

A NOTE ON THE ZEROS OF JENSEN POLYNOMIALS

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ABSTRACT. Sufficient conditions for the Jensen polynomials of the derivatives of a real entire function to be hyperbolic are obtained. The conditions are given in terms of the growth rate and zero distribution of the function. As a consequence some recent results on Jensen polynomials, relevant to the Riemann hypothesis, are extended and improved.

1. Introduction

This paper is concerned with the zeros of entire functions. An entire function f is said to be *real* if $f^{(m)}(0) \in \mathbb{R}$ for all m . A real polynomial is said to be *hyperbolic* if it has real zeros only. The Riemann Xi-function is given by

$$\Xi(iz) = \frac{1}{2} \left(z^2 - \frac{1}{4} \right) \pi^{-\frac{z}{2} - \frac{1}{4}} \Gamma \left(\frac{z}{2} + \frac{1}{4} \right) \zeta \left(\frac{1}{2} + z \right).$$

It is an even real entire function, its zero set is contained in the infinite strip $\mathbb{S} = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1/2\}$, and the Riemann hypothesis is the conjecture that all the zeros of Ξ are real. It is well known that $(-1)^m \Xi^{(2m)}(0) > 0$ for all non-negative integers m .

For non-negative integers d and n , the polynomials $J^{d,n}$ and $P^{d,n}$ are given by

$$J^{d,n}(z) = \sum_{j=0}^d \binom{d}{j} \gamma(n+j) z^j \quad \text{and} \quad P^{d,n}(z) = \sum_{j=0}^d \binom{d}{j} \gamma(n+j) H_{d-j}(z),$$

where

$$\gamma(m) = \frac{(-1)^m m!}{(2m)!} \Xi^{(2m)}(0) \quad (m = 0, 1, \dots)$$

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and H_0, H_1, \dots are the Hermite polynomials [4, 5, 11]. Recall that the d -th Hermite polynomial is given by

$$H_d(z) = d! \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{(-1)^k}{k!(d-2k)!2^k} z^{d-2k}.$$

We also consider the polynomials $\tilde{J}^{d,n}$ and $\tilde{P}^{d,n}$ defined by

$$\tilde{J}^{d,n}(z) = \sum_{j=0}^d \binom{d}{j} \Xi^{(n+j)}(0) z^j \quad \text{and} \quad \tilde{P}^{d,n}(z) = \sum_{j=0}^d \binom{d}{j} \Xi^{(n+j)}(0) H_{d-j}(z).$$

As we shall see in the sequel, the polynomials $J^{d,n}$, $P^{d,n}$, $\tilde{J}^{d,n}$ and $\tilde{P}^{d,n}$ are the Jensen or Appell polynomials of the Riemann Xi-function itself or certain real entire functions related with it. According to a theorem of Pólya and Schur (Theorem 2.1 below), each of the following four statements is equivalent to the Riemann hypothesis.

- (i) All the polynomials $J^{d,n}$ are hyperbolic.
- (ii) All the polynomials $\tilde{J}^{d,n}$ are hyperbolic.
- (iii) All the polynomials $P^{d,n}$ are hyperbolic.
- (iv) All the polynomials $\tilde{P}^{d,n}$ are hyperbolic.

Recently, several authors proved that hyperbolic ones form a large portion of the polynomials $J^{d,n}$ and $P^{d,n}$. In 2019, Griffin et al. [5] proved that for every positive integer d , some renormalization of the sequence $\langle J^{d,n} \rangle_{n=0}^\infty$ converges to the Hermite polynomial H_d locally uniformly in the complex plane. As a consequence, if d is a positive integer, then $J^{d,n}$ is hyperbolic for all sufficiently large n . The result is improved by Theorem 1.1 in [4] which states that there is a constant $c > 0$ such that $J^{d,n}$ is hyperbolic for $d \geq 1$ and $n \geq ce^{d/2}$. As for the polynomials $P^{d,n}$, O’Sullivan [11] showed that for all d sufficiently large, $P^{d,n}$ is hyperbolic whenever $n/\log^2 n \geq d^{3/4}/2$. On the other hand, Chasse [1] proved that if $T \geq 1/2$, Ξ has only real zeros in the rectangle $\{z \in \mathbb{S} : |\operatorname{Re} z| \leq T\}$ and $d \leq T^2$, then $J^{d,n}$ is hyperbolic for every n . Currently, it is known that all the zeros of Ξ that lie in the rectangle $\{z \in \mathbb{S} : |\operatorname{Re} z| \leq 3 \cdot 10^{12}\}$ are real [12]. Furthermore, Theorem 2.3 below implies that if $J^{d,n}$ (or $\tilde{J}^{d,n}$) is hyperbolic, then all the zeros of $P^{d,n}$ (or $\tilde{P}^{d,n}$) are real and simple. In particular, $J^{d,n}$ and $P^{d,n}$ are hyperbolic for $d \leq 9 \cdot 10^{24}$ and for all n .

