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C*-ALGEBRAIC SCHUR PRODUCT THEOREM, PÓLYA-SZEGŐ-RUDIN QUESTION AND NOVAK'S CONJECTURE

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Abstract. Striking result of Vybíral [51] says that Schur product of positive matrices is bounded below by the size of the matrix and the row sums of Schur product. Vybíral used this result to prove the Novak's conjecture. In this paper, we define Schur product of matrices over arbitrary C*-algebras and derive the results of Schur and Vybíral. As an application, we state C*-algebraic version of Novak's conjecture and solve it for commutative unital C*-algebras. We formulate Pólya-Szegő-Rudin question for the C*-algebraic Schur product of positive matrices.

1. Introduction

Given matrices $A \coloneqq [a_{j,k}]_{1\leq j,k\leq n}$ and $B \coloneqq [b_{j,k}]_{1\leq j,k\leq n}$ in the matrix ring $M_n(\mathbb{K})$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, the *Schur/Hadamard/pointwise product* of A and B is defined as

$$A \circ B \coloneqq \left[a_{j,k}b_{j,k}\right]_{1\leq j,k\leq n}. \tag{1}$$

Recall that a matrix $A \in M_n(\mathbb{K})$ is said to be positive (also known as self-adjont positive semidefinite) if it is self-adjoint and

$$\langle Ax, x\rangle \geq 0, \quad \forall x \in \mathbb{K}^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product (which is left linear right conjugate linear) on $\mathbb{K}^n$ (to move with the tradition of 'operator algebra', by 'positive' we only consider self-adjoint matrices). In this case we write $A \succeq 0$ and we write $A \succeq B$ if all of A, B and $A-B$ are positive. It is a century old result that whenever $A, B \in M_n(\mathbb{K})$ are positive, then their Schur product $A\circ B$ is positive. Schur originally proved this result in his famous 'Crelle' paper [46] and today there are varieties of proofs of this theorem. For a comprehensive look on Hadamard products we refer the reader to [18–20, 38, 48, 55].

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Once we know that the Schur product of two positive matrices is positive, then next step is to ask for a lower bound for the product, if exists. There are series of papers obtaining lower bounds for Schur product of positive correlation matrices [32, 54], positive invertible matrices [1, 2, 8, 9, 22, 31, 50, 52] but for arbitrary positive matrices there are a couple of recent results by Vybíral [51] which we mention now. To state the results we need some notations. Given a matrix $M \in M_n(\mathbb{K})$, by $\overline{M}$ we mean the matrix obtained by taking conjugate of each entry of M. Conjugate transpose of a matrix M is denoted by M^* and M^{T} denotes its transpose. Notation $\operatorname{diag}(M)$ denotes the vector consisting of the diagonal of matrix in the increasing subscripts. Matrix E_n denotes the n by n matrix in $M_n(\mathbb{K})$ with all one's. Given a vector $x \in \mathbb{K}^n$, by $\operatorname{diag}(x)$ we mean the n by n diagonal matrix obtained by putting i'th co-ordinate of x as (i, i) entry.

Theorem 1.1 ([51]). *Let $A \in M_n(\mathbb{K})$ be a positive matrix. Let $M = AA^*$ and $y \in \mathbb{K}^n$ be the vector of row sums of A. Then*

$$M \succeq \frac{1}{n} yy^*.$$

Theorem 1.2 ([51]). *Let $M, N \in M_n(\mathbb{K})$ be positive matrices. Let $M = AA^*$, $N = BB^*$ and $y \in \mathbb{K}^n$ be the vector of row sums of $A \circ B$. Then*

$$M \circ N \succeq (A \circ B)(A \circ B)^* \succeq \frac{1}{n} yy^*.$$

Immediate consequences of Theorem 1.2 are the following.

Corollary 1.3 ([51]). *Let $M \in M_n(\mathbb{K})$ be a positive matrix. Then*

$$M \circ \overline{M} \succeq \frac{1}{n}(diag\ M)(diag\ M)^T$$

and

$$M \circ M \succeq \frac{1}{n}(diag\ M)(diag\ M)^*.$$

Corollary 1.4 ([51]). *Let $M \in M_n(\mathbb{R})$ be a positive matrix such that all diagonal entries are one's. Then*

$$M \circ M \succeq \frac{1}{n} E_n.$$

Vybíral used Corollary 1.4 to solve two decades old Novak's conjecture which states as follows.

