

FOLIATIONS FROM LEFT ORDERS

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ABSTRACT. We describe a construction which takes as an input a left order of the fundamental group of a manifold, and outputs a (singular) foliation of this manifold which is analogous to a taut foliation. We investigate this construction in detail in dimension 2, and exhibit connections to various problems in dimension 3.

Introduction

A group is called left-orderable (or simply LO) if it admits a left-invariant linear order (a linear order $<$ on G so that $a < b$ implies $ca < cb$ for all $a, b, c \in G$). We call such an order simply a *left order*. This seemingly purely algebraic notion is actually related to quite different aspects of the group. We can only share a tiny part of the vast literature on orderable groups here; compare e.g. the book of Mura and Rhemtulla [12] for a discussion of early results, or [6] and references therein for deep connections between topology and orderability of groups has been discovered

For us, a crucial insight will be that orderability has a dynamical interpretation: a countable group G is LO if and only if it admits a faithful action on the real line by orientation-preserving homeomorphisms. To learn more about the dynamical point of view of orderable groups, one can consult [13] for instance.

For a closed orientable irreducible 3-manifold M , Zhao [18] very recently presented a construction of foliation \mathcal{F} in $M \setminus \text{int}(B^3)$, where B^3 is a 3-dimensional ball embedded in M with certain transverse structure. \mathcal{F} is analogous to a taut foliation in the sense that there is a finite collection of arcs and simple closed curves which transversely meet every leaf of \mathcal{F} .

In this paper, we give an alternative construction of such foliations under the assumption that M has a strongly essential 1-vertex triangulation. An one-vertex triangulation of a manifold is called *essential* if no edge-loop is

Received September 2, 2021; Accepted November 2, 2021.

2010 *Mathematics Subject Classification*. Primary 57R30, 06F15.

Key words and phrases. Left-invariant order, orderable group, geodesic lamination, foliation.

Hyungrlyul Baik was partially supported by Samsung Science & Technology Foundation grant No. SSTF-BA1702-01, and Sebastian Hensel was partially supported by the DFG SPP 2026 “Geometry at Infinity”.

null-homotopic, and *strongly essential* if in addition no two edges are homotopic (fixing the vertex). In low dimensions, the such triangulations are abundant; in fact it has been conjectured that every closed orientable irreducible 3-manifold admits a strongly essential triangulation. A result of Hodgson-Rubinstein-Segerman-Tillmann [8] established this conjecture for a large class of 3-manifolds (namely, either a closed Haken 3-manifold, a non-compact hyperbolic 3-manifold, or a closed 3-manifold with a Riemannian metric of constant negative or zero curvature).

Our main result is the following construction (which works in arbitrary dimension) and produces singular foliations from orders.

Theorem. *Suppose M admits an essential triangulation in M with a unique vertex v and $\pi_1(M)$ is LO. Then to each left order \mathcal{O} on $\pi_1(M)$ we associate a singular foliation $\mathcal{F}_{\mathcal{O}}$ which has a unique singular point at v and it is normal to each top-dimensional simplex of \mathcal{T} .*

See Section 1 for details of the construction. Our foliation is also analogous to a taut foliation in the same sense as the one constructed by Zhao is. In fact, since the foliation is normal to each top-dimensional simplex of the triangulation, the set of edges of the triangulation is a collection of simple closed curves which meet transversely every leaf of the foliation. From this reason, when our foliation is non-singular, it is indeed an honest taut foliation.

The singular foliation is called a **fLOiation**, suggesting that it is a foliation coming from a left order.

We investigate this construction thoroughly in dimension 2. In particular, we give a complete classification of fLOiations in the 2-torus in Section 2. In the case of hyperbolic surfaces, such a classification seems much harder. However, we can prove the following first steps

Theorem. *Let \mathcal{O} be a left order on $\pi_1(S_g)$ for $g \geq 2$. Then, each leaf of $\mathcal{F}_{\mathcal{O}}$ lifts to a quasigeodesic in \mathbb{H}^2 . In particular, $\mathcal{F}_{\mathcal{O}}$ can be pulled tight to a geodesic lamination $\lambda_{\mathcal{O}}$. This assignment is continuous with respect to the topology on orders introduced by Sikora in [16], and the Hausdorff topology on the space of geodesic laminations.*

Given this theorem, one can first try to characterise which geodesic laminations can be obtained as straightenings of fLOiations. We give some positive and negative results in Section 2: in particular, minimal orientable geodesic laminations are possible.

In Section 3, we study the case of dimension 3, and we give a necessary and sufficient condition for a fLOiation in a 3-manifold to be an honest regular foliation. The exact nature of fLOiations in dimension 3 is still very mysterious and we pose a lot of questions in this section. Some of the questions are related to so-called the L-space conjecture, proposed in the seminal paper of Boyer-Gordon-Watson [2]. It states that a rational homology 3-sphere Y is not an L-space if and only if its fundamental group $G = \pi_1(Y)$ is LO. Although the

conjecture is still widely open, it suggests a deep connection between topology of a manifold and algebraic property of its fundamental group and has been studied by many authors, including [3, 7]. Later a connection to foliations is also made. Ozsváth and Szabó [14] showed that a rational homology 3-sphere Y has a taut foliation then it is not an L-space (see also [1, 10]), and the converse was conjectured by Juhász in [9]. From this point of view, one can ask if in general the existence of a taut foliation in a manifold is equivalent for its fundamental group being LO. We hope that further investigation on fLOiation in the future would shed some light in this direction too.

Acknowledgements. The construction of fLOiations was inspired by a conversation that the first author had with Bill Thurston, Dylan Thurston, and Sergio Fenley in Spring 2012, and the name fLOiation was suggested by Bram Petri. We thank them greatly.