The purpose of this paper is to extend and improve the results mentioned above.

Theorem 1.1. *For every $c > 1$ there is a positive integer d_0 such that $J^{d,n}$, $P^{d,n}$, $\tilde{J}^{d,n}$ and $\tilde{P}^{d,n}$ are hyperbolic whenever $d \geq d_0$ and $n \geq d^c$.*

Theorem 1.2. *If $T \geq 1/2$, Ξ has only real zeros in the rectangle $\{z \in \mathbb{S} : |\operatorname{Re} z| \leq T\}$ and $d \leq 1+4T^2$, then the polynomials $\tilde{J}^{d,n}$ and $\tilde{P}^{d,n}$ are hyperbolic for all n .*

These are consequences of some general theorems on real entire functions with restricted zeros, which are stated and proved in Sections 2 and 3 below. It should be remarked that our proof gives no information on the polynomials considered such as their relation to the Hermite polynomials, except that they are hyperbolic.

In Section 2, we review some known results on the zeros of real polynomials which are needed in our discussion, and obtain Theorems 2.8 and 2.11 which imply Theorems 1.2 and 1.1, respectively. Theorem 2.11 is a consequence of Obreschkoff's theorem and a technical result (Theorem 2.9). It is proved in Section 3.

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2. The zeros of real polynomials

Let f be an entire function. We denote the zero set of f by $\mathcal{Z}(f)$, that is, $\mathcal{Z}(f) = \{z \in \mathbb{C} : f(z) = 0\}$. (If f is identically equal to 0, we put $\mathcal{Z}(f) = \emptyset$.) The order $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r; f)}{\log r},$$

where $M(r; f) = \max_{|z|=r} |f(z)|$. It is well known and easy to see that $\rho(f') = \rho(f)$. If f is even, f_0 denotes the function such that $f(z) = f_0(z^2)$ for all z . In this case, f_0 is also an entire function, $\mathcal{Z}(f_0) = \{z^2 : z \in \mathcal{Z}(f)\}$ and $\rho(f_0) = \rho(f)/2$. Furthermore, if f is real, then so is f_0 .

A real entire function f is said to be of *genus 1** if it can be represented as

$$f(z) = cz^m e^{-\alpha z^2 + \beta z} \prod_j \left(1 - \frac{z}{a_j}\right) e^{z/a_j},$$

where c and β are real constants, m is a non-negative integer, $\alpha \geq 0$, $a_j \neq 0$ for all j , and $\sum |a_j|^{-2} < \infty$. In this case, we have $\rho(f) \leq 2$, and there is a sequence $\langle P_k \rangle$ of real polynomials such that $P_k \rightarrow f$ locally uniformly in the complex plane and $\mathcal{Z}(P_k) \subset \mathcal{Z}(f) \cup \mathbb{R}$ for all k . Furthermore, Hadamard's factorization theorem implies that every real entire function of order less than 2 is of genus 1*. The *Laguerre-P olya class* \mathcal{LP} is the collection of real entire functions f for which there is a sequence $\langle P_k \rangle$ of hyperbolic polynomials such that $P_k \rightarrow f$ locally uniformly in the complex plane. Since the product of two hyperbolic polynomials and the derivative of a hyperbolic polynomial are hyperbolic, the Laguerre-P olya class is closed under product and differentiation. If f is a real entire function of genus 1*, then it is clear that $f \in \mathcal{LP}$ if and only if $\mathcal{Z}(f) \subset \mathbb{R}$. Actually, P olya proved that every function in the Laguerre-P olya class is of genus 1* [13]. Therefore $f \in \mathcal{LP}$ if and only if $\mathcal{Z}(f) \subset \mathbb{R}$ and f is of

genus 1*. Since Ξ is a real entire function of order 1, the Riemann hypothesis is equivalent to the statement that $\Xi^{(n)} \in \mathcal{LP}$ for all n .