Theorem 1.5 ([16, 36, 37], Novak's conjecture). *The matrix*

$$\left[\prod_{l=1}^{d} \frac{1+\cos(x_{j,l}-x_{k,l})}{2} - \frac{1}{n}\right]_{1\leq j,k\leq n}$$

is positive for all $n, d \geq 2$ and all choices of $x_j = (x_{j,1}, \dots, x_{j,d}) \in \mathbb{R}^d$, $\forall 1 \leq j \leq n$.

Theorem 1.2 is also used in the study of random variables, numerical integration, trigonometric polynomials and tensor product problems, see [15, 51].

The purpose of this paper is to introduce the Schur product of matrices over C*-algebras, obtain some fundamental results and to state some problems. A very handy tool which we use is the theory of Hilbert C*-modules. This was first introduced by Kaplansky [26] for commutative C*-algebras and later by Paschke [39] and Rieffel [42] for non commutative C*-algebras. The theory attained a greater height from the work of Kasparov [6, 21, 27]. For an introduction to the subject Hilbert C*-modules we refer [30, 34].

Definition 1 ([26, 39, 42])**.** Let $\mathcal{A}$ be a C*-algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) *Hilbert C*-module* if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that the following hold.

(i) $\langle x, x\rangle \geq 0$, $\forall x \in \mathcal{E}$. If $x \in \mathcal{E}$ satisfies $\langle x, x\rangle = 0$, then $x = 0$.
(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\forall x, y, z \in \mathcal{E}$.
(iii) $\langle ax, y\rangle = a\langle x, y\rangle$, $\forall x, y \in \mathcal{E}$, $\forall a \in \mathcal{A}$.
(iv) $\langle x, y\rangle = \langle y, x\rangle^*$, $\forall x, y \in \mathcal{E}$.
(v) $\mathcal{E}$ is complete with respect to the norm $\|x\| \coloneqq \sqrt{\|\langle x, x\rangle\|}$, $\forall x \in \mathcal{E}$.

We are going to use the following inequality.

Lemma 1.6 ([39], Cauchy-Schwarz inequality for Hilbert C*-modules)**.** *If $\mathcal{E}$ is a Hilbert C*-module over $\mathcal{A}$, then*

$$\langle x, y\rangle\langle y, x\rangle \leq \|\langle y, y\rangle\|\langle x, x\rangle, \quad \forall x, y \in \mathcal{E}.$$

We encounter the following *standard Hilbert C*-module* in this paper. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{A}^n$ be the left module over $\mathcal{A}$ with respect to natural operations. Modular $\mathcal{A}$-inner product on $\mathcal{A}^n$ is defined as

$$\langle (x_j)_{j=1}^n, (y_j)_{j=1}^n\rangle \coloneqq \sum_{j=1}^n x_j y_j^*, \quad \forall (x_j)_{j=1}^n, (y_j)_{j=1}^n \in \mathcal{A}^n.$$

Hence the norm on $\mathcal{A}^n$ becomes

$$\|(x_j)_{j=1}^n\| \coloneqq \left\|\sum_{j=1}^n x_j x_j^*\right\|^{\frac{1}{2}}, \quad \forall (x_j)_{j=1}^n \in \mathcal{A}^n.$$

This paper is organized as follows. In Section 2 we define Schur/Hadamard/pointwise product of two matrices over C*-algebras (Definition 2). This is not a direct mimic of Schur product of matrices over scalars. After the definition of Schur product, we derive Schur product theorem for matrices over commutative C*-algebras (Theorem 2.2), σ-finite W*-algebras or AW*-algebras (Theorem 2.7). Followed by these results, we ask Pólya-Szegő-Rudin question for positive matrices over C*-algebras (Question 2.8). We then develop the paper following the developments by Vybíral in [51] to the setting of C*-algebras. In Section 3 we first derive lower bound for positive matrices over C*-algebras (Theorem

3.1) and using that we derive lower bounds for Schur product (Theorem 3.2 and Corollaries 3.3, 3.4). We later state C*-algebraic version of Novak's conjecture (Conjecture 4.2). We solve it for commutative unital C*-algebras (Theorem 4.3). Finally we end the paper by asking Question 4.5.