We also thank Danny Calegari, Nathan Dunfield, Ursula Hamenstädt, and anonymous referee for helpful comments and discussions.

1. Two constructions of fLOiations

In this section, we describe our constructions of singular foliations from orders. We begin with a general construction that works in any dimension. Then, we briefly describe an alternate, equivalent construction in dimension 2, which is slightly easier to visualize. Finally, we discuss how the construction can be adapted to work with multi-vertex triangulations, and (certain) partial orders.

1.1. General construction of fLOiations

Throughout, we let M be a manifold, and \mathcal{T} a fixed triangulation with a single vertex. Under the assumption that $\pi_1(M)$ is left-orderable, there is a faithful action

$$\rho : \pi_1(M) \rightarrow \text{Homeo}(\mathbb{R})$$

with the additional property that the orbit map of 0 is injective (see Proposition 2.1 in [13] for instance). We call such ρ *allowed*.

We now describe a construction of a singular foliation from \mathcal{T} and ρ (and discuss dependence later). Let \tilde{M} be the universal cover of M , and lift the triangulation \mathcal{T} from M to a triangulation $\tilde{\mathcal{T}}$ of \tilde{M} . We equip M and \tilde{M} with PL structures based on these triangulations. This makes the covering map PL, which in turn implies that the deck group $G = \pi_1(M)$ acts as PL transformations.

Denote the vertex of \mathcal{T} by v , and let v_0 be one of its lifts. The preimage of v under covering is Gv_0 . We inductively define a map $F : \tilde{M}$ to \mathbb{R} satisfying the following:

- For each vertex gv_0 , we have $F(gv_0) = \rho(g)(0)$.

- For edge e , pick an orientation such that the value of F on the endpoint is greater than the starting point. Suppose the end point is gv_0 . Now, if $g = e$, then we define F to be linear on e . If $g \neq 1$, then for any $x \in e$ we set $F(x) = \rho(g)(F(g^{-1}x))$.
- For d -dimensional cells, we extend F from the edges to the cell in such a way that the level sets are linear $d - 1$ polytopes.

Lemma 1.1. *The level sets of F are invariant under the action of the deck group $\pi_1(M)$.*

Proof. First, we observe that by definition, for any x in the 1-skeleton of \tilde{M} , and any $g \in \pi_1(M)$, we have

$$gF(x) = F(gx).$$

Hence, the deck group action preserves the level sets of F restricted to the 1-skeleton.

Now, let Δ be any higher dimensional simplex. The intersection of a level set $F^{-1}(x)$ with Δ is the polyhedron in Δ whose vertices are exactly those points in the 1-skeleton Δ^1 which lie in $F^{-1}(x)$. Since the deck group action acts as PL transformations, and preserves the level sets of F on the 1-skeleton, this implies that the deck group action preserves all level sets. \square

Let $\tilde{\mathcal{F}}$ be the singular foliation defined by the level sets of F . By the previous lemma, it descends to a singular foliation $\mathcal{F} = \mathcal{F}(\mathcal{T}, \rho)$ on M .

The rest of this subsection is concerned with a discussion of the dependence on ρ . In particular, we would like to obtain a construction which depends only on an order of $\pi_1(M)$, not an action on \mathbb{R} . The first step is the following easy observation.

Lemma 1.2. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is any orientation-preserving homeomorphism which fixes 0, then $\mathcal{F}(\mathcal{T}, \rho) = \mathcal{F}(\mathcal{T}, f\rho)$, where $f\rho$ denotes the representation where each element is postcomposed with f .*

Proof. Denote by F and F' the functions defining the covers of $\mathcal{F}(\mathcal{T}, \rho)$, $\mathcal{F}(\mathcal{T}, f\rho)$. One then observes that

$$F(x) = t \quad \Leftrightarrow \quad F'(x) = f(t).$$

Namely, this is true by definition on the 1-skeleton of the triangulation on \tilde{M} , and follows with an argument as in the previous lemma for all higher dimensional simplices. Thus, F and F' have the same level sets. \square

In order to define the singular foliation from an order, we therefore aim to construct an (allowed) action ρ from the order, without choices. As a first step, we begin with an order-preserving map

$$i : \pi_1(M) \rightarrow \mathbb{R}$$

such that the identity e is sent to 0. We now promote i to an allowed action

$$\rho_i : \pi_1(M) \rightarrow \text{Homeo}^+(\mathbb{R})$$

in the following way:

- (1) Define the action of $g \in \pi_1(M)$ on $i(\pi_1(M))$ by $g(i(h)) = i(gh)$.
- (2) Extend the action above to an action on \mathbb{R} by homeomorphisms linearly in all complementary intervals $\mathbb{R} - i(\pi_1(M))$.

Hence, we can now define a foliation $\mathcal{F}(\mathcal{T}, i) = \mathcal{F}(\mathcal{T}, \rho_i)$, and need to discuss how the construction depends on i .

