof. ($\Longleftarrow$) If G is one of the groups (1)–(7) in Lemma 2.3, then the orders of non-normal subgroups of G are p^k and p^{k+1} by Lemma 2.3. Since $|G'| = p$, $H^G = HG'$ for any non-normal subgroup H of G. It follows that $\Delta(G) = \{p^{k+1}, p^{k+2}\}$. Thus G is a Δ_2-group.

($\Longrightarrow$) Let $\Delta(G) = \{p^k, p^l\}$, where $k < l$. Assume that H is a non-normal subgroup of G. Since $|G'| = p$ and $HG' \trianglelefteq G$, $H^G = HG'$. It follows that $|H| \in \{p^{k-1}, p^{l-1}\}$. Thus, the orders of the non-normal subgroups of G are p^{k-1} and p^{l-1}. By [4, Theorem 1.10], we have $l = k + 1$. The conclusion follows by Lemma 2.3. $\square$

Theorem 2.5. *If G is a Δ_2-p-groups, then $d(G) \leq 5$.*

Proof. We prove this conclusion by induction on $|G'|$. If $|G'| = p$, then $d(G) \leq 5$ by Theorem 2.4. Assume the conclusion holds for $|G'| \leq p^k$. Now assume

$|G'| = p^{k+1}$. Take a normal subgroup N of order p of G contained in G'. Then $|(G/N)'| = p^k$. By Lemma 2.2, G/N is a Δ_1-group or a Δ_2-group. If G/N is a Δ_1-group, then $d(G/N) \leq 5$ by Lemma 2.1. If G/N is a Δ_2-group, then, since $|(G/N)'| = p^k$, $d(G/N) \leq 5$ by induction hypothesis. Since $N \leq G'$, $d(G) \leq 5$. The conclusion follows. □

3. The case of $d(G) = 2$

In this section, we prove the following:

Theorem 3.1. *Let G be a Δ_2-p-group with $d(G) = 2$. Then $|G| \leq p^5$ or G is isomorphic to one of the following groups*:

(1) $\mathrm{M}_p(n, n+1)$, *a metacyclic minimal nonabelian p-group*;
(2) $\langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle$, $n \geq m \geq 2$, *a metacyclic p-group.*

We will prove Theorem 3.1 in two cases: G is metacyclic or non-metacyclic. Moreover, we classify metacyclic Δ_2-p-groups.

For $x_1, x_2 \in G$, the commutator of x_1, x_2 is defined by $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. For commutators of length greater than 2, we keep the left-normed convention, that is, $[x_1, x_2, \cdots, x_n] = [[\cdots[[x_1, x_2], x_3]\cdots, x_{n-1}], x_n]$. The following result is useful in our proof.

Lemma 3.2 ([6, Lemma 3])**.** *Let G be a metabelian p-group and a, b be elements of G. Then, for all positive integers m and n,*

$$[a^m, b^n] = \prod_{i=1}^{m}\prod_{j=1}^{n}[ia, jb]^{\binom{m}{i}\binom{n}{j}},$$

where $[ia, jb] = [a, b, \underbrace{a, \ldots, a}_{i-1}, \underbrace{b, \ldots, b}_{j-1}]$.

Lemma 3.3 ([3, Theorem 10])**.** *Let p be an odd prime and G a finite p-group. Assume that N is a normal subgroup of order p of G contained in G' and $G/N \cong \mathrm{M}_p(n, m)$, then G is one of the following non-isomorphic groups*:

(1) $\langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle$, $n \geq 2, m \geq 2$;
(2) $\langle a, b \mid a^{p^{n+1}} = 1, b^{p^m} = a^{p^n}, [a, b] = a^{p^{n-1}} \rangle$, $m > n \geq 2$.

Lemma 3.4. *Let G be a metacyclic p-group. If G is a Δ_2-p-group with $|G'| = p^2$, then $G \cong \langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle$, $n \geq m \geq 2$.*

Proof. Take $N \trianglelefteq G$ such that $N \leq G'$ and $|N| = p$. Let $\overline{G} = G/N$. By Lemma 2.2, $\overline{G}$ is a Δ_1-group or a Δ_2-group. Notice that $|\overline{G}'| = p$ and $\overline{G}$ is metacyclic. If $\overline{G}$ is a Δ_1-group, then $\overline{G} \cong \mathrm{M}_p(n, m)$ by Lemma 2.1, where $n \geq m$. If $\overline{G}$ is a Δ_2-group, then, by checking those groups in Lemma 2.3, $\overline{G} \cong \mathrm{M}_p(k, k+1)$, where $k \geq 2$.

Assume $\overline{G} \cong \mathrm{M}_p(n,m)$, where $n \geq m$. By Lemma 3.3, G is isomorphic to the group (1) or (2) listed in Lemma 3.3. Since $n \geq m$,

$$G \cong \langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a,b] = a^{p^{n-1}} \rangle, \text{ where } n \geq m \geq 2.$$

We get the required group.

Assume that $\overline{G} \cong \mathrm{M}_p(k, k+1)$. Then, by Lemma 3.3, G is isomorphic to

$$\langle a, b \mid a^{p^{k+1}} = b^{p^{k+1}} = 1, [a,b] = a^{p^{k-1}} \rangle, \ k \geq 2$$

or

$$\langle a, b \mid a^{p^{k+1}} = 1, b^{p^{k+1}} = a^{p^k}, [a,b] = a^{p^{k-1}} \rangle, \ k \geq 2.$$

For the first group, we have $\langle b \rangle$, $\langle b^p \rangle$ and $\langle ab^p \rangle$ are non-normal in G, and

$$\langle b \rangle^G = \langle b, a^{p^{k-1}} \rangle, \ \langle b^p \rangle^G = \langle b^p, a^{p^k} \rangle \ \text{ and } \ \langle ab^p \rangle^G = \langle ab^p, a^{p^{k-1}} \rangle,$$

$$|\langle b \rangle^G| = p^{k+3}, \ |\langle b^p \rangle^G| = p^{k+1} \text{ and } |\langle ab^p \rangle^G| = p^{k+2}.$$

Thus $\{p^{k+3}, p^{k+1}, p^{k+2}\} \subseteq \Delta(G)$, which is contrary to $|\Delta(G)| = 2$.