2. C*-algebraic Schur product, Schur product theorem and Pólya-Szegő-Rudin question

We first recall the basics in the theory of matrices over C*-algebras. More information can be found in [35, 53]. Let $\mathcal{A}$ be a unital C*-algebra and n be a natural number. Set $M_n(\mathcal{A})$ is defined as the set of all n by n matrices over $\mathcal{A}$ which becomes an algebra with respect to natural matrix operations. The involution of an element $A \coloneqq [a_{j,k}]_{1\leq j,k\leq n} \in M_n(\mathcal{A})$ is defined as $A^* \coloneqq [a^*_{k,j}]_{1\leq j,k\leq n}$. Then $M_n(\mathcal{A})$ becomes a *-algebra. *Gelfand-Naimark-Segal theorem* says that there exists a unique universal representation $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space, $\pi : M_n(\mathcal{A}) \to M_n(\mathcal{B}(\mathcal{H}))$ is an isometric *-homomorphism. Using this, the norm on $M_n(\mathcal{A})$ is defined as

$$\|A\| \coloneqq \|\pi(A)\|, \quad \forall A \in M_n(\mathcal{A})$$

which makes $M_n(\mathcal{A})$ as a C*-algebra (where $\mathcal{B}(\mathcal{H})$ is the C*-algebra of all continuous linear operators on $\mathcal{H}$ equipped with the operator-norm).

We define C*-algebraic Schur product as follows.

Definition 2. Let $\mathcal{A}$ be a C*-algebra. Given $A\coloneqq[a_{j,k}]_{1\leq j,k\leq n}, B\coloneqq[b_{j,k}]_{1\leq j,k\leq n} \in M_n(\mathcal{A})$, we define the *C*-algebraic Schur/Hadamard/pointwise product* of A and B as

$$A \circ B \coloneqq \frac{1}{2}\left[a_{j,k}b_{j,k} + b_{j,k}a_{j,k}\right]_{1\leq j,k\leq n}. \tag{2}$$

Whenever the C*-algebra is commutative, then (2) becomes

$$A \circ B = \left[a_{j,k}b_{j,k}\right]_{1\leq j,k\leq n}.$$

In particular, Definition 2 reduces to the definition of classical Schur product given in Equation (1). From a direct computation, we have the following result.

Theorem 2.1. *Let $\mathcal{A}$ be a unital C*-algebra and let $A, B, C \in M_n(\mathcal{A})$. Then*

(i) $A \circ B = B \circ A$.
(ii) $(A \circ B)^* = A^* \circ B^*$. *In particular, if A and B are self-adjoint, then $A \circ B$ is self-adjoint.*
(iii) $A \circ (B + C) = A \circ B + A \circ C$.
(iv) $(A + B) \circ C = A \circ C + B \circ C$.

One of the most important difference of Definition 2 from the classical Schur product is that the product may not be associative, i.e., $(A\circ B)\circ C \neq A\circ(B\circ C)$ in general.

Similar to the scalar case, $A := [a_{j,k}]_{1\leq j,k\leq n} \in M_n(\mathcal{A})$ is said to be positive if it is self-adjoint and

$$\langle Ax, x\rangle \geq 0, \quad \forall x \in \mathcal{A}^n,$$

where $\geq$ is the partial order on the set of all positive elements of $\mathcal{A}$. In this case we write $A \succeq 0$. It is well-known in the theory of C*-algebras that the set of all positive elements in a C*-algebra is a closed positive cone. We then have that the set of all positive matrices in $M_n(\mathcal{A})$ is a closed positive cone. Here comes the first version of C*-algebraic Schur product theorem.