Corollary 1.3. *Any two (ambiently) isotopic embeddings i, i' yield the same foliations $\mathcal{F}(\mathcal{T}, i) = \mathcal{F}(\mathcal{T}, i')$.*

Proof. If i, i' are ambiently isotopic, there is a homeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ so that $i' = f \circ i$. In fact, we may assume that f is affine on each complementary interval in $\mathbb{R} - i(\pi_1(M))$. By definition, we then have

$$\rho_{i'} = f \rho_i,$$

and thus $\mathcal{F}(\mathcal{T}, i) = \mathcal{F}(\mathcal{T}, i')$ by Lemma 1.2. □

As a last step, we promote the construction to start with the order itself, not the embedding i . To this end, given any totally ordered countable set $S = \{s_0, s_1, \dots\}$, we define a “minimal embedding” i_0 of S into \mathbb{R} inductively as follows:

- (1) $i_0(s_0) = 0$.
- (2) For any $n > 0$, if $s_n > s_p, \forall p < n$, let $i_0(s_n) = \max_{p < n} i_0(s_p) + 1$; if $s_n < s_p, \forall p < n$, let $i_0(s_n) = \min_{p < n} i_0(s_p) - 1$; if $s_a < s_n < s_b$ and s_a and s_b are the closest elements to s_n in the finite sets $\{s_0, \dots, s_{n-1}\}$, from below and from above respectively, then $i_0(s_n) = \frac{i_0(s_a) + i_0(s_b)}{2}$.

In fact, we can describe the difference between the foliations defined by minimal embedding and arbitrary ones.

Lemma 1.4. *Given any totally ordered countable set $S = \{s_0, s_1, \dots\}$, any other order preserving map from S to \mathbb{R} can be “collapsed” into i_0 . In other words, two different choices of i differ only in the addition and removal of disjoint intervals, and the application of an ambient isotopy.*

Proof. It is enough to construct a continuous, monotone (but generally not strictly monotone) map from \mathbb{R} to itself that sends $i(G)$ to $i_0(G)$. This can be done in the following two steps:

- (1) On $i(G)$, let $f(i(g)) = i_0(g)$.
- (2) Now we extend it to $\overline{i(G)}$. Suppose $x \in \overline{i(G)} \setminus i(G)$, then either there is a sequence in $i(G)$ converging to x from above but no from below, or there is a sequence from below but not above, or there are sequences converging to it from both above and below. In the previous two cases f shall send x to the limit of the image of those converging sequences under f . □

Corollary 1.5. *If i, i' are any two order-preserving embeddings of $\pi_1(M)$ into \mathbb{R} , then the foliations $\mathcal{F}(\mathcal{T}, i), \mathcal{F}(\mathcal{T}, i')$ differ by inserting or removing “strips” $L \times (0, 1)$ of parallel leaves.*

Proof. By Corollary 1.3 and Lemma 1.4 it suffices to analyse the effect of modifying i by inserting or collapsing an interval (not containing any points from the embedding i). Now, from the construction it is immediate that this corresponds to blowing up a leaf into a band of parallel leaves (or collapsing such a strip). \square

In general, it is not clear to us how the choice of \mathcal{T} influences the foliation. In low dimensions, where different triangulations can easily be connected combinatorial, it seems likely that the choice of triangulation is immaterial, but the general case is less clear.

Question 1.6. Describe the relation between $\mathcal{F}(\mathcal{T}, \mathcal{O})$ and $\mathcal{F}(\mathcal{T}', \mathcal{O})$ for different 1-vertex triangulations.

1.2. Alternative construction of the fLOiations

In this subsection, we describe an alternative, more explicit construction of fLOiations in dimension 2.

Consider a triangle Δ with 3 edges α, β and γ in the 2-skeleton of the quotient complex $\tilde{M}/\pi_1(M)$ above, oriented so that all three edges represents positive elements in $\pi_1(M)$, and labelled such that α followed by β is homotopic to γ . For each of the three edges $e = \alpha, \beta$ or γ , we parameterize it with an interval $[0, -i(e^{-1})]$ as follows: firstly, because $\pi_1(M)$ acts freely on the preimages of e , we can identify e with the preimage that ends with the point v_0 . Now the identification between this edge and $[0, -i(e^{-1})]$ is done via the continuous map F (defining the fLOiation, as above) composed with multiplication by -1 .

Following leaves of \mathcal{F} identifies β with a subinterval $[0, -i(\beta^{-1})]$ of γ . In the parameterization of edges described above, this identification is simply the identity. In the same way, α is identified with another subinterval $[-i(\beta^{-1}), -i(\gamma^{-1})]$ of γ , via the map $x \mapsto -\beta^{-1}(-x)$ (in the parametrisations given above).

Hence, we could alternatively define \mathcal{F} by choosing the parametrisations of the edges first, and joining them with line segments according to the rule in the previous paragraph.

1.3. Generalization to multi-vertex triangulations

The above construction can be generalized to the case when the triangulation has more than one vertex: let \tilde{M} be the universal cover, D a fundamental domain of the deck group action, V the set of vertices of \tilde{M} . Choose a partial ordering on V satisfying that $gv < v$ if and only if $g < e$. If i is an order preserving embedding of V into \mathbb{R} , then a faithful action of G on \mathbb{R} can be defined as above by $g(i(v)) = i(gv)$ on $i(V)$ and extended linearly to \mathbb{R} . Now consider a map F from \tilde{M} to \mathbb{R} , defined as i on the 0-skeleton, and extend to higher dimensional cells as follows:

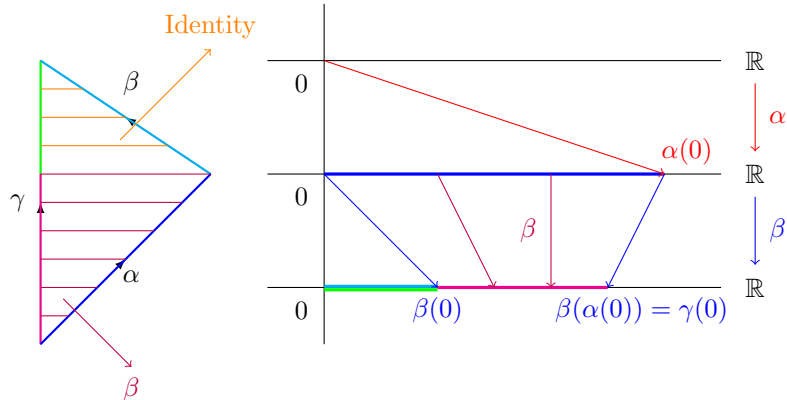


FIGURE 1. Foliation on a 2-simplex

- For 1-cell e , pick an orientation such that the value of F on the end point is greater than the starting point. Now, if $e \in D$, then F is linear on e . If $e = ge_0$, $e_0 \in D$, then for any $x \in e$, $F(x) = gF(g^{-1}x)$.
- For d -dimensional cells, we extend the function from the 1-faces to the cell in such a way that the level sets are all linear $d - 1$ polytopes.