For the second group, we have $\langle b \rangle$, $\langle b^p a^{-1} \rangle$ and $\langle b^{p^2} a^{-p} \rangle$ are non-normal in G, and

$$\langle b \rangle^G = \langle b, a^{p^{k-1}} \rangle, \ \langle b^p a^{-1} \rangle^G = \langle b^p a^{-1}, a^{p^{k-1}} \rangle \ \text{ and } \ \langle b^{p^2} a^{-p} \rangle^G = \langle b^{p^2} a^{-p}, a^{p^k} \rangle,$$

$$|\langle b \rangle^G| = p^{k+3}, \ |\langle b^p a^{-1} \rangle^G| = p^{k+2} \text{ and } |\langle b^{p^2} a^{-p} \rangle^G| = p^k.$$

Thus $\{p^{k+3}, p^{k+2}, p^k\} \subseteq \Delta(G)$, which is contrary to $|\Delta(G)| = 2$. □

Lemma 3.5. *Let G be a metacyclic Δ_2-p-group. Then $|G'| \leq p^2$.*

Proof. Suppose that the result is false and G is a counterexample with the smallest order. Take $N \trianglelefteq G$ such that $N \leq G'$ and $|N| = p$. Let $\overline{G} = G/N$. Then $|\overline{G}'| \geq p^2$ and $\overline{G}$ is a Δ_1-group or a Δ_2-group by Lemma 2.2. Since G is metacyclic, $d(G) = 2$. Hence $d(\overline{G}) = 2$. If $\overline{G}$ is a Δ_1-group, then $|\overline{G}'| = p$ by Lemma 2.1. This is contrary to $|\overline{G}'| \geq p^2$. It follows that $\overline{G}$ is a Δ_2-group. By the minimality of G, we have $|\overline{G}'| = p^2$. By Lemma 3.4,

$$\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^{n+1}} = \bar{b}^{p^m} = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^{n-1}} \rangle, \ n \geq m \geq 2.$$

Since G is metacyclic, $G' = \langle a^{p^{n-1}} \rangle$ and $N = \langle a^{p^{n+1}} \rangle$. Thus $a^{p^{n+2}} = 1$. Let $b^{p^m} = a^{ip^{n+1}}$. Replacing b by $ba^{-ip^{n-m+1}}$, we have $b^{p^m} = 1$ and

$$G = \langle a, b \mid a^{p^{n+2}} = 1, b^{p^m} = 1, [a,b] = a^{jp^{n-1}} \rangle, \quad \text{where} \quad (p,j) = 1.$$

If $m = 2$, then, by Lemma 3.2, we have

$$1 = [a, b^{p^2}] = [a,b]^{\binom{p^2}{1}} [a,b,b]^{\binom{p^2}{2}} = a^{jp^{n+1}}.$$

Hence, $a^{p^{n+1}} = 1$, which is a contradiction. Thus, $m \geq 3$. Now we have $\langle b \rangle$, $\langle b^p \rangle$ and $\langle b^{p^2} \rangle$ are non-normal in G. By calculating their normal closure respectively, we have $\{p^{m+3}, p^{m+1}, p^{m-1}\} \subseteq \Delta(G)$. This is a contradiction. □

Theorem 3.6. *Let G be a metacyclic Δ_2-p-group. Then G is isomorphic to* $\mathrm{M}_p(n, n+1)$ *or* $\langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle$*, where* $n \geq m \geq 2$*.*

Proof. By Lemma 3.5, $|G'| \leq p^2$. If $|G'| = p^2$, then G is the second group in the theorem by Lemma 3.4. If $|G'| = p$, then G is one of the groups in Theorem 2.4. Since G is metacyclic, $G \cong \mathrm{M}_p(n, n+1)$. □

Now, we consider the non-metacyclic Δ_2-p-groups with $d(G) = 2$.

Lemma 3.7. *Let G be a non-metacyclic Δ_2-p-group with $d(G) = 2$. If there exists a normal subgroup N of order p of G contained in G' such that G/N is a Δ_1-group, then $c(G) \leq 3$ and* $\exp(G') = p$*.*

Proof. Since $d(G) = 2$, G/N is isomorphic to $\mathrm{M}_p(n, m)$ or $\mathrm{M}_p(1, 1, 1)$ by Lemma 2.1. If $G/N \cong \mathrm{M}_p(n, m)$, then G is metacyclic by Lemma 3.3. This is a contradiction. Hence G/N is isomorphic to $\mathrm{M}_p(1, 1, 1)$. We may assume

$$G/N = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^p = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c}, [\bar{a}, \bar{c}] = [\bar{b}, \bar{c}] = 1 \rangle.$$

Let $N = \langle x \rangle$. It is clear that $G' = \langle c, x \rangle$ and G' is abelian. Obviously, $a^p \in N \subseteq Z(G)$. Hence $[a^p, b] = 1$. On the other hand, by $|G_3| \leq p$ and Lemma 3.2,

$$[a^p, b] = [a, b]^p [a, b, a]^{\binom{p}{2}} = [a, b]^p = (cx^i)^p = c^p.$$

Hence, $c^p = 1$. It follows that $G' = \langle c, x \rangle \cong \mathrm{C}_p^2$ and $\exp(G') = p$. Since $|G'| = p^2$, $c(G) \leq 3$. □

Theorem 3.8. *If G is a non-metacyclic Δ_2-p-group with $d(G) = 2$, then $c(G) \leq 3$ and* $\exp(G') = p$*.*

Proof. Suppose that the theorem is false. Let G be a counterexample with the smallest order. Let $\Delta(G) = \{p^k, p^l\}$, where $k < l$. Since G is non-metacyclic, there exists a normal subgroup N of G with the type (p, p) by [2, Lemma 1.4]. We prove the theorem by three steps.

Step 1. $N \leq Z(G)$.

If not, then there exists a subgroup H of order p of N such that H is non-normal in G. Thus $N = H^G$ and $p^2 \in \Delta(G)$. Since $H^G = H[H, G]$, $|[H, G]| = p$. Let $K = [H, G]$. Then $|\Delta(G/K)| = 1$. Since G is non-metacyclic and $d(G) = 2$, G/K is isomorphic to $\mathrm{M}_p(1, 1, 1)$ by Lemma 2.1 and Lemma 3.3.

For convenience assume

$$G/K = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^p = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c}, [\bar{a}, \bar{c}] = [\bar{b}, \bar{c}] = 1 \rangle.$$

Let $K = \langle x \rangle$. It is clear that $G' = \langle c, x \rangle$ and G is metabelian. It follows from $\bar{a}^p = 1$ that $a^p \in Z(G)$ and $[a^p, b] = 1$. On the other hand, by $|G_3| \leq p$ and Lemma 3.2,

$$[a^p, b] = [a, b]^p [a, b, a]^{\binom{p}{2}} = [a, b]^p = (cx^i)^p = c^p.$$

Hence, $c^p = 1$. It follows that $G' = \langle c, x \rangle \cong \mathrm{C}_p^2$ and $\exp(G') = p$. Since $d(G) = 2$ and $G' \cong \mathrm{C}_p^2$, $c(G) = 3$. This contradicts with G being a counterexample. Hence $N \leq Z(G)$.