For $d = 0, 1, 2, \dots$ the d -th Appell polynomial $A(f; d)$ and Jensen polynomial $J(f; d)$ of an entire function f are defined by

$$A(f; d)(z) = \sum_{k=0}^d \binom{d}{k} f^{(k)}(0) z^{d-k} \quad \text{and} \quad J(f; d)(z) = \sum_{k=0}^d \binom{d}{k} f^{(k)}(0) z^k,$$

respectively. Since $A(f; d)(z) = z^d J(f; d)(1/z)$, we see that $A(f; d)$ is hyperbolic if and only if $J(f; d)$ is hyperbolic. If P is a polynomial, we write

$$\sum_{k=0}^{\deg P} \frac{f^{(k)}(0)}{k!} P^{(k)} = f(D)P.$$

Thus $f(D)P$ is obtained by applying the differential operator

$$f(D) = f(0) + f'(0)D + \frac{f''(0)}{2!}D^2 + \dots + \frac{f^{(k)}(0)}{k!}D^k + \dots$$

to the polynomial P . If g is another entire function and $h = fg$, then it is clear that

$$h(D)P = f(D)(g(D)P) = g(D)(f(D)P).$$

For notational clarity, we write $e^{-z^2/2} = G(z)$, and denote the monic monomial of degree d by M^d , that is, $M^d(z) = z^d$. With this notation, we have $G \in \mathcal{LP}$,

$$A(f; d) = f(D)M^d \quad \text{and} \quad H_d = A(G; d) = e^{-D^2/2}M^d.$$

In 1914, Pólya and Schur obtained a necessary and sufficient condition for a real entire function to be in the Laguerre-Pólya class in terms of its Jensen polynomials [15].

Theorem 2.1 (Pólya-Schur). *Let f be a real entire function. Then $f \in \mathcal{LP}$ if and only if $J(f; d)$ is hyperbolic for every d .*

Remark 2.2. Actually, they proved the theorem under the assumption that f is merely a formal power series. For a modern and self-contained proof, see [9].

Since Ξ is even, we have

$$\Xi(z) = \sum_{m=0}^{\infty} \frac{\Xi^{(2m)}(0)}{(2m)!} z^{2m} \quad \text{and} \quad \Xi_0(z) = \sum_{m=0}^{\infty} \frac{\Xi^{(2m)}(0)}{(2m)!} z^m.$$

The Riemann hypothesis is equivalent to the inclusion $\mathcal{Z}(\Xi_0) \subset [0, \infty)$. But $\Xi_0^{(n)}$ has no zeros in $(-\infty, 0)$ for all n , because $(-1)^m \Xi^{(2m)}(0) > 0$ for all m . Hence the Riemann hypothesis holds if and only if $\Xi_0^{(n)} \in \mathcal{LP}$ for all n . We write

$$\Theta(z) = \Xi_0(-z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Xi^{(2m)}(0)}{(2m)!} z^m.$$

It is straightforward to see that

$$J^{d,n} = J(\Theta^{(n)}; d) \text{ and } \tilde{J}^{d,n} = J(\Xi^{(n)}; d).$$

Therefore each of the statements (i) and (ii) in Section 1 is equivalent to the Riemann hypothesis. On the other hand, we have

$$P^{d,n} = \Theta^{(n)}(D)H_d = A(\Theta^{(n)}\mathbf{G}; d) = e^{-D^2/2}A(\Theta^{(n)}; d)$$

and

$$\tilde{P}^{d,n} = \Xi^{(n)}(D)H_d = A(\Xi^{(n)}\mathbf{G}; d) = e^{-D^2/2}A(\Xi^{(n)}; d).$$

Hence each of the statements (iii) and (iv) in Section 1 is equivalent to the Riemann hypothesis.

If $\lambda > 0$, the differential operator $e^{-\lambda D^2}$ preserves the hyperbolicity of real polynomials in a strong way, and this may be the reason why the polynomials $P^{d,n}$ introduced by O'Sullivan are easier to handle than those of Griffin et al.

Theorem 2.3 (De Bruijn). *Suppose that $\lambda \geq 0$, P is a non-constant real polynomial and $\Delta = \max\{|\operatorname{Im} z| : P(z) = 0\}$. If $\Delta^2 - 2\lambda < 0$, then all the zeros of $e^{-\lambda D^2}P$ are real and simple, otherwise $\mathcal{Z}(e^{-\lambda D^2}P) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \sqrt{\Delta^2 - 2\lambda}\}$.*

Remark 2.4. The first version of this theorem was proved by de Bruijn [2]. A general version is proved in [6].

Corollary 2.5. *Suppose that d and n are non-negative integers and $J^{d,n}$ (or $\tilde{J}^{d,n}$) is hyperbolic. Then all the zeros of $P^{d,n}$ (or $\tilde{P}^{d,n}$) are real and simple.*

For $\delta > 0$ we set $S(\delta) = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \delta|z|\}$. The following theorem of Obreschkoff generalizes the classical Hermite-Poulain theorem.