Arguing as in the one-vertex case, we now see that the level sets of F are regular unless they pass through one of the vertices, and are invariant under deck transformations. Hence, it covers a singular foliation of M as before.

1.4. Generalization to partial ordering

The construction for fLOiation can also be generalized to certain partial orders. More specifically, suppose that we are given a partial order $<$ on $\pi_1(M)$ satisfying the property that “ $a < b$ and $b < a$ are both not true” is an equivalence relation \sim . Instead of strongly essential, we now assume that the fundamental group elements corresponding to edges of \mathcal{T} do not lie in the same equivalence class as the unit.

Note that there is a total ordering on G/\sim induced by this partial ordering, hence one can embed G/\sim into \mathbb{R} , which provides a map from G to \mathbb{R} , and then carry out the same construction as above. Since the group elements corresponding to edges are not equivalent to the unit under \sim , the result is again a singular foliation.

This construction will be useful in trying to understand fLOiation for orders which are constructed from partial orders. For later use, we record the following:

Remark 1.7. Let i be an embedding of G/\sim into \mathbb{R} , and let J be a union of disjoint intervals of \mathbb{R} . Define an equivalence relation \sim' by declaring $a \sim' b$ if and only if $i(a)$ and $i(b)$ lies in the same connected component of J . Let j be

the embedding of G/\sim' to \mathbb{R} obtained from i by composing with a map that collapses each connected component of J .

If \sim' also satisfy the condition that fundamental group elements corresponding to edges do not lie in the same equivalence class, then the fLOiation defined j is obtained from the fLOiation defined by i by collapsing the preimage of each connected component of J into a single leaf.

2. Surfaces

In this section we discuss fLOiation on surfaces, where more precise questions can be answered.

2.1. Torus examples

Here we discuss examples of the construction on the two-dimensional torus T . We identify \mathbb{R}^2 with the universal cover of $S^1 \times S^1$ in the usual way, and \mathbb{Z}^2 with the deck group as translations.

In this situation, there is a geometric way to construct orders on $\pi_1(T)$. Namely, choose an (oriented) line $L \subset \mathbb{R}^2$ through the origin of \mathbb{R}^2 , and denote by V the halfspace to the left of L . If L has irrational slope, then we declare all elements of $\pi_1(T)$ in V to be positive. If L has rational slope, we need to furthermore cut V at L^\perp , and declare one of the halves to be positive. Any order on \mathbb{Z}^2 is (up to isomorphism) one of these two classes (orders on \mathbb{Z}^n are classified in [15, 17]. See Section 2 of [11] also).

If L has irrational slope, then we let $\pi : \mathbb{R}^2 \rightarrow L$ denote orthogonal projection. Then under an identification of L with \mathbb{R} , the restriction $\pi|_{\mathbb{Z}^2} : \mathbb{Z}^2 \rightarrow L = \mathbb{R}$ yields a bi-invariant order on \mathbb{Z}^2 , via the action as translations. Now, as the defining function F for the definition of the fLOiation we can simply take the projection π – which implies that the resulting fLOiation is the (irrational slope) foliation by lines orthogonal to L .

If L has rational slope, then the orthogonal projection $\pi : \mathbb{R}^2 \rightarrow L$ has a kernel $K \simeq \mathbb{Z}$. The choice of the halfspace in V is equivalent to the choice of an order on K , and the order is then the lexicographic order defined by these two orders. We can thus modify $\pi : \mathbb{Z}^2 \rightarrow L$ to the desired function F by “blowing up” the maps $a+K \rightarrow \pi(a)$ into injective maps to small intervals. In particular, observe that we can do this so that the resulting F still has connected level sets (compare Remark 1.7).

Now, consider any element $k \in K$. Observe that then k^n is bounded (in the order) by any element which maps under π to a positive number. This implies that the geodesic in \mathbb{R}^2 corresponding to k converges to the leaf of the fLOiation corresponding to $\lim_{n \rightarrow \infty} F(k^n)$. Swapping perspective, this shows that the fLOiation (on T) has a leaf which spirals into the image of this geodesic (since level sets are connected). Hence, the fLOiation on T has a closed leaf in the homotopy class defined by k .

Summarizing, we have the following result connecting algebraic and topological properties.

Proposition 2.1. *Let O be an order on $\pi_1(T) = \mathbb{Z}^2$, and F_O the corresponding fLOiation. Then F_O has a closed leaf if and only if O is not Archimedean.*

2.2. Geodesic laminations on hyperbolic surfaces

In this section, we focus on closed hyperbolic surfaces Σ of genus $g \geq 2$, and Δ be a one-vertex triangulation of Σ , all edges of which are essential simple closed curves. Consider the universal cover \mathbb{H}^2 of Σ , and the preimage $\tilde{\Delta}$ of Δ in \mathbb{H}^2 . Let γ be a side of a triangle in Δ . The preimage of γ in \mathbb{H}^2 consists of countably many embedded infinite paths.