Step 2. $N \leq \Phi(G)$.

If not, then there exists $b \in N \backslash \Phi(G)$. Since $d(G) = 2$, $G = \langle b, c\rangle$ for some $c \in G$. By Step 1, $b \in Z(G)$. It follows that G is abelian, a contradiction. Hence, $N \leq \Phi(G)$.

Step 3. Final contradiction.

Let $N = N_1 \times N_2$. Then $|\Delta(G/N_i)| = 1$ or 2 for $i = 1, 2$. If $|\Delta(G/N_i)| = 1$, then $c(G) \leq 3$ and $\exp(G') = p$ by Lemma 3.7. If $|\Delta(G/N_i)| = 2$, we also have $c(G/N_i) \leq 3$ and $\exp((G/N_i)') = p$ by the minimality of G. It follows that $c(G) \leq 3$ and $\exp(G) = p$ since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$. This contradicts with G being a counterexample. □

Theorem 3.9. *If G is a non-metacyclic Δ_2-p-group with $d(G) = 2$, then G' is elementary abelian and $|G'| \leq p^3$. If $G' \cong \mathrm{C}_p^2$, then $|G| = p^4$; if $G' \cong \mathrm{C}_p^3$, then $|G| = p^5$.*

Proof. By Theorem 3.8, $c(G) \leq 3$. Hence, $G_4 = 1$ and G' is abelian. Let $G = \langle a, b\rangle$. Then $G' = \langle [a, b], [a, b, a], [a, b, b]\rangle$. By Theorem 3.8 again, $\exp(G') = p$. It follows that G' is elementary abelian and $|G'| \leq p^3$.

Assume $G' \cong \mathrm{C}_p^2$. Let N be a normal subgroup of order p of G contained in G'. Then $(G/N)' \cong \mathrm{C}_p$. By Lemma 2.2, G/N is a Δ_1-group or a Δ_2-group.

We assert that G/N is a Δ_1-group. If not, then G/N is a Δ_2-group. Since $(G/N)' \cong \mathrm{C}_p$ and G is non-metacyclic, $G/N \cong \mathrm{M}_p(2, 1, 1)$ or $\mathrm{M}_p(2, 2, 1)$ by Lemma 2.4. Clearly, $\Delta(G/N) = \{p^2, p^3\}$ or $\{p^3, p^4\}$. It follows by Lemma 2.2(2) that $\Delta(G) = \{p^3, p^4\}$ or $\{p^4, p^5\}$.

Let $[a, b] = c$. Then $G' = \langle c, [c, a], [c, b]\rangle$. By Theorem 3.8, $c^p = 1$. Since $G' \cong \mathrm{C}_p^2$, $[c, a] \neq 1$ or $[c, b] \neq 1$. It follows that $\langle c\rangle$ is non-normal in G. Clearly, $\langle c\rangle^G = G'$. It follows that $p^2 \in \Delta(G)$, which is a contradiction. Therefore, G/N is a Δ_1-group. Since $d(G) = 2$ and G is non-metacyclic, $G/N \cong \mathrm{M}_p(1, 1, 1)$ by Lemma 2.1. It follows that $|G| = p^4$.

Assume $G' \cong \mathrm{C}_p^3$. Let N be a normal subgroup of order p of G contained in G'. Then $(G/N)' \cong \mathrm{C}_p^2$. By Lemma 2.2, G/N is a Δ_1-group or a Δ_2-group. Since $(G/N)' \cong \mathrm{C}_p^2$, G/N is not a Δ_1-group by Lemma 2.1. Hence, G/N is a Δ_2-group. By the argument above previous three paragraphs, $|G/N| = p^4$. Therefore, $|G| = p^5$. □

The proof of Theorem 3.1. If G is metacyclic, then, by Theorem 3.6, G is isomorphic to $\mathrm{M}_p(n, n+1)$ or $\langle a, b \mid a^{p^{n+1}} = b^{p^m} = 1, [a, b] = a^{p^{n-1}}\rangle$, where $n \geq m \geq 2$. Now, we assume that G is non-metacyclic. If $|G'| = p$, then G is isomorphic to $\mathrm{M}_p(2, 1, 1)$ or $\mathrm{M}_p(2, 2, 1)$ by Theorem 2.4. Hence, $|G| \leq p^5$. If $|G'| > p$, then $|G| \leq p^5$ by Theorem 3.9. This completes the proof. □

4. The case of $d(G) \geq 3$

Assume G is a Δ_2-p-groups with $d(G) \geq 3$. In this section, we give some properties of G and the relationship among $|G|$, $|G'|$ and $d(G)$. In particular, we prove G is a $\mathcal{C}_a$-group. A finite p-group G is called a $\mathcal{C}_a$-group if G/H^G is abelian for every non-normal subgroup H of G. This concept was introduced and studied by Zhang et al. in [10].

Lemma 4.1. *Let G be a Δ_2-p-group with $d(G) \geq 3$. If there exists a normal subgroup N of G of order p contained in G' such that G/N is a Δ_1-group, then $c(G) = 2$ and $\exp(G') = p$.*

Proof. By the hypothesis, G/N is isomorphic to one of the groups (2)–(4) in Lemma 2.1. We discuss in two cases:

Case 1. $\overline{G}$ is isomorphic to the group (2) in Lemma 2.1.

That is, $\overline{G} \cong \mathrm{M}_p(1,1,1) * \mathrm{C}_{p^n}$. For convenience assume

$$\overline{G} \cong \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{p^n} = \bar{b}^p = \bar{c}^p = 1, [\bar{b}, \bar{c}] = \bar{a}^{p^{n-1}}, [\bar{a}, \bar{b}] = [\bar{a}, \bar{c}] = 1 \rangle.$$

Let $N = \langle x \rangle$. It is clear that $G' = \langle a^{p^{n-1}}, x \rangle$ and G is metabelian. Obviously, $b^p \in Z(G)$ and $[b^p, c] = 1$. On the other hand, by $|G_3| \leq p$ and Lemma 3.2,

$$[b^p, c] = [b, c]^p [b, c, b]^{\binom{p}{2}} = [b, c]^p = (a^{p^{n-1}} x^i)^p = a^{p^n}.$$

Hence, $a^{p^n} = 1$. It follows that $G' = \langle a^{p^{n-1}}, x \rangle \cong \mathrm{C}_p^2$ and $\exp(G') = p$.

Thus, by Lemma 3.2 again, we have

$$[a^p, b] = [a, b]^p = 1 \text{ and } [a^p, c] = [a, c]^p = 1.$$

Hence, $a^p \in Z(G)$. It follows that $G' \leq Z(G)$ and $c(G) = 2$.

Case 2. $\overline{G}$ is isomorphic to the group (3) or (4) in Lemma 2.1.