Theorem 2.6 (Obreschkoff). *Suppose that P and Q are real polynomials, $\delta > 0$, $\mathcal{Z}(P) \subset S(\delta)$, Q is hyperbolic and $\deg Q \leq \delta^{-2}$. Then $P(D)Q$ is hyperbolic.*

Proof. See [10]. □

Corollary 2.7. *Suppose that $\delta > 0$, f is of genus 1^* and $\mathcal{Z}(f) \subset S(\delta)$. Then $J(f; d)$ is hyperbolic for $d \leq \delta^{-2}$.*

Proof. Since f is of genus 1^* , there is a sequence $\langle P_k \rangle$ of real polynomials such that $P_k \rightarrow f$ locally uniformly in the complex plane, and $\mathcal{Z}(P_k) \subset \mathcal{Z}(f) \cup \mathbb{R}$ for all k . We have $\mathcal{Z}(f) \cup \mathbb{R} \subset S(\delta)$. Hence Obreschkoff's theorem implies that $A(P_k; d) = P_k(D)M^d$ is hyperbolic for every k and for $d \leq \delta^{-2}$. Since $P_k \rightarrow f$ locally uniformly in the complex plane, it follows that $A(P_k; d) \rightarrow A(f; d)$ locally uniformly in the complex plane for every d . Therefore $A(f; d)$ is hyperbolic for $d \leq \delta^{-2}$, and the result follows. □

The Gauss-Lucas theorem states that if f is a polynomial, C is a convex set in the complex plane and $\mathcal{Z}(f) \subset C$, then $\mathcal{Z}(f') \subset C$. It is trivial to extend the theorem to the case where f is an entire function of order less than 1. The set $S(\delta)$ is not convex in general, however the following analogue of the Gauss-Lucas theorem holds.

Theorem 2.8. *Suppose that $0 < \delta \leq 2^{-1/2}$, f is an even entire function of genus 1^* , and that $\mathcal{Z}(f) \subset S(\delta)$. Then $\mathcal{Z}(f^{(n)}) \subset S(\delta)$ for all n .*

Proof. Let $\delta_0 = 2\delta\sqrt{1-\delta^2}$ and $S(\delta_0)^+ = \{z \in S(\delta_0) : \operatorname{Re} z \geq 0\}$. Then $S(\delta_0)^+$ is convex. Since $0 < \delta \leq 2^{-1/2}$, it follows that $z \in S(\delta)$ if and only if $z^2 \in S(\delta_0)^+$. In particular, we have $\mathcal{Z}(f_0) \subset S(\delta_0)^+$.

Suppose for the moment that f is a polynomial. Then so is f_0 , and the Gauss-Lucas theorem implies that $\mathcal{Z}(f'_0) \subset S(\delta_0)^+$. Hence $\mathcal{Z}(f') \subset S(\delta)$, because $f'(z) = 2zf'_0(z^2)$. We have

$$f''(z) = 2(f'_0(z^2) + 2z^2f''_0(z^2)).$$

If we write $g(z) = z(f'_0(z))^2$, then g is a polynomial and $\mathcal{Z}(g) \subset S(\delta_0)^+$. Again, it follows from the Gauss-Lucas theorem that $\mathcal{Z}(g') \subset S(\delta_0)^+$, that is, all the roots of the equation

$$f'_0(z)(f'_0(z) + 2zf''_0(z)) = 0$$

lie in $S(\delta_0)^+$. Therefore $\mathcal{Z}(f'') \subset S(\delta)$. Since f'' is even, the result follows from an induction.

In the general case, there is a sequence $\langle P_k \rangle$ of even polynomials such that $P_k \rightarrow f$ locally uniformly in the complex plane and $\mathcal{Z}(P_k) \subset S(\delta)$ for all k . Hence the result holds in this case too. \square

Proof of Theorem 1.2. Suppose that $T \geq 1/2$, all the zeros of Ξ that lie in the rectangle $\{z \in \mathbb{S} : |\operatorname{Re} z| \leq T\}$ are real, and that $d \leq 1 + 4T^2$. Put $\delta = (1 + 4T^2)^{-1/2}$. Then we have $0 < \delta \leq 2^{-1/2}$,

$$\mathcal{Z}(\Xi) \subset \{z \in \mathbb{S} : |\operatorname{Re} z| > T\} \cup \mathbb{R} \subset S(\delta),$$

and $d \leq \delta^{-2}$. Since Ξ is even and of order 1, Theorem 2.8 implies that $\mathcal{Z}(\Xi^{(n)}) \subset S(\delta)$ for all n . Therefore $\tilde{J}^{d,n}(= J(\Xi^{(n)}; d))$ is hyperbolic for all n , by Corollary 2.7. Finally, Corollary 2.5 implies that the polynomial $\tilde{P}^{d,n}$ is hyperbolic for every n . \square

The following theorem, which plays a crucial role in our proof of Theorem 2.11, is proved in Section 3.