Theorem 2.2 (Commutative C*-algebraic version of Schur product theorem)**.** *Let $\mathcal{A}$ be a commutative unital C*-algebra. If $M, N \in M_n(\mathcal{A})$ are positive, then their Schur product $M \circ N$ is also positive.*

Proof. Let $x \in \mathcal{A}^n$ and define $L := (M^{\frac{1}{2}})^{\mathrm{T}}(\text{diag } x)(N^{\frac{1}{2}})^{\mathrm{T}}$. First note that $M \circ N$ is self-adjoint. Using the commutativity of C*-algebra, we get

$$\begin{aligned}\langle (M \circ N)x, x\rangle &= x^*(M \circ N)x = \mathrm{Tr}((\text{diag } x^*)M(\text{diag } x)N^{\mathrm{T}})\\ &= \mathrm{Tr}((\text{diag } x^*)M(\text{diag } x)(N^{\frac{1}{2}})^{\mathrm{T}}(N^{\frac{1}{2}})^{\mathrm{T}})\\ &= \mathrm{Tr}((N^{\frac{1}{2}})^{\mathrm{T}}(\text{diag } x^*)M(\text{diag } x)(N^{\frac{1}{2}})^{\mathrm{T}})\\ &= \mathrm{Tr}((N^{\frac{1}{2}})^{\mathrm{T}}(\text{diag } x^*)(M^{\frac{1}{2}})^{\mathrm{T}}(M^{\frac{1}{2}})^{\mathrm{T}}(\text{diag } x)(N^{\frac{1}{2}})^{\mathrm{T}})\\ &= \mathrm{Tr}(L^*L) \geq 0.\end{aligned}$$

Since x was arbitrary, the result follows. □

Note that we used commutativity of the C*-algebra in the proof of Theorem 2.2 to ensure $x^*(M \circ N)x = \mathrm{Tr}((\text{diag } x^*)M(\text{diag } x)N^{\mathrm{T}})$. It is easy to see that for non commutative C*-algebras, this equality fails.

In the sequel, we use the following notation. Given $M \in M_n(\mathcal{A})$, we define

$$\begin{aligned}(M^\circ)^n &:= M \circ \cdots \circ M \quad (n \text{ times}), \forall n \geq 1,\\ (M^\circ)^0 &:= I \quad (\text{identity matrix in } M_n(\mathcal{A})).\end{aligned}$$

Corollary 2.3. *Let $\mathcal{A}$ be a commutative unital C*-algebra. Let $M \in M_n(\mathcal{A})$ be positive. If $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ is any polynomial with coefficients from $\mathcal{A}$ with all $a_0, \dots, a_n$ are positive elements of $\mathcal{A}$, then the matrix*

$$a_0I + a_1M + a_2(M^\circ)^2 + \cdots + a_n(M^\circ)^n \in M_n(\mathcal{A})$$

is positive.

Proof. This follows from Theorem 2.2 and Mathematical induction. □

Remark 2.4. Note that we used commutativity of C*-algebra in the proof of Theorem 2.2 and thus it can not be carried over to non commutative C*-algebras.

Theorem 2.2 leads us to seek a similar result for non commutative C*-algebras. At present we don't know Schur product theorem for positive matrices over arbitrary C*-algebras. For the purpose of definiteness, we state it as an open problem.

Question 2.5. *Let $\mathcal{A}$ be a C*-algebra. Given positive matrices $M, N \in M_n(\mathcal{A})$, whether $M \circ N$ is positive? In other words, classify those C*-algebras $\mathcal{A}$ such that $M \circ N$ is positive whenever $M, N \in M_n(\mathcal{A})$ are positive.*

To make some progress to Question 2.5, we give an affirmative answer for certain classes of C*-algebras (von Neumann algebras). To do so we need spectral theorem for matrices over C*-algebras. First let us recall two definitions.

Definition 3 ([33])**.** A W*-algebra is called *σ-finite* if it contains no more than a countable set of mutually orthogonal projections.

Definition 4 ([25])**.** A C*-algebra $\mathcal{A}$ is called an *AW*-algebra* if the following conditions hold.

(i) Any set of orthogonal projections has supremum.
(ii) Any maximal commutative self-adjoint subalgebra of $\mathcal{A}$ is generated by its projections.