Assume without loss of generality

$$\overline{G} \cong \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{p^2} = \bar{b}^{p^2} = \bar{c}^p = 1, [\bar{a}, \bar{b}] = 1, [\bar{a}, \bar{c}] = \bar{b}^{\mu p}, [b, c] = \bar{a}^p \bar{b}^{wp} \rangle,$$

where $\mu = 1$ or ν, ν is a quadratic non-residue modulo p, $1 \leq w \leq \frac{p-1}{2}$ and $\mu + \frac{w^2}{4}$ is a quadratic non-residue modulo p.

Let $N = \langle x \rangle$. It is clear that $G' = \langle a^p, b^p, x \rangle$, $G_3 \leq N$. Hence, G is metabelian. It follows from $\bar{c}^p = 1$ that $c^p \in Z(G)$ and $[a, c^p] = 1$. On the other hand, by $G_3 \leq N$ and Lemma 3.2,

$$[a, c^p] = [a, c]^p [a, c, c]^{\binom{p}{2}} = [a, c]^p = (b^{\mu p} x^i)^p = b^{\mu p^2}.$$

Hence $b^{p^2} = 1$. Similarly, $a^{p^2} = 1$. It follows that $G' = \langle a^p, b^p, x \rangle \cong \mathrm{C}_p^3$ and $\exp(G') = p$.

Thus, by Lemma 3.2 again, we have

$$[a^p, b] = [a, b]^p = 1 \text{ and } [a^p, c] = [a, c]^p = 1.$$

Hence, $a^p \in Z(G)$. Similarly, $b^p \in Z(G)$. It follows that $G' \leq Z(G)$ and $c(G) = 2$. □

Theorem 4.2. *If G is a Δ_2-p-groups with $d(G) \geq 3$, then $c(G) = 2$ and $\exp(G') = p$. In particular, $\Phi(G) \leq Z(G)$.*

Proof. Suppose that the theorem is false. Let G be a counterexample with the smallest order. Assume $\Delta(G) = \{p^k, p^l\}$, where $k < l$. Since $d(G) \geq 3$, there exists a normal subgroup N of G with the type (p, p) by [2, Lemma 1.4]. We prove the results by following steps.

(1) $N \leq Z(G)$.

If not, then there exists a subgroup H of N of order p such that H is non-normal in G. Thus $N = H^G$ and $k = 2$. Since $H^G = H[H, G]$, $|[H, G]| = p$. Let $K = [H, G]$. Then $|\Delta(G/K)| = 1$. It follows by Lemma 4.1 that $c(G) = 2$ and $\exp(G') = p$. This is a contradiction. Hence $N \leq Z(G)$.

(2) $N \leq \Phi(G)$.

If not, then G has a maximal subgroup M such that $G = MN$. Since $N \cong \mathrm{C}_p \times \mathrm{C}_p$ and $N \leq Z(G)$, there exists a subgroup K of order p of N such that $G = M \times K$. Since $G/K \cong M$, $|\Delta(M)| = 1$ or 2 by Lemma 2.2.

If $|\Delta(M)| = 1$, then $c(M) = 2$ and $\exp(M') = p$ by Lemma 2.1. Since $G = M \times K$, $G' = M' \leq Z(M) \leq Z(G)$. It follows that $c(G) = 2$ and $\exp(G') = p$. This is a contradiction. Hence $|\Delta(M)| = 2$.

Let H_1 and H_2 be non-normal subgroups of M such that $|H_1^M| \neq |H_2^M|$. Without loss generality assume that $|H_1^M| < |H_2^M|$. Since $G = M \times K$, $H_1^G = H_1^M$ and $H_2^G = H_2^M$. It is clear that H_2K is non-normal in G and $(H_2K)^G = H_2^G K = H^M K$. It follows that $|(H_2K)^G| > |H_2^G|$. Thus,

$$\{|H_1^G|, |H_2^G|, |(H_2K)^G|\} \subseteq \Delta(G),$$

which is contradict to $|\Delta(G)| = 2$. Hence, $N \leq \Phi(G)$.

(3) Final contradiction.

Let $N = N_1 \times N_2$. Then $|\Delta(G/N_i)| = 1$ or 2 for $i = 1, 2$. If $|\Delta(G/N_i)| = 1$, then $c(G/N_i) = 2$ and $\exp((G/N_i)') = p$ by Lemma 2.1. If $|\Delta(G/N_i)| = 2$, we also have $c(G/N_i) = 2$ and $\exp((G/N_i)') = p$ by the minimality of G. It follows that $c(G) = 2$ and $\exp(G) = p$ since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$. This is a contradiction.

In particular, take any two elements a, b of G. Since $c(G) = 2$ and $\exp(G') = p$, $[a^p, b] = [a, b]^p = 1$. By the arbitrariness of a, b, we have $\mho_1(G) \leq Z(G)$. It follows that $\Phi(G) \leq Z(G)$. □

Theorem 4.3. *If G is a Δ_2-p-groups with $d(G) \geq 3$, then G is a $\mathcal{C}_a$-group,*

Proof. By [10, Lemma 4.1], it is enough to show that $H^G = HG'$ for any non-normal subgroup H of G. If $|G'| = p$, then $H < H^G \leq HG'$ for any non-normal subgroup H. Since $|HG' : H| = p$, $H^G = HG'$. Assume $|G'| \geq p^2$. We prove the result by induction on $|G'|$.

Assume that $|G'| = p^2$. We assert that $\langle a\rangle^G = \langle a\rangle G'$ for any non-normal cyclic subgroup $\langle a\rangle$ of G. If not, then $\langle a\rangle^G = \langle a\rangle[a, G] < \langle a\rangle G'$. It is clear that $|[a, G]| = p$. Let $[a, G] = \langle x\rangle$. Then there exists $b \in G$ such that $[a, b] = x$. Let $G' = \langle x, y\rangle$. By Theorem 4.2, there exists $c \in G$ such that $[a, c] = 1$ and $G' = \langle [a, G], [c, G]\rangle$. Then

$$\langle a\rangle^G = \langle a\rangle[a, G],\ \langle a, y\rangle^G = \langle a\rangle G',\ \langle a, c\rangle^G = \langle a, c\rangle G'.$$

It is not difficult to verify that the orders of $\langle a\rangle^G, \langle a,y\rangle^G$ and $\langle a,c\rangle^G$ are pairwise unequal. This is contrary to $|\Delta(G)| = 2$. Hence the assertion holds. By [10, Lemma 4.1], G is a $\mathcal{C}_a$-group.