Theorem 2.9. *Suppose that f is a transcendental real entire function, $a \geq 0$, $0 \leq b < 1$, $\rho(f) < 2(1 - b)$, and that*

$$\mathcal{Z}(f) \subset \left\{ z \in \mathbb{C} : |\operatorname{Im} z| \leq (a + |\operatorname{Re} z|)^b \right\}.$$

Then for every $B > 0$ and for every $c > \rho(f)$ there is a positive integer n_0 such that

$$\mathcal{Z}(f^{(n)}) \subset \left\{ z \in \mathbb{C} : |\operatorname{Re} z| > Bn^{1/c} \right\} \cup \mathbb{R} \quad (n \geq n_0).$$

Remark 2.10. (1) The assumption implies that $\rho(f) < 2$. Hence the result supports Pólya’s principle given in [14] which reads as follows. *The real axis seems to exert an influence on the complex (non-real) zeros of $f^{(n)}$; it seems to attract these zeros when the order is less than 2, and it seems to repel them when the order is greater than 2.* (2) Some refined versions of Theorem 2.9 are given in [7, 8].

Theorem 2.11. *Suppose that f is a transcendental real entire function, c is a positive constant, $\rho(f) < \min\{2, c\}$ and $\mathcal{Z}(f) \subset \mathbb{S}$. Then there is a positive integer d_0 such that $J(f^{(n)}; d)$ is hyperbolic whenever $d \geq d_0$ and $n \geq d^{c/2}$. If f is even, then there is a positive integer d_1 such that $J(f_0^{(n)}; d)$ is hyperbolic whenever $d \geq d_1$ and $n \geq d^{c/2}$.*

Proof. First of all, f satisfies the assumption of Theorem 2.9 with $b = 0$. Hence there is a positive integer n_0 such that

$$\mathcal{Z}(f^{(n)}) \subset \left\{ z \in \mathbb{C} : |\operatorname{Re} z| > 2^{-1}n^{1/c} \right\} \cup \mathbb{R} \quad (n \geq n_0).$$

Since $\mathcal{Z}(f) \subset \mathbb{S}$ and $\rho(f) < 2$, there is a sequence $\langle P_k \rangle$ of real polynomials such that $P_k \rightarrow f$ locally uniformly in the complex plane and $\mathcal{Z}(P_k) \subset \mathbb{S}$ for all k . It follows that $P'_k \rightarrow f'$ locally uniformly in the complex plane, and the Gauss-Lucas theorem implies that $\mathcal{Z}(P'_k) \subset \mathbb{S}$ for all k . Hence $\mathcal{Z}(f') \subset \bar{\mathbb{S}}$. Since $\rho(f') = \rho(f)$, we see that $\mathcal{Z}(f^{(n)}) \subset \bar{\mathbb{S}}$ for all n , by an induction. Therefore

$$\mathcal{Z}(f^{(n)}) \subset \left\{ z \in \bar{\mathbb{S}} : |\operatorname{Re} z| > 2^{-1}n^{1/c} \right\} \cup \mathbb{R} \subset S(\delta(n)) \quad (n \geq n_0),$$

where

$$\delta(n) = \left(1 + n^{2/c} \right)^{-1/2}.$$

Suppose that d and n are positive integers such that $n \geq d^{c/2} \geq n_0$. Then we have $n \geq n_0$ and $d < 1 + n^{2/c} = \delta(n)^{-2}$. Hence it follows from Corollary 2.7 that $J(f^{(n)}; d)$ is hyperbolic.

To prove the second part, suppose that f is even. Then we have $\rho(f_0) < \min\{1, c/2\}$,

$$\mathcal{Z}(f_0) \subset \mathbb{S}_0 = \{z^2 : z \in \mathbb{S}\} = \{z \in \mathbb{C} : (\operatorname{Im} z)^2 < 4^{-1} + \operatorname{Re} z\},$$

and the last set is contained in $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq (4^{-1} + |\operatorname{Re} z|)^{1/2}\}$. By Theorem 2.9, there is a positive integer n_1 such that

$$\mathcal{Z}(f_0^{(n)}) \subset \left\{ z \in \mathbb{C} : |\operatorname{Re} z| > n^{2/c} \right\} \cup \mathbb{R} \quad (n \geq n_1).$$