Lemma 2.2. *There is a constant $K > 0$ so that each component $\tilde{\gamma}$ of the preimage of an edge γ is a K -quasigeodesic.*

Proof. The vertices of $\tilde{\Delta}$ are labelled by elements of the deck group $\pi_1(\Sigma)$. A component $\tilde{\gamma}$ as in the lemma contains vertices of the form $c \cdot g^i, i \in \mathbb{Z}$ for some $c \in \pi_1(\Sigma)$, and g the label of the edge γ . Now recall that $\pi_1(\Sigma)$ is a hyperbolic group, and any infinite cyclic subgroup of a hyperbolic group is undistorted [4]. Thus, $c \cdot g^i, i \in \mathbb{Z}$ is a quasigeodesic in $\pi_1(\Sigma)$, and therefore \mathbb{H}^2 . Hence, so is $\tilde{\gamma}$ (as the segments connecting consecutive vertices are lifts of γ and therefore have the same, finite length). The uniformity statement follows as the triangulation only has finitely many edges. □

We call a preimage $\tilde{\gamma}$ of an edge γ as in the lemma a *wall (of type γ)*. The complement of $\tilde{\gamma}$ in \mathbb{H}^2 thus consists of two (quasiconvex) halfspaces. We denote their boundary intervals by $I(\tilde{\gamma})_i, i = 1, 2$. They intersect in the boundary points of $\tilde{\gamma}$.

The first main result of this section is the following

Proposition 2.3. *Suppose that $L : [0, \infty) \rightarrow \mathbb{H}^2$ is an embedded path so that each intersection with a wall is transverse and in at most one point, and which intersects infinitely many walls. Then L has a well-defined endpoint in the circle at infinity.*

If $L : (-\infty, \infty) \rightarrow \mathbb{H}^2$ intersects each wall it intersects transversely, and in at most one point, and both the positive and negative ray intersect infinitely many walls, then the corresponding endpoints are distinct.

Proof. Let $0 \leq t_1 < t_2 < \dots$ be all positive real numbers so that $L(t_i)$ lies in a wall w_i . By the assumption that L intersects each wall in at most one point, the path $L : [t_i, \infty) \rightarrow \mathbb{H}^2$ is contained in one of the halfspaces C_i defined by w_i ; we let I_i be the boundary interval of that halfspace. To show the proposition, it suffices to show that $\bigcap_i I_i$ consists of a single point. Namely, since the w_i are quasigeodesics with uniform constants, this would imply that the intersection of the closures \overline{C}_i of all C_i also consists of a single point at infinity.

Since Δ has only finitely many edges, there are indices $n_1 < n_2 < \dots$ so that all w_{n_i} are lifts $\tilde{\gamma}_i$ of the same edge γ of Δ . The corresponding halfspaces C_{n_i} are nested (since the $\tilde{\gamma}_i$ are all disjoint). Assuming that $\bigcap_i I_i$ is not a single point, we conclude that $\bigcap_i I_{n_i}$ is also not a single point.

Thus, $\bigcap_i I_{n_i} = [\alpha, \beta]$ for some circular arc $[\alpha, \beta] \subset S^1$, and the endpoints of $\tilde{\gamma}_i$ converge to α, β .

Suppose that γ corresponds to the element $g \in \pi_1(\Sigma)$ in the deck group. Then there is a lift $\tilde{\gamma}$ whose endpoints agree with the endpoints of the axis of g . Since all $\tilde{\gamma}_i$ are lifts of γ , we have that $\tilde{\gamma}_i = g_i \tilde{\gamma}$.

Thus, we conclude that the endpoints of the axes of the conjugates $g_i g g_i^{-1}$ (which agree with the endpoints of $\tilde{\gamma}_i$) converge to $\alpha \neq \beta$. But this is impossible for a cocompact Fuchsian group.

It remains to show the last claim about distinct endpoints. To this end, note that under the assumptions there will be numbers $r < 0 < s$ so that $L(r), L(s)$ are both on walls which are lifts of the same edge γ . This implies that the corresponding walls are disjoint, and in fact the same is true for the boundary intervals of the halfspaces which $L(t), t \geq s$ and $L(t), t \leq r$ are contained in. This shows that the endpoints are distinct. \square

We now aim to improve this argument to study 2-dimensional fLOiations, and their dependence on the order. The first is the following consequence of Proposition 2.3.

Corollary 2.4. *Let \mathcal{F} be a fLOiation of Σ determined by the 1-vertex triangulation Δ and an embedding $\rho : \pi_1(\Sigma, v) \rightarrow \text{Homeo}^+(\mathbb{R})$. Then each leaf of \mathcal{F} lifts in $\tilde{\Sigma}$ to a bi-infinite path with well-defined endpoints at infinity.*

Proof. Given Proposition 2.3 we need to check that walls are intersected transversely, in at most one point, and infinitely often in both directions. Transversality of intersections is clear from the construction.

Next, we argue that no wall is intersected more than once by a leaf. Namely, suppose that a leaf segment l_0 would intersect the same wall w in two points. This would imply that along the wall w , two points map to the same point in \mathbb{R} under F (as leaves are level sets). On the other hand, the wall consists of images $cg^i(e)$, $i \in \mathbb{Z}$ for some edge e and group elements c, g (where c corresponds to some point on the wall, and g is the group element corresponding to the edge e). By construction of the function F defining the fLOiation, it restricts on $cg^i(e)$ as an injective function taking values in $[\rho(CG^i), \rho(CG^{i+1})]$, where ρ is the embedding of G into \mathbb{R} . Hence, the restriction of F to the wall is injective, showing the claim.