Assume that $|G'| \geq p^3$. Let $\langle a\rangle$ be any non-normal subgroup of G. We assert that $|\langle a\rangle^G \cap G'| \geq p^2$. If not, then $|\langle a\rangle^G \cap G'| = p$. Since

$$\langle a\rangle^G \cap G' = \langle a\rangle[a,G] \cap G' = (\langle a\rangle \cap G')[a,G],$$

$|[a,G]| = p$ and $\langle a\rangle \cap G = 1$. Let $[a,G] = \langle x\rangle$. Since $|G'| \geq p^3$, there exist $y,z \in G'\backslash\langle a\rangle^G$. Then

$$\langle a\rangle^G = \langle a,x\rangle,\ \langle a,y\rangle^G = \langle a,x,y\rangle G',\ \langle a,y,z\rangle^G = \langle a,x,y,z\rangle.$$

It is not difficult to verify that the orders of $\langle a\rangle^G, \langle a,y\rangle^G$ and $\langle a,y,z\rangle^G$ are pairwise unequal. This is contrary to $|\Delta(G)| = 2$. Hence $|\langle a\rangle^G \cap G'| \geq p^2$.

Take a normal subgroup N of G of order p contained in $\langle a\rangle^G \cap G'$. Let $\overline{G} = G/N$. Then $\overline{G}$ is a Δ_1-group or a Δ_2-group. If $\overline{G}$ is a Δ_1-group, then $\overline{\langle a\rangle}^{\overline{G}} = \overline{\langle a\rangle}\overline{G}'$. If $\overline{G}$ is a Δ_2-group, then $\overline{\langle a\rangle}^{\overline{G}} = \overline{\langle a\rangle}\overline{G}'$ by induction hypotheses. Since $\overline{\langle a\rangle}^{\overline{G}} = \langle a\rangle^G/N$ and $\overline{\langle a\rangle}\overline{G}' = \langle a\rangle G'/N$, $\langle a\rangle^G = \langle a\rangle G'$. By [10, Lemma 4.1], G is a $\mathcal{C}_a$-group. □

Remark 4.4. If G is a Δ_2-p-group with $d(G) = 2$, then G is not always a $\mathcal{C}_a$-group. For example, let $G = \langle a,b \mid a^{p^{n+1}} = b^{p^m} = 1, [a,b] = a^{p^{n-1}}\rangle$, where $n \geq m \geq 2$ and $H = \langle b^p\rangle$. By Lemma 3.4, G is a Δ_2-p-group. By a simple computation, we have $H^G = \langle b^p, a^{p^n}\rangle$ and $HG' = \langle b^p, a^{p^{n-1}}\rangle$. Thus $H^G \neq HG'$. That is, G is not a $\mathcal{C}_a$-group.

4.1. The case of $d(G) = 3$

Theorem 4.5. *If G is a Δ_2-p-groups with $d(G) = 3$, then $|G'| \leq p^3$.*

Proof. Let $G = \langle a,b,c\rangle$. Then $G' = \langle [a,b]^g, [a,c]^g, [b,c]^g \mid g \in G\rangle$. By Theorem 4.2, $G' \leq Z(G)$ and $\exp(G') = p$. It follows that $G' = \langle [a,b],[a,c],[b,c]\rangle$ is elementary abelian and Hence $|G'| \leq p^3$. □

Lemma 4.6. *Let G be a Δ_2-group. If there exists a normal subgroup N of order p of G contained in G' such that $G/N \cong \mathrm{M}_p(1,1,1) * \mathrm{C}_{p^n}$, then $|G| \leq p^6$.*

Proof. Let $N = \langle x\rangle$. Then

$$G = \langle a,b,c \mid a^p = x^{i_1}, b^p = x^{i_2}, c_{p^n} = x^{i_3}, [a,b] = c^{p^{n-1}}x^{i_4},$$
$$[a,c] = x^{i_5}, [b,c] = x^{i_6}\rangle.$$

It is clear that $G' = \langle c^{p^{n-1}}, x\rangle$. By Theorem 4.2, $\exp(G') = p$. It follows that $c^{p^n} = 1$.

We assert that $a^p \neq 1$ or $b^p \neq 1$. If not, then $a^p = b^p = 1$. Clearly, $G' \cong C_{p^2}$. It follows that $[a,c] \neq 1$ or $[b,c] \neq 1$. By the symmetry of a and b, we can assume that $[a,c]=x\neq 1$. Replacing b by ba^{-i_6}, we have $[b,c]=1$. Thus,

$$G=\langle a,b,c \mid a^p=b^p=x^p=1, c_{p^n}=1, [a,b]=c^{p^{n-1}}x^{i_4}, [a,c]=x, [b,c]=1\rangle.$$

It is not difficult to verify that $\langle a\rangle$, $\langle b\rangle$ and $\langle b,c\rangle$ are non-normal in G. Now

$$\begin{aligned}
&|\langle a\rangle^G|=|\langle a,[a,b],[a,c]\rangle|=|\langle a,c^{p^{n-1}}x^{i_4},x\rangle|=p^3,\\
&|\langle b\rangle^G|=|\langle b,[a,b],[b,c]\rangle|=|\langle b,c^{p^{n-1}}x^{i_4}\rangle|=p^2,\\
&|\langle b,c\rangle^G|=|\langle b,c,[a,b],[b,c],[a,c]\rangle|=|\langle b,c,x\rangle|=p^{n+2}\geq p^4.
\end{aligned}$$

It follows that $|\Delta(G)|\geq 3$. This is a contradiction. Therefore, $a^p\neq 1$ or $b^p\neq 1$.

Without loss of generality, assume that $a^p\neq 1$. Then $a^{p^2}=1$ and $b^p=a^{i_2 p}$. Replacing b with ba^{-i_2}, we have $b^p=1$.

We assert that $[b,c]\neq 1$. If not, then $[b,c]=1$. Hence $[a,c]=a^{i_5 p}\neq 1$. Thus,

$$G=\langle a,b,c \mid a^{p^2}=b^p=c^{p^n}=1, [a,b]=c^{p^{n-1}}x^{i_4}, [a,c]=a^{i_5 p}, [b,c]=1\rangle.$$

By a similar argument as above, we get that $\langle a\rangle$, $\langle b\rangle$ and $\langle b,c\rangle$ are non-normal in G and $|\Delta(G)|\geq 3$. This is contrary to $|\Delta(G)|=2$. Hence, $[b,c]=a^{i_6 p}\neq 1$.