Furthermore, the Gauss-Lucas theorem implies that $\mathcal{Z}(f_0^{(n)}) \subset \mathbb{S}_0$ for all n . Therefore we have

$$\mathcal{Z}(f_0^{(n)}) \subset \left\{ z \in \mathbb{S}_0 : \operatorname{Re} z > n^{2/c} \right\} \cup \mathbb{R} \subset S(\delta_0(n)) \quad (n \geq n_1),$$

where

$$\delta_0(n) = \frac{\sqrt{n^{2/c} + 4^{-1}}}{n^{2/c} + 2^{-1}}.$$

It is trivial to see that $\delta_0(n)^{-2} > n^{2/c}$ for all n . Hence $J(f_0^{(n)}; d)$ is hyperbolic whenever $n \geq d^{c/2} \geq n_1$. \square

Remark 2.12. Since our argument deals with entire functions of order less than 2, it refines the proof of Theorem 3.6 in [1].

Proof of Theorem 1.1. Suppose that $c > 1$. By Theorem 2.11, there is a positive integer d_0 such that the polynomials $J(\Xi^{(n)}; d)$ and $J(\Xi_0^{(n)}; d)$ are hyperbolic whenever $d \geq d_0$ and $n \geq d^{c/2}$. We have $\tilde{J}^{d,n} = J(\Xi^{(n)}; d)$ and $J^{d,n}(z) = J(\Theta^{(n)}; d)(z) = J(\Xi_0^{(n)}; d)(-z)$, and Corollary 2.5 implies that the polynomials $P^{d,n}$ and $\tilde{P}^{d,n}$ are hyperbolic whenever $J^{d,n}$ and $\tilde{J}^{d,n}$ are hyperbolic. This completes the proof. \square

3. Proof of Theorem 2.9

We begin with some known theorems of Jensen and Gontcharoff that are needed in our proof of Theorem 2.9.

Theorem 3.1 (Jensen). *Suppose that f is a non-constant real entire function of genus 1^* and z_1 is a non-real zero of f' . Then there is a non-real zero z_0 of f such that*

$$|z_1 - \operatorname{Re} z_0| \leq |\operatorname{Im} z_0|.$$

Proof. See [7, p. 822]. \square

Corollary 3.2. *Suppose that f is a real entire function of order less than 2, n is a positive integer, $f^{(n)}(z_n) = 0$ and $\operatorname{Im} z_n > 0$. Then there are $z_0, z_1, \dots, z_{n-1} \in \mathbb{C}$ such that $f^{(k)}(z_k) = 0$ for $k = 0, 1, \dots, n - 1$, $\operatorname{Im} z_0 \geq \operatorname{Im} z_1 \geq \dots \geq \operatorname{Im} z_n$,*

$$(1) \quad \begin{aligned} & |\operatorname{Re} z_0 - \operatorname{Re} z_1| + |\operatorname{Re} z_1 - \operatorname{Re} z_2| + \dots + |\operatorname{Re} z_{n-1} - \operatorname{Re} z_n| \\ & \leq \sqrt{n((\operatorname{Im} z_0)^2 - (\operatorname{Im} z_n)^2)} \end{aligned}$$

and

$$(2) \quad \begin{aligned} & |z_0 - z_1| + |z_1 - z_2| + \dots + |z_{n-1} - z_n| \\ & \leq \operatorname{Im} z_0 - \operatorname{Im} z_n + \sqrt{n((\operatorname{Im} z_0)^2 - (\operatorname{Im} z_n)^2)}. \end{aligned}$$

Proof. Since the order of an entire function is not changed by differentiation, the functions $f, f', \dots, f^{(n)}$ are of genus 1^* . By Jensen's theorem, there are complex numbers z_0, z_1, \dots, z_{n-1} such that $\text{Im } z_k > 0, f^{(k)}(z_k) = 0$ and $|z_{k+1} - \text{Re } z_k| \leq \text{Im } z_k$ for $k = 0, 1, \dots, n - 1$. We have $\text{Im } z_{k+1} \leq \text{Im } z_k$ and

$$|\text{Re } z_k - \text{Re } z_{k+1}| \leq \sqrt{(\text{Im } z_k)^2 - (\text{Im } z_{k+1})^2}$$

for all k . Hence the Cauchy-Schwarz inequality implies that (1) holds. Finally, (2) follows from (1), the triangle inequality and the fact that $\text{Im } z_{k+1} \leq \text{Im } z_k$ for all k . □

Theorem 3.3 (Gontcharoff). *Suppose that f is an entire function, n is a positive integer, $z, z_0, z_1, \dots, z_{n-1} \in \mathbb{C}$ and $f^{(k)}(z_k) = 0$ for $k = 0, 1, \dots, n - 1$. Then*

$$|f(z)| \leq \frac{M}{n!} (|z - z_0| + |z_0 - z_1| + \dots + |z_{n-2} - z_{n-1}|)^n,$$

where M is the maximum of $|f^{(n)}|$ on the convex hull of the set $\{z, z_0, z_1, \dots, z_{n-1}\}$.