Theorem 2.6 ([14,33], Spectral theorem for Hilbert C*-modules)**.** *Let $\mathcal{A}$ be a σ-finite W*-algebra or an AW*-algebra. If $M \in M_n(\mathcal{A})$ is normal, then there exists a unitary matrix $U \in M_n(\mathcal{A})$ such that UMU^* is a diagonal matrix.*

Theorem 2.7 (Non commutative C*-algebraic Schur product theorem)**.** *Let $\mathcal{A}$ be a σ-finite W*-algebra or an AW*-algebra and $M, N \in M_n(\mathcal{A})$ be positive. Let $U = [u_{j,k}]_{1\leq j,k\leq n}, V = [v_{j,k}]_{1\leq j,k\leq n} \in M_n(\mathcal{A})$ be unitary such that*

$$M = U \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} U^*, \quad N = V \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} V^*$$

for some $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \mathcal{A}$. If all $\lambda_j, \mu_k, u_{l,m}, v_{r,s}$, $1 \leq j, k, l, m, r, s \leq n$ commute with each other, then the Schur product $M \circ N$ is also positive.

Proof. Let $\{u_1, \dots, u_n\}$ be columns of U and $\{v_1, \dots, v_n\}$ be columns of V. Then

$$M = \sum_{j=1}^{n} \lambda_j u_j u_j^*, \quad N = \sum_{k=1}^{n} \mu_k v_k v_k^*,$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of M, $\{u_1, \dots, u_n\}$ is an orthonormal basis for $\mathcal{A}^n$, $\mu_1, \dots, \mu_n$ are eigenvalues of N and $\{v_1, \dots, v_n\}$ is an orthonormal basis for $\mathcal{A}^n$ (they exist from Theorem 2.6). Definition 2 of Schur product says that $M \circ N$ is self-adjoint. It is well known in the theory of C*-algebras that sum of positive elements in a C*-algebra is positive and the product of two commuting positive elements is positive. This observation, Theorem 2.1 and the following calculation shows that $M \circ N$ is positive:

$$M \circ N = \left(\sum_{j=1}^{n} \lambda_j u_j u_j^*\right) \circ \left(\sum_{k=1}^{n} \mu_k v_k v_k^*\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j \mu_k (u_j u_j^*) \circ (v_k v_k^*)$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j \mu_k (u_j \circ v_k)(u_j \circ v_k)^* \succeq 0.$$

□

Since the *spectral theorem fails for matrices over C*-algebras* (see [11,23,24]), proof of Theorem 2.7 can not be executed for arbitrary C*-algebras.

Given certain order structure, one naturally considers functions (in a suitable way) which preserve the order. For matrices over C*-algebras, we formulate this in the following definition.

Definition 5. Let $\mathcal{B}$ be a subset of a C*-algebra $\mathcal{A}$ and n be a natural number. Define $\mathcal{P}_n(\mathcal{B})$ as the set of all n by n positive matrices with entries from $\mathcal{B}$. Given a function $f : \mathcal{B} \to \mathcal{A}$, define a function

$$\mathcal{P}_n(\mathcal{B}) \ni A := \left[a_{j,k}\right]_{1\leq j,k\leq n} \mapsto f[A] := \left[f(a_{j,k})\right]_{1\leq j,k\leq n} \in M_n(\mathcal{A}).$$

A function $f : \mathcal{B} \to \mathcal{A}$ is said to be a *positivity preserver in all dimensions* if

$$f[A] \in \mathcal{P}_n(\mathcal{A}), \quad \forall A \in \mathcal{P}_n(\mathcal{B}), \quad \forall n \in \mathbb{N}.$$

A function $f : \mathcal{B} \to \mathcal{A}$ is said to be a *positivity preserver in fixed dimension n* if

$$f[A] \in \mathcal{P}_n(\mathcal{A}), \quad \forall A \in \mathcal{P}_n(\mathcal{B}).$$

We now have the important C*-algebraic Pólya-Szegő-Rudin open problem.