Replacing a by ab^{-i_6}, we get $[a,c]=1$. Thus,

$$G=\langle a,b,c \mid a^{p^2}=b^p=c^{p^n}=1, [a,b]=c^{p^{n-1}}x^{i_4}, [a,c]=1, [b,c]=a^{i_6 p}\rangle.$$

Assume that $n\geq 4$. Similarly, $\langle a\rangle$, $\langle b,c^{p^{n-2}}\rangle$ and $\langle b,c^{p^{n-3}}\rangle$ are non-normal in G. It is not difficult to verify that

$$|\langle a\rangle^G|=p^3,\ |\langle b,c^{p^{n-2}}\rangle^G|=p^4,\ |\langle b,c^{p^{n-3}}\rangle^G|=p^5.$$

It follows that $|\Delta(G)|\geq 3$. This is a contradiction. Hence, $n\leq 3$. Thus $|G|\leq p^6$. □

Lemma 4.7. *Let G be a Δ_2-group. If there exists a normal subgroup N of order p of G contained in G' such that $G/N\cong \mathrm{M}_p(2,1,1)*\mathrm{C}_{p^n}$, then $|G|\leq p^7$.*

Proof. Let $N=\langle x\rangle$. Then

$$\begin{aligned}
G=\langle a,b,c \mid a^{p^2}=x^{i_1}, b^p=x^{i_2}, c^{p^n}=x^{i_3}, [a,b]=c^{p^{n-1}}x^{i_4},\\
[a,c]=x^{i_5}, [b,c]=x^{i_6}\rangle.
\end{aligned}$$

It is clear that $G'=\langle c^{p^{n-1}},x\rangle$. By Theorem 4.2, $\exp(G')=p$. Hence, $G'\cong \mathrm{C}_p^2$. It follows that $c^{p^n}=1$ and $[a,c]\neq 1$ or $[b,c]\neq 1$. Thus, $\langle c\rangle$ is non-normal in G. It is clear that $\langle c\rangle^G=\langle c,x\rangle$ and $|\langle c\rangle^G|=p^{n+1}$.

Since $\Delta(G/N)=\{p^2,p^3\}$ and $|N|=p$, $\Delta(G)=\{p^3,p^4\}$ by Lemma 2.2(2). Thus, $p^{n+1}\leq p^4$. It follows that $n\leq 3$ and $|G|\leq p^7$. □

Using the similar argument as that in Lemma 4.7, we have:

Lemma 4.8. *Let G be a Δ_2-p-group. If there exists a normal subgroup N of order p of G contained in G' such that $G/N\cong \mathrm{M}_p(2,2,1)*\mathrm{C}_{p^n}$, then $|G|\leq p^8$.*

Lemma 4.9. *Let G be a Δ_2-p-group. If there exists a normal subgroup N of order p of G contained in G' such that $G/N \cong \mathrm{M}_p(n,m) \times \mathrm{C}_p$, where $n \geq m$, then $|G| \leq p^8$.*

Proof. Since $\Delta(G/N) = \{p^{m+1}, p^{m+2}\}$ and $|N| = p$, $\Delta(G) = \{p^{m+2}, p^{m+3}\}$ by Lemma 2.2(2). Let $N = \langle x \rangle$. Then

$$G = \langle a, b, c \mid a^{p^n} = x^{i_1}, b^{p^m} = x^{i_2}, c^p = x^{i_3}, [a,b] = a^{p^{n-1}} x^{i_4}, [a,c] = x^{i_5}, [b,c] = x^{i_6} \rangle.$$

It is clear that $G' = \langle a^{p^{n-1}}, x \rangle$. By Theorem 4.2, $\exp(G') = p$. Hence, $G' \cong \mathrm{C}_p^2$. It follows that $a^{p^n} = 1$ and $[a,c] \neq 1$ or $[b,c] \neq 1$.

We assert that $c^{p^2} = 1$. If not, then $c^p = 1$. It follows that $\langle c \rangle^G = \langle c, x \rangle$ has order p^2. Thus $p^2 \in \Delta(G)$. This is a contradiction. Hence, $c^{p^2} = 1$.

Assume $b^{p^m} = 1$. If $[a,c] \neq 1$, then $\langle a, b^p \rangle$ is non-normal in G. If $[b,c] \neq 1$, then $\langle a^p, b \rangle$ is non-normal G. In either case, we have $|\langle a, b^p \rangle^G| = |\langle a^p, b \rangle^G| = p^{n+m} \in \Delta(G)$. Since $\Delta(G) = \{p^{m+2}, p^{m+3}\}$, $n \leq 3$. On the other hand, $n \geq m$. It follows that $|G| = p^{n+m+2} \leq p^8$.

Assume $b^{p^m} \neq 1$. Then $b^{p^m} = c^{pi_2}$, where $p \nmid i_2$. Thus $o(cb^{-i_2^{-1}p^{m-1}}) = p$. It is clear that $|\langle a^p, cb^{-i_2^{-1}p^{m-1}} \rangle^G| = p^{n+1}$. Since $\Delta(G) = \{p^{m+2}, p^{m+3}\}$, $n \leq m+2$. Noticing $|\langle c \rangle^G| = p^3$, we get $m = 1$. Hence $n \leq 3$. It follows that $|G| = p^{n+m+2} \leq p^6$. The lemma is proved. □

Theorem 4.10. *If G is a Δ_2-p-group with $d(G) = 3$ and $|G'| = p^2$, then $|G| \leq p^8$.*

Proof. Take $N \trianglelefteq G$ such that $N \leq G'$ and $|N| = p$. Clearly, $N \leq Z(G)$. Let $\overline{G} = G/N$. By Lemma 2.2(1), $\overline{G}$ is a Δ_1-group or a Δ_2-group. By Lemma 2.1 and Theorem 2.4, $\overline{G}$ is isomorphic to one of the following groups: $\mathrm{M}_p(1,1,1) * \mathrm{C}_{p^n}$, $\mathrm{M}_p(2,1,1) * \mathrm{C}_{p^n}$, $\mathrm{M}_p(2,2,1) * \mathrm{C}_{p^n}$, $\mathrm{M}_p(n,m) \times \mathrm{C}_p$ or $\mathrm{M}_p(1,1,1) \times \mathrm{C}_p$.

If $\overline{G} \cong \mathrm{M}_p(1,1,1) \times \mathrm{C}_p$, then $|G| \leq p^5$. For other cases, it follows by Lemma 4.6, Lemma 4.7, Lemma 4.8 and Lemma 4.9 that $|G| \leq p^8$. □

Theorem 4.11. *If G is a Δ_2-p-group with $d(G) = 3$ and $|G'| = p^3$, then $|G| \leq p^7$.*

Proof. Let $G = \langle a_1, a_2, a_3 \rangle$. By Theorem 4.2, $c(G) = 2$ and $\exp(G') = p$. Hence, $G' = \langle [a_1,a_2], [a_1,a_3], [a_2,a_3] \rangle \cong \mathrm{C}_p^3$. It follows that $\langle a_i \rangle (i = 1,2,3)$ are non-normal in G.