Finally, observe that since every wall is intersected at most once, and every leaf intersects infinitely many edges of the triangulation in both directions, leaves also intersect infinitely many walls in both directions. \square

As a consequence, every surface fLOiation has a well-defined straightening. We denote by

$$S : \{\rho : \pi_1(\Sigma) \rightarrow \text{Homeo}^+(\mathbb{R}) \text{ allowed}\} \rightarrow \mathcal{GL}(\Sigma)$$

the map which assigns to an allowed representation ρ the straightening of the associated fLOiation, seen as a geodesic lamination on Σ . Observe also that by results in Section 1, the image $S(\rho)$ only depends on the order defined by ρ , not ρ itself.

Note that the map S is in fact the composition of the maps

$$\{\rho : \pi_1(\Sigma) \rightarrow \text{Homeo}^+(\mathbb{R}) \text{ allowed}\} \rightarrow \{\text{fLOiations}\} \rightarrow \mathcal{GL}(\Sigma),$$

where the first map is our construction in Section 1 and the second map is the straightening which is allowed by Corollary 2.4. Abusing notation, we also just use S to denote the second map in the composition.

Next, we will discuss the dependence of a fLOiation on the embedding $\rho : \pi_1(\Sigma, v) \rightarrow \text{Homeo}^+(\mathbb{R})$. The main ingredient is the following version of Proposition 2.3.

Proposition 2.5. *Let $D \subset \mathbb{H}^2$ be a fundamental domain for the $\pi_1(\Sigma)$ -action. For each $\epsilon > 0$ there is a compact set $K \subset \mathbb{H}^2$ so that the following holds.*

Suppose that $L_1, L_2 : [0, \infty) \rightarrow \mathbb{H}^2$ are embedded paths which intersect D , and which satisfy the prerequisites of Proposition 2.3. Further suppose that for each wall w which intersects K , the path L_1 intersects w if and only if the path L_2 intersects w .

Then the endpoints of L_1, L_2 guaranteed by Proposition 2.3 have distance at most ϵ (with respect to the standard angular metric on S^1).

Proof. By cocompactness of the action of $\pi_1(\Sigma)$ on \mathbb{H}^2 , there are walls w_1, \dots, w_N so that the endpoints of each w_i cut off an interval I_i of size $< \epsilon$ in S^1 , and each point in S^1 lies in at least one such. Now, choose a compact set K which intersects each w_i . By possibly choosing the w_i further from the fundamental domain D , we may assume that there is no quasigeodesic with the constants from Proposition 2.3 which can intersect D and have both endpoints in I_i .

Suppose that L_1, L_2 satisfy the conditions of the proposition. Then, each L_i is, by Proposition 2.3 and the choices above, a quasigeodesic whose endpoints lie in different of these intervals. Hence, L_i has an endpoint in I_j if and only if it intersects w_j . Now, by assumption, this means that L_1, L_2 have endpoints in the same intervals I_r, I_s . This shows the proposition. \square

Corollary 2.6. *The map S is continuous, for the topologies of pointwise convergence on $\text{Hom}(\pi_1(\Sigma, v), \text{Homeo}^+(\mathbb{R}))$ and the Hausdorff topology on geodesic lamination space.*

Proof. Let D be a fundamental domain for the action of $\pi_1(\Sigma, v)$ on \mathbb{H}^2 , and let ρ_0 be some allowed representation.

Given any allowed representation ρ , denote by $F_\rho : \mathbb{H}^2 \rightarrow \mathbb{R}$ the function defining the fLOiation. Observe that, by construction, for any compact set

$K \subset \mathbb{H}^2$, and any $\epsilon > 0$, if ρ is close enough to ρ_0 , then F_ρ, F_{ρ_0} are ϵ -close on K . In particular, intersections of level sets with K can be guaranteed to be arbitrarily close to each other in the Hausdorff topology.

This implies that, if $x \in D$ is a point so that the leaf L_x of $\tilde{\mathcal{F}}_{\rho_0}$ through x is nonsingular, and w is any given wall w , then for any representation ρ sufficiently close to ρ_0 , the leaf of $\tilde{\mathcal{F}}_\rho$ through x is also nonsingular, and intersects w .

Hence as $\rho \rightarrow \rho_0$, the straightened fLOiations $S(\tilde{\mathcal{F}}_\rho)$ have leaves converging to $S(L_x)$. Since we can take a finite number of leaves L_{x_1}, \dots, L_{x_n} whose closure (on Σ) is the whole fLOiation, this implies that the straightened fLOiations $S(\tilde{\mathcal{F}}_\rho)$ converge to a lamination which contains $S(\tilde{\mathcal{F}}_{\rho_0})$.

Finally, suppose that this containment is proper. Then there is sequence of straightened leaves $S(L_i)$ of $\tilde{\mathcal{F}}_{\rho_i}$ which converge to a geodesic g which is not contained in $S(\tilde{\mathcal{F}}_{\rho_0})$. Since laminations are closed, this implies that there is a pair of walls w_1, w_2 intersected by g so that no leaf of $S(\tilde{\mathcal{F}}_{\rho_0})$ intersects both. Up to passing to a subsequence we may assume that L_i are leaves through points $y_i \in D$ which converge to $y \in D$. But now, if the leaf L_y of $\tilde{\mathcal{F}}_{\rho_0}$ does not intersect w_1 or w_2 , then the same will be true for the leaves L_i by the above, which is a contradiction. \square

Corollary 2.7. *For the canonical embedding and extension to allowed actions described in Section 1, the assignment of the straightened fLOiation $S(\mathcal{F}(\rho_{\mathcal{O}}))$ to an order \mathcal{O} on $\pi_1(\Sigma)$ is continuous.*

2.3. The image of the straightening map

We end this section with an explicit construction of laminations on hyperbolic surfaces obtained by straightening fLOiations.