Question 2.8 (Pólya-Szegő-Rudin question for C*-algebraic Schur product of positive matrices)**.** *Let $\mathcal{B}$ be a subset of a* (*commutative*) *C*-algebra $\mathcal{A}$ and $\mathcal{P}_n(\mathcal{B})$ be as in Definition 5.*

(i) *Characterize f such that f is a positivity preserver for all $n \in \mathbb{N}$.*
(ii) *Characterize f such that f is a positivity preserver for fixed n.*

Answer to (i) in Question 2.8 in the case $\mathcal{A} = \mathbb{R}$ (which is due to Pólya and Szegő [40]) is known from the works of Schoenberg [45], Vasudeva [49], Rudin [43], Christensen and Ressel [7]. Further the answer to Question 2.8(i) in the case $\mathcal{A} = \mathbb{C}$ (which is due to Rudin [43]) is also known from the work of Herz [13]. There are certain partial answers to (ii) in Question 2.8 from the works of Horn [17], Belton, Guillot, Khare, Putinar, Rajaratnam and Tao [3–5, 12, 29]. Corollary 2.3 and the observation that the set of all positive matrices in $M_n(\mathcal{A})$ is a closed set gives a partial answer to (i) in Question 2.8.

Theorem 2.9. *Let $\mathcal{A}$ be a commutative unital C*-algebra. Let the power series $f(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n$ over $\mathcal{A}$ be convergent on a subset $\mathcal{B}$ of $\mathcal{A}$. If all a_n's are positive elements of $\mathcal{A}$, then the matrix*

$$f[A] = \sum_{n=0}^{\infty} a_n (A^\circ)^n \in M_m(\mathcal{A})$$

is positive for all positive $A \in M_m(\mathcal{A})$, for all $m \in \mathbb{N}$. In other words, a convergent power series over a commutative unital C-algebra with positive elements as coefficients is a positivity preserver in all dimensions.*

3. Lower bounds for C*-algebraic Schur product

Our first result is on the lower bound of positive matrices over C*-algebras.

Theorem 3.1. *Let $\mathcal{A}$ be a unital C*-algebra (need not be commutative) and $A \in M_n(\mathcal{A})$ be a positive matrix. Let $M = AA^*$ and $y \in \mathcal{A}^n$ be the vector of row sums of A. Then*

$$M \succeq \frac{1}{n} yy^*,$$

i.e.,

$$\langle Mx, x\rangle \geq \frac{1}{n}\langle x, x\rangle, \quad \forall x \in \mathcal{A}^n. \tag{3}$$

Proof. Set

$$A \coloneqq \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \in M_n(\mathcal{A}),$$

$$x \coloneqq \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{A}^n, \quad y \coloneqq \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{A}^n.$$

Since y is the vector of row sums of A, we have

$$y_j = \sum_{k=1}^{n} a_{j,k}, \quad \forall 1 \leq j \leq n.$$

Consider

$$\langle Mx, x\rangle = \langle AA^*x, x\rangle = \langle A^*x, A^*x\rangle = \left\langle \begin{pmatrix} \sum_{k=1}^n a_{k,1}^* x_k \\ \sum_{k=1}^n a_{k,2}^* x_k \\ \vdots \\ \sum_{k=1}^n a_{k,n}^* x_k \end{pmatrix}, \begin{pmatrix} \sum_{l=1}^n a_{l,1}^* x_l \\ \sum_{l=1}^n a_{l,2}^* x_l \\ \vdots \\ \sum_{l=1}^n a_{l,n}^* x_l \end{pmatrix} \right\rangle$$

$$= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{k,j}^* x_k \right) \left(\sum_{l=1}^{n} a_{l,j}^* x_l \right)^* = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{k,j}^* x_k x_l^* a_{l,j}$$

which is the left side of Inequality (3). Set

$$e_n := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathcal{A}^n, \quad z := \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} := \begin{pmatrix} \sum_{k=1}^n a_{k,1}^* x_k \\ \sum_{k=1}^n a_{k,2}^* x_k \\ \vdots \\ \sum_{k=1}^n a_{k,n}^* x_k \end{pmatrix} \in \mathcal{A}^n.$$