We can assume that $\langle a_i \rangle \cap \langle a_j \rangle = 1$ for $i \neq j$ and $o(a_1) \leq o(a_2) \leq o(a_3)$. In fact, if $\langle a_i \rangle \cap \langle a_j \rangle \neq 1$ for $i \neq j$, then $a_i^{p^s} = a_j^{lp^t}$ for some s, t. Without losing generality, we can assume that $s \leq t$. By Theorem 4.2, $G' \leq Z(G)$ and $\exp(G') = p$. Thus $(a_i a_j^{-lp^{t-s}})^{p^s} = 1$. Replacing a_i by $a_i a_j^{-lp^{t-s}}$, we have $\langle a_i \rangle \cap \langle a_j \rangle = 1$.

We prove $|G| \leq p^7$ by the following assertions.

(1) $\langle a_i\rangle \cap G' \neq 1$ for $i = 1, 2, 3$. Moreover, $G' \leq \mho_1(G)$.

Since $\langle a_i\rangle$ is non-normal in G, $\langle a_i\rangle^G = \langle a_i\rangle G'$ by Theorem 4.3. Clearly, $\langle a_i\rangle^G = \langle a_i\rangle[\langle a_i\rangle, G]$. By the modular law, $G' = (\langle a_i\rangle \cap G')[\langle a_i\rangle, G]$. Since $[\langle a_i\rangle, G] \cong \mathrm{C}_p^2$ and $G' \cong \mathrm{C}_p^3$, $\langle a_i\rangle \cap G' \neq 1$. Since $\langle a_i\rangle \cap \langle a_j\rangle = 1$ for $i \neq j$ and $G' \cong \mathrm{C}_p^3$,

$$G' = \langle \langle a_1\rangle \cap G', \langle a_2\rangle \cap G', \langle a_3\rangle \cap G'\rangle.$$

Clearly, $\langle a_i\rangle \cap G' \leq \langle a_i^p\rangle$. Thus, $G' \leq \mho_1(G)$.

(2) $o(a_1) \geq p^2$, $o(a_3) \leq p^3$ and $o(a_2) \leq p^2$.

By (1), $\langle a_1\rangle \cap G' \neq 1$. It follows that $o(a_1) \geq p^2$. If $o(a_3) \geq p^4$, then $\langle a_1\rangle$, $\langle a_1, a_3^p\rangle$ and $\langle a_1, a_3^{p^2}\rangle$ are non-normal in G, and the orders of $\langle a_1\rangle^G, \langle a_1, a_3^p\rangle^G$ and $\langle a_1, a_3^{p^2}\rangle^G$ are pairwise unequal. Hence $|\Delta(G)| \geq 3$. This is a contradiction. Similarly, if $o(a_2) \geq p^3$, then $|\Delta(G)| \geq 3$, a contradiction.

(3) $|G| \leq p^7$.

By (2) we have $o(a_1) = o(a_2) = p^2$, $o(a_3) \leq p^3$. Thus, $|\mho_1(G)| \leq p^4$. By (1), $\Phi(G) = \mho_1(G)$. Since $d(G) = 3$, $|G| \leq p^7$. □

4.2. The case of $d(G) \geq 4$

Theorem 4.12. *Assume G is a Δ_2-p-group with $d(G) \geq 4$. If $|G'| = p^2$, then $|G| = p^6$ for $d(G) = 4$; and $|G| = p^7$ for $d(G) = 5$.*

Proof. Take $N \trianglelefteq G$ such that $N \leq G'$ and $|N| = p$. Clearly, $N \leq Z(G)$. Let $\overline{G} = G/N$. By Lemma 2.2, $\overline{G}$ is a Δ_1-group or a Δ_2-group. If $\overline{G}$ is a Δ_1-group, then $d(\overline{G}) \leq 3$ by Lemma 2.1. This is contrary to $d(G) \geq 4$. It follows that $\overline{G}$ is a Δ_2-group. We prove the theorem in two cases.

Case 1. $d(G) = 4$.

Since $|G'| = p^2$, $|\overline{G}'| = p$. Since $d(G) = 4$, $d(\overline{G}) = 4$. It follows by Lemma 2.4 that $\overline{G} \cong \mathrm{M}_p(1,1,1) * \mathrm{M}_p(m,1)$, $(\mathrm{M}_p(1,1,1) * \mathrm{C}_{p^n}) \times \mathrm{C}_p$ or $\mathrm{M}_p(1,1,1) * \mathrm{M}_p(1,1,1)$. If $\overline{G} \cong \mathrm{M}_p(1,1,1) * \mathrm{M}_p(1,1,1)$, then $|\overline{G}| = p^5$. It is clear that $|G| = p^6$.

Assume that $\overline{G} \cong \mathrm{M}_p(1,1,1) * \mathrm{M}_p(m,1)$. It is enough to show that $m = 2$. Let

$$\overline{G} = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \mid \bar{a}^p = \bar{b}^p = \bar{c}^{p^m} = \bar{d}^p = 1, [\bar{a}, \bar{b}] = [\bar{c}, \bar{d}] = \bar{c}^{p^{m-1}}, [\bar{a}, \bar{c}] = [\bar{a}, \bar{d}] = [\bar{b}, \bar{c}] = [\bar{b}, \bar{d}] = 1\rangle$$

and $N = \langle x\rangle$. By Theorem 4.2, $G' = \langle c^{p^{m-1}}, x\rangle \cong \mathrm{C}_p^2$ and $c^{p^m} = 1$. By the symmetry of a and b, without loss of generality assume that $a^p = 1$.

If $\langle c\rangle \trianglelefteq G$, then $[a, c], [b, c] \in \langle c\rangle \cap N = 1$. It follows that $[a, c] = [b, c] = 1$. Obviously, $\langle a\rangle^G = \langle a, [a, b], [a, c], [a, d]\rangle = \langle a, [a, b], [a, d]\rangle$. Since $\Delta(G) = \{p^3, p^4\}$, $[a, d] \neq 1$. Let $[a, d] = x$. Then $\langle a, c\rangle^G = \langle a, c, x\rangle$ has order p^{m+2}. By $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, it follows that $m = 2$.

Now assume that $\langle c\rangle$ is non-normal in G. If $[a, c] = 1$, then $\langle a, c\rangle^G = \langle a, c, x\rangle$ has order p^{m+2}. By $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, it follows that $m = 2$. Assume that $[a, c] = x \neq 1$. By a suitable replacement, we can assume that $[b, c] = 1$.