Proof. Since $f^{(k)}(z_k) = 0$ for $k = 0, 1, \dots, n - 1$, we have

$$f(z) = \int_{z_0}^z \int_{z_1}^{v_1} \dots \int_{z_{n-1}}^{v_{n-1}} f^{(n)}(v_n) dv_n \dots dv_2 dv_1.$$

Put

$$\begin{aligned} t_0 &= |z_{n-1} - z_{n-2}| + |z_{n-2} - z_{n-3}| + \dots + |z_2 - z_1| + |z_1 - z_0|, \\ t_1 &= |z_{n-1} - z_{n-2}| + |z_{n-2} - z_{n-3}| + \dots + |z_2 - z_1|, \\ &\vdots \\ t_{n-2} &= |z_{n-1} - z_{n-2}|, \\ t_{n-1} &= 0 \quad \text{and} \\ t &= t_0 + |z - z_0|, \end{aligned}$$

and parameterize the polygonal path with vertices $z, z_0, z_1, \dots, z_{n-1}$ by $v : [0, t] \rightarrow \mathbb{C}$ such that $v(t) = z, v(t_k) = z_k$ for $k = 0, 1, \dots, n - 1$, and $|v'(s)| = 1$ for $s \neq t_0, t_1, \dots, t_{n-1}$. Then we have

$$\begin{aligned} |f(z)| &\leq \int_{t_0}^t \int_{t_1}^{s_1} \dots \int_{t_{n-1}}^{s_{n-1}} |f^{(n)}(v(s_n))| ds_n \dots ds_2 ds_1 \\ &\leq M \int_{t_0}^t \int_{t_1}^{s_1} \dots \int_{t_{n-1}}^{s_{n-1}} ds_n \dots ds_2 ds_1 \\ &\leq M \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} ds_n \dots ds_2 ds_1 \\ &= \frac{M}{n!} t^n \end{aligned}$$

$$= \frac{M}{n!} (|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|)^n. \quad \square$$

Remark 3.4. This proof is identical with the one that is given in [3, pp. 11–12].

Proof of Theorem 2.9. We prove the theorem by very rough estimations based on Corollary 3.2 and Theorem 3.3. Suppose, to obtain a contradiction, that there are $B > 0$ and $c > \rho(f)$ such that $f^{(n)}$ has non-real zeros in $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq Bn^{1/c}\}$ for infinitely many positive integers n . Let E denote the set of such integers. Since $\rho(f) < 2(1 - b)$, we may assume that $\rho(f) < c \leq 2(1 - b)$. Otherwise replace c with $2(1 - b)$. We put $B_1 = ((a + 1)^b + B^{1-b})^{\frac{1}{1-b}}$. It is obvious that $B_1 > 1$ and $B_1 > B$.

We first show that

$$(3) \quad |f(z)| \leq \frac{M(5B_1n^{1/c}; f^{(n)})}{n!} (5B_1n^{1/c})^n \quad (|z| \leq 1; n \in E).$$

Suppose that n is a positive integer, $f^{(n)}(z_n) = 0$, $\operatorname{Im} z_n > 0$ and $|\operatorname{Re} z_n| \leq Bn^{1/c}$. Then Corollary 3.2 implies that there are $z_0, z_1, \dots, z_{n-1} \in \mathbb{C}$ such that the inequalities (1) and (2) hold, $f^{(k)}(z_k) = 0$ and $\operatorname{Im} z_k > 0$ for $k = 0, 1, \dots, n - 1$. To proceed further, we need a lemma.

Lemma 3.5. *We have*

$$(4) \quad \operatorname{Im} z_0, |\operatorname{Re} z_0|, \sqrt{n}(a + |\operatorname{Re} z_0|)^b \leq B_1n^{1/c}.$$

Proof. Since $f(z_0) = 0$, we have

$$(5) \quad \operatorname{Im} z_0 \leq (a + |\operatorname{Re} z_0|)^b \leq \sqrt{n}(a + |\operatorname{Re} z_0|)^b.$$

We prove the second and third inequalities of (4) simultaneously.