Let α be the real part of a holomorphic 1-form ω on a closed Riemann surface Σ with one single zero, so that any non zero element of $H_1(\Sigma)$ paired with α is non-zero. We further assume that there is no closed singular leaf in the foliation induced by α .

Recall that ω induces a piecewise Euclidean structure on the surface, with the only singular point being the unique zero. Choose a geodesic triangulation \mathcal{T} of Σ for this piecewise Euclidean structure, so that the zero is the unique vertex (such a triangulation always exist, for example, one can always choose the Delaunay triangulation), and as always lift the triangulation to a triangulation $\tilde{\mathcal{T}}$ on $\tilde{\Sigma}$ via the covering map.

Let $\pi_1(\Sigma) = G_0 > G_1 > G_2 > \dots$ be the lower central series, then there is an embedding i_0 from $H_1(\Sigma) = G_0/G_1$ to \mathbb{R} which send the order induced by α to the order on \mathbb{R} .

Now we try to build an embedding of $\pi_1(\Sigma)$ into \mathbb{R} so that the induced order on $\pi_1(\Sigma)$ is left invariant:

Proposition 2.8. *Let S be a closed hyperbolic surface. Then, given any left order on $H_1(\Sigma) = G_0/G_1$, there exists a left order on $G_0 = \pi_1(\Sigma)$ extending*

the partial ordering induced by the given ordering on G_0/G_1 (which we call a lexicographic ordering on G_0). Furthermore, suppose $i_0 : H_1(\Sigma) \rightarrow \mathbb{R}$ is an embedding that sends the given left order on $H_1(\Sigma)$ to the order in \mathbb{R} , then there is a embedding $i : G_0 \rightarrow \mathbb{R}$, such that i_0 can be obtained by collapsing intervals in the image of i . (We call i a blow-up of i_0 .)

Proof. Firstly, for each k , fix a left order of G_k/G_{k+1} . Then, start with the embedding i_0 above. Replace each image of i_0 with an interval, such that the total length of such intervals is no more than 2^{-1} , and send G_1/G_2 to the interval in a way that is order preserving, which defines a map $i_1 : G/G_2 \rightarrow \mathbb{R}$. Repeat the process to get i_2, i_3, \dots , where the new intervals added at the j -th step has total length no more than 2^{-j} . Then i is the limit of i_j as $j \rightarrow \infty$. \square

We now have maps $i_0, i : G \rightarrow \mathbb{R}$. The map i_0 is not an embedding, but it satisfies the conditions discussed in Section 1.4, and so defines a fLOiation.

Proposition 2.9. *The foliation defined by i_0 is identical to the foliation on Σ induced by α .*

Proof. By construction, ω as well as α are constant forms on each triangle on \tilde{T} . Hence, the integral of α is linear on each triangle, in particular they would be linear on edges starting at a given lifting of the vertex v_0 . This implies that the integral of α is exactly the function F in Section 1, hence its level set is this new foliation. \square

Now, by Remark 1.7, the foliation defined by i differs from the one defined by i_0 by blowing up certain leaves into strips. Due to the assumption on α , each connected component of the preimage of any inserted interval during the blowup can contain only one vertex in $\tilde{\Sigma}$. Hence, the foliation on this connected component can only be isotopic to the foliation on a tubular neighborhood of the singular leaf of $Re(z^k dz)$ (which we shall call a “star-shaped ribbon graph”).

Hence, the foliation formed from i can be obtained by “blowing up” the foliation corresponding to α , by replacing the singular leave with a star-shaped ribbon graph. This ribbon graph will become a n -gon after straightening just like the singular leaf of the foliation corresponding to α .

In particular, the straightening of \mathcal{F}_i agrees with the straightening of the vertical foliation of α .

In conclusion, we have the following:

Proposition 2.10. *Suppose α is an Abelian differential with a single zero on Σ , so that the foliation defined by the real part of α has no closed leaves. Then the straightening of this foliation lies in the image of S .*

Above proposition naturally lead us to the following question.

Question 2.11. Characterise (in terms of properties of the order O on $\pi_1(\Sigma_g)$) when the fLOiation F_O has a closed leaf.

We also remark the following, which follows since the exact choice of how to extend the partial order i_0 to i is immaterial:

Corollary 2.12. *The map S is not injective.*

3. Dimension 3

In this section, we focus in the case of dimension 3. First, we note that a construction similar to the one given in Section 2 for surfaces can be done to the mapping torus M of surface maps. This can be done via the central series: $G = \pi_1(M) > \pi_1(\Sigma) > \dots$, where Σ is the surface, and $\pi_1(\Sigma) > \dots$ is the lower central series of $\pi_1(\Sigma)$. The one-vertex triangulation is formed from a one vertex triangulation of S , and the restriction of the resulting foliation on Σ is as constructed above.

Next we ask when the foliation \mathcal{F} constructed in Section 1 is nonsingular.

In [5], Calegari showed that orientations on the edges of a triangulation which satisfy certain conditions gives information about the 3-manifold. Let $M, \mathcal{T}, \mathcal{F}$ be as in Section 1. We have orientations on the edges of \mathcal{T} given by a left order on $\pi_1(M)$. In the section, we study when this choice of orientations satisfies the conditions that Calegari considered.