If $b^p = 1$, then $\langle b, c\rangle^G = \langle b, c, x\rangle$ has order p^{m+2}. By $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, it follows that $m = 2$. Now assume that $b^{p^2} = 1$ and $[a, c] = b^p$. Let $[c, d] = c^{p^{m-1}} b^{ip}$ and $d^p = b^{jp}$. Then $(da^i b^{-j})^p = 1$ and $[c, da^i b^{-j}] = c^{p^{m-1}}$. Since $\langle c, da^i b^{-j}\rangle^G = \langle c, da^i b^{-j}, x\rangle$, $|\langle c, da^i b^{-j}\rangle^G| = p^{m+2}$. Again by $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, we have $m = 2$.

Assume that $\overline{G} \cong (\mathrm{M}_p(1, 1, 1) * \mathrm{C}_{p^m}) \times \mathrm{C}_p$. Then, by a similar argument as that of $\overline{G} \cong \mathrm{M}_p(1, 1, 1) * \mathrm{M}_p(m, 1)$, we have $m = 2$. The detail is omitted.

Case 2. $d(G) = 5$.

Since $d(G) = 5$, $\overline{G} \cong \mathrm{M}_p(1, 1, 1) * \mathrm{M}_p(1, 1, 1) * \mathrm{C}_{p^m}$ by Lemma 2.4. We only need to prove that $m = 2$. Let

$$\overline{G} = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e} \mid \bar{a}^p = \bar{b}^p = \bar{c}^p = \bar{d}^p = \bar{e}^{p^m} = 1, [\bar{a}, \bar{b}] = [\bar{c}, \bar{d}] = \bar{e}^{p^{m-1}}, [\bar{a}, \bar{c}] = [\bar{a}, \bar{d}] = [\bar{a}, \bar{e}] = [\bar{b}, \bar{c}] = [\bar{b}, \bar{d}] = [\bar{b}, \bar{e}] = [\bar{c}, \bar{e}] = [\bar{d}, \bar{e}] = 1\rangle$$

and $N = \langle x\rangle$. By Theorem 4.2(i), $e^{p^m} = 1$ and $G' = \langle e^{p^{m-1}}, x\rangle \cong \mathrm{C}_p^2$. By the symmetry of a, b and c, d, without loss of generality, assume that $a^p = c^p = 1$.

If $\langle e\rangle \trianglelefteq G$, then $[a, e] \in \langle e\rangle \cap N = 1$. It follows that $[a, e] = 1$. Since $\langle a\rangle^G = \langle a, [a, b], [a, c], [a, d], [a, e]\rangle = \langle a, [a, b], [a, c], [a, d]\rangle \leq \langle a, [a, b]\rangle N$ and $\Delta(G) = \{p^3, p^4\}$, $[a, c] \neq 1$ or $[a, d] \neq 1$. Without loss of generality, assume that $[a, c] = x$. Then $\langle a, e\rangle^G = \langle a, e, x\rangle$ has order p^{m+2}. By $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, it follows that $m = 2$.

Now assume that $\langle e\rangle$ is non-normal in G. If $[a, e] = 1$, then $\langle a, e\rangle^G = \langle a, e, x\rangle$ has order p^{m+2}. By $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, it follows that $m = 2$. Assume that $[a, e] = x$. By suitable replacement, we can assume that $[b, e] = 1$. If $b^p = 1$, then $\langle b, e\rangle^G = \langle b, e, x\rangle$ has order p^{m+2}. By $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, it follows that $m = 2$. Now assume that $b^{p^2} = 1$ and $[a, e] = b^p$. Let $[c, e] = b^{ip}$. Then $[ca^{-i}, e] = 1$ and $(ca^{-i})^p = 1$. Since $\langle e, ca^{-i}\rangle^G = \langle e, ca^{-i}, x\rangle$, $|\langle e, ca^{-i}\rangle^G| = p^{m+2}$. Again by $\Delta(G) = \{p^3, p^4\}$ and $m \geq 2$, we have $m = 2$. □

Corollary 4.13. *Assume G is a Δ_2-p-groups with $d(G) \geq 4$. If $|G'| \geq p^2$, then $G' = \Phi(G)$.*

Proof. We prove the corollary by induction on $|G'|$. If $|G'| = p^2$, then $G' = \Phi(G)$ by Lemma 4.12. Now assume that $|G'| > p^2$. Take a normal subgroup N of G contained in G' such that $|G' : N| = p^2$. Let $\overline{G} = G/N$. By Lemma 2.2, $|\Delta(\overline{G})| \leq 2$. Since $d(G) \geq 4$, $|\Delta(\overline{G})| \neq 1$ by Lemma 2.1. It follows that $\overline{G}$ is a Δ_2-group. It is clear that $d(\overline{G}) \geq 4$ and $|\overline{G}'| = p^2$. By Lemma 4.12, $\overline{G}' = \Phi(\overline{G})$. Since $\overline{G}' = G'/N$ and $\Phi(\overline{G}) = \Phi(G)/N$, it follows that $G' = \Phi(G)$. □

Theorem 4.14. *Assume G is a Δ_2-p-group. Then $|G'| \leq p^4$ if $d(G) = 4$, and $|G'| \leq p^5$ if $d(G) = 5$.*

Proof. Let $\langle a\rangle$ be any non-normal cyclic subgroup of G. Then $\langle a\rangle^G = \langle a\rangle G'$ by Theorem 4.3. By Corollary 4.13 and Theorem 4.2(i), $G' = \Phi(G) \leq Z(G)$. Hence $a \notin G'$. If $d(G) = 4$, then there exist b, c, d in G such that $G = \langle a, b, c, d\rangle$.

Since $c(G) = 2$, $\langle a\rangle^G = \langle a, [a, b], [a, c], [a, d]\rangle$. By Theorem 4.2(ii), $\exp(G') = p$. It follows that $G' \leq \Omega_1(\langle a\rangle^G)$. It is clear that $|\Omega_1(\langle a\rangle^G)| \leq p^4$. Therefore, $|G'| \leq p^4$. Using the similar argument as that of $d(G) = 4$, we get $|G'| \leq p^5$ when $d(G) = 5$. □

Notice that $G' = \Phi(G)$ by Corollary 4.13. Since $|G/\Phi(G)| = p^{d(G)}$, $|G| = p^{d(G)}|G'|$. Thus, from Theorem 4.14 we get the following:

Theorem 4.15. *Assume G is a Δ_2-p-group with $|G'| \geq p^2$. If $d(G) = 4$, then $|G| = p^7$ if $|G'| = p^3$; and $|G| = p^8$ if $|G'| = p^4$. If $d(G) = 5$, then $|G| = p^8$ if $|G'| = p^3$; $|G| = p^9$ if $|G'| = p^4$; and $|G| = p^{10}$ if $|G'| = p^5$.*

Now, from Theorems 2.4-2.5, Theorem 3.1, Theorem 4.5, Theorems 4.10-4.12, Theorem 4.15 we get the main result, Theorem 1.1, in this paper.

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