Suppose, for the moment, that $|\operatorname{Re} z_0| \leq 1$. Then we have $|\operatorname{Re} z_0| < B_1n^{1/c}$ and

$$\begin{aligned} \sqrt{n}(a + |\operatorname{Re} z_0|)^b &\leq \sqrt{n}(a + 1)^b + (Bn^{1/c})^{1-b} \\ &\leq \left(\sqrt{n}(a + 1)^b + (Bn^{1/c})^{1-b} \right)^{\frac{1}{1-b}} \\ &= n^{1/c} \left(n^{-\frac{1-b}{c}} \sqrt{n}(a + 1)^b + B^{1-b} \right)^{\frac{1}{1-b}} \\ &\leq B_1n^{1/c}. \end{aligned}$$

Next, we consider the case where $|\operatorname{Re} z_0| > 1$. From (1) and (5), it follows that

$$\begin{aligned} |\operatorname{Re} z_0| &\leq |\operatorname{Re} z_n| + \sqrt{n} \operatorname{Im} z_0 \\ &\leq |\operatorname{Re} z_n| + \sqrt{n}(a + |\operatorname{Re} z_0|)^b. \end{aligned}$$

Since $b < 1$, there is a real number $A \geq |\operatorname{Re} z_0|$ such that

$$A = |\operatorname{Re} z_n| + \sqrt{n}(a + A)^b.$$

We have $A \geq \max\{1, |\operatorname{Re} z_0|, |\operatorname{Re} z_n|, \sqrt{n}(a + A)^b\}$, and it is easy to see that

$$A = \left(|\operatorname{Re} z_n| A^{-b} + \sqrt{n} (aA^{-1} + 1)^b \right)^{\frac{1}{1-b}}.$$

Hence

$$\begin{aligned} \sqrt{n}(a + |\operatorname{Re} z_0|)^b &\leq \sqrt{n}(a + A)^b \\ &\leq A \\ &= \left(|\operatorname{Re} z_n| A^{-b} + \sqrt{n} (aA^{-1} + 1)^b \right)^{\frac{1}{1-b}} \\ &\leq \left(|\operatorname{Re} z_n|^{1-b} + \sqrt{n} (a + 1)^b \right)^{\frac{1}{1-b}} \\ &\leq \left((Bn^{1/c})^{1-b} + \sqrt{n} (a + 1)^b \right)^{\frac{1}{1-b}} \\ &\leq n^{1/c} \left(B^{1-b} + (a + 1)^b \right)^{\frac{1}{1-b}} \\ &= B_1 n^{1/c}. \end{aligned}$$

Since $|\operatorname{Re} z_0| \leq A$, the proof is complete. □

By (2) and (4), we have

$$\begin{aligned} &|z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| \\ &\leq (1 + \sqrt{n}) \operatorname{Im} z_0 \\ &\leq 2\sqrt{n} (a + |\operatorname{Re} z_0|)^b \\ &\leq 2B_1 n^{1/c}, \end{aligned}$$

and (4) implies that $|z_0| \leq 2B_1 n^{1/c}$. Hence we have $|z_k| \leq 4B_1 n^{1/c}$ for $k = 1, \dots, n$ and

$$|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-1} - z_n| \leq 1 + 4B_1 n^{1/c} < 5B_1 n^{1/c}$$

for $|z| \leq 1$. Therefore Gontcharoff's theorem implies that (3) holds.

Choose a real number d such that $\rho(f) < d < c$. Then $e^{-r^d} M(r; f) \rightarrow 0$ as $r \rightarrow \infty$; hence we may assume that

$$M(r; f) \leq e^{r^d} \quad (r \geq 0).$$

For notational simplicity, we write $5B_1 = \alpha$, so that

$$|f(z)|^{1/n} \leq \alpha n^{1/c} \left(\frac{M(\alpha n^{1/c}; f^{(n)})}{n!} \right)^{1/n} \quad (|z| \leq 1; n \in E).$$

If $n > 1$, then $\alpha n^{1/d} - \alpha n^{1/c} > 0$, and Cauchy's integral formula implies that

$$\alpha n^{1/c} \left(\frac{M(\alpha n^{1/c}; f^{(n)})}{n!} \right)^{1/n} \leq \alpha n^{1/c} \left(\frac{M(\alpha n^{1/d}; f)}{(\alpha n^{1/d} - \alpha n^{1/c})^n} \right)^{1/n}$$

$$\begin{aligned}
&= \alpha n^{1/c} \frac{M(\alpha n^{1/d}, f)^{1/n}}{\alpha n^{1/d} - \alpha n^{1/c}} \\
&\leq \frac{\left(e^{(\alpha n^{1/d})^d}\right)^{1/n}}{n^{(c-d)/cd} - 1} \\
&= \frac{e^{\alpha^d}}{n^{(c-d)/cd} - 1}.
\end{aligned}$$

Since the last expression tends to 0 as $n \rightarrow \infty$, we conclude that $f(z) = 0$ for $|z| \leq 1$. This is the desired contradiction. \square

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