We begin by recalling some terms introduced in [5]. Let M be a closed 3-manifold, and let \mathcal{T} be a triangulation of M . For each vertex v , we define the maximal subgraphs $o(v)$ and $i(v)$ of $link(v)$ whose vertices are, respectively, the outgoing and the incoming vertices from and to v . A *direction* on M is a choice of orientation for each edge in the 1-skeleton \mathcal{T} . A direction is a *local orientation* if it satisfies the conditions

1. for each vertex v the graphs $o(v)$ and $i(v)$ are nonempty and connected
2. the direction restricts to a total ordering on the vertices of each tetrahedron
3. the 1-skeleton is recurrent as a directed graph. That is, there is an increasing path from each vertex to each other vertex.

Suppose M, \mathcal{T} are as in the previous section. The direction is given so that each edge represents a positive element of $\pi_1(M)$ with respect to the linear order to begin with. Since there is a total order on the entire group $\pi_1(M)$ which restricts to the direction on the edges of \mathcal{T} , the condition 2 in the definition of a local orientation is automatically satisfied. Condition 3 is also vacuously true, since there is only one vertex.

To have a local orientation, now it suffices to check that condition 1 is also satisfied. What we show is the following.

Proposition 3.1. *The singular foliation \mathcal{F} is a regular foliation if and only if the choice of orientations on the edges of \mathcal{T} given by the left order is a local orientation.*

Proof. To show this, we consider the link of the vertex v . Since M is a manifold and \mathcal{T} is a triangulation, the $link(v)$ is actually a triangulated sphere. Let τ

denote the triangulation of $\text{link}(v)$. We color the edges of τ in the following way:

- one colors an edge e in blue if the vertices connected by e represents the local edges have the same orientation.
- otherwise, one colors e in black.

It is straightforward to see that this coloring is well-defined. Each tetrahedron in \mathcal{T} gives four triangles of τ , and obviously two of them have all edges colored in blue, and each of other two has one blue edge and two black edges. Let n be the number of tetrahedra in \mathcal{T} . Then, there are $4n$ blue edges and $2n$ black edges in τ .

On each tetrahedron Υ of \mathcal{T} , among connected components of $\mathcal{F} \cap \Upsilon$, there are two distinguished triangles which meet the vertex at the corner which are not local sink nor local source. Some faces of τ meets these distinguished triangles, and we record the intersection as a mid-curve colored in red. On the 2-skeleton of τ , these form a union of closed red curves so that each face intersects at most one red curve.

Note that from the construction it is clear that both $o(v)$ and $i(v)$ are nonempty. First we show that the red locus has only one connected component if and only if $o(v)$ and $i(v)$ are connected. Observe that the complement of the interior of the carrier of the red curves (union of the triangles intersecting a red curve) has only blue edges. There are more than one connected components of the red locus if and only if this complement has at least three connected components. Each connected component is entirely contained in either $o(v)$ or $i(v)$. Hence this means either $o(v)$ or $i(v)$ is not connected. To finish the proof, we need to show that the red locus has only one connected component if and only if the foliation \mathcal{F} is non-singular. Note that each leaf has a natural cell-decomposition where each cell is a connected component of the intersection between the leaf and a tetrahedron. Hence each cell is either a triangle or a quadrilateral. Let L be the leaf containing the vertex v . Note that all the cells of L containing v as a vertex are triangles which has a piece of red curve. These are distinguished triangles in tetrahedra described above. Let U be the union of all these triangles incident to v . Then components of $U \setminus v$ precisely corresponds to the components of the red locus. Now simply note that \mathcal{F} is non-singular if and only if $U \setminus v$ has only one component. Now the proposition is proved. \square

Using the above proposition and the result of [5], we can conclude that when \mathcal{F} is a regular foliation (equivalently, the order-induced edge orientation on \mathcal{T} is a local orientation), then the universal cover of M admits a measured foliation.

Lemma 3.2. *The direction on the 1-skeleton of \mathcal{T} given by the linear order has the property that every directed loop is homotopically essential.*

Proof. This is a direct consequence of the fact that the positive cone for a linear order is a semi-group, i.e., a product of positive elements is again a positive element. \square

Theorem 3.3. *If the singular foliation \mathcal{F} is a regular foliation, then \widetilde{M} admits a measured foliation which is normal with respect to the lift of \mathcal{T} .*

Proof. This now directly follows from Theorem 5.1 of [5]. \square

From our construction, a left order on $\pi_1(M)$ gives both an edge orientation and a singular foliation. Sometime an edge orientation is good enough to get a foliation. For example, Dunfield introduced a notion of foliar orientation in [7] and it is a sufficient condition for the manifold to admit a taut foliation. In the case a given order on $\pi_1(M)$ is bi-invariant, i.e., both left and right invariant, the induced edge orientation necessarily admits a sink edge so cannot be foliar. Hence it is not a priori clear if the manifold admits a taut foliation or not. On the other hand, we still have a fLOiation in this case, and there is no obvious obstruction for the edge orientation to be a local orientation. Once it is a local orientation, by Proposition 3.1, the fLOiation is a regular foliation and actually it is taut since the collection of edges is a collection of simple closed curves which meet every leaf of the foliation transversely. Hence, this could provide a larger class of examples of edge orientations than foliar orientations which guarantee the existence of a taut foliation in the manifold (hence related to the L-space conjecture). In short, we propose the following question.

Question 3.4. When is the edge orientation induced by a bi-invariant order a local orientation?

Above question might be easier to answer when we have more information about bi-invariant orders of $\pi_1(M)$.

Problem 3.5. When the manifold is either Seifert-fibered or Sol, then there is a known necessary and sufficient condition for π_1 to be bi-orderable by Boyer-Rolfsen-Wiest. Answer Question 3.4 in this case.

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