We now consider the right side of Inequality (3) and use Lemma 1.6 to get

$$\begin{aligned}
\frac{1}{n}\langle yy^* x, x\rangle &= \frac{1}{n}\langle y^* x, y^* x\rangle \\
&= \frac{1}{n}\left\langle \begin{pmatrix} y_1^* & y_2^* & \dots & y_n^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1^* & y_2^* & \dots & y_n^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \\
&= \frac{1}{n}\left(\sum_{k=1}^{n} y_k^* x_k \right) \left(\sum_{l=1}^{n} y_l^* x_l \right)^* \\
&= \frac{1}{n}\left(\sum_{k=1}^{n} y_k^* x_k \right) \left(\sum_{l=1}^{n} x_l^* y_l \right) \\
&= \frac{1}{n}\left(\sum_{k=1}^{n} \sum_{r=1}^{n} a_{k,r}^* x_k \right) \left(\sum_{l=1}^{n} x_l^* \sum_{s=1}^{n} a_{l,s} \right) \\
&= \frac{1}{n}\left(\sum_{k=1}^{n} \sum_{r=1}^{n} a_{k,r}^* x_k \right) \left(\sum_{l=1}^{n} \sum_{s=1}^{n} x_l^* a_{l,s} \right) \\
&= \frac{1}{n}\left(\sum_{r=1}^{n} \sum_{k=1}^{n} a_{k,r}^* x_k \right) \left(\sum_{s=1}^{n} \sum_{l=1}^{n} x_l^* a_{l,s} \right) \\
&= \frac{1}{n}\left(\sum_{r=1}^{n} \left(\sum_{k=1}^{n} a_{k,r}^* x_k \right) .1 \right) \left(\sum_{s=1}^{n} 1. \left(\sum_{l=1}^{n} a_{l,s}^* x_l \right)^* \right) \\
&= \frac{1}{n}\left\langle \begin{pmatrix} \sum_{k=1}^n a_{k,1}^* x_k \\ \sum_{k=1}^n a_{k,2}^* x_k \\ \vdots \\ \sum_{k=1}^n a_{k,n}^* x_k \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} \sum_{k=1}^n a_{k,1}^* x_k \\ \sum_{k=1}^n a_{k,2}^* x_k \\ \vdots \\ \sum_{k=1}^n a_{k,n}^* x_k \end{pmatrix} \right\rangle \\
&= \frac{1}{n}\langle z, e_n\rangle\langle e_n, z\rangle \le \frac{1}{n}\|\langle e_n, e_n\rangle\|\langle z, z\rangle = \langle z, z\rangle
\end{aligned}$$

$$= \sum_{j=1}^{n} z_j z_j^* = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{k,j}^* x_k \right) \left(\sum_{l=1}^{n} a_{l,j}^* x_l \right)^*$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{k,j}^* x_k x_l^* a_{l,j} = \langle Mx, x \rangle$$

which is the required inequality. □

Theorem 3.2. *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Let $M, N \in M_n(\mathcal{A})$ be positive matrices. Let $M = AA^*$, $M = BB^*$ and $y \in \mathcal{A}^n$ be the vector of row sums of $A \circ B$. Then*

$$M \circ N \succeq (A \circ B)(A \circ B)^* \succeq \frac{1}{n} yy^*.$$

Proof. Let $\{A_1, \dots, A_n\}$ be columns of A and $\{B_1, \dots, B_n\}$ be columns of B. Then using commutativity and Theorem 3.1 we get

$$M \circ N = (AA^*) \circ (BB^*)$$
$$= \left(\sum_{j=1}^{n} A_j A_j^* \right) \circ \left(\sum_{k=1}^{n} B_k B_k^* \right)$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} ((A_j A_j^*) \circ (B_k B_k^*))$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} (A_j \circ B_k)(A_j \circ B_k)^*$$
$$\succeq \sum_{j=1}^{n} (A_j \circ B_j)(A_j \circ B_j)^* = (A \circ B)(A \circ B)^* \succeq \frac{1}{n} yy^*.$$

□

Corollary 3.3. *Let $M \in M_n(\mathcal{A})$ be a positive matrix. Then*

$$M \circ M \succeq \frac{1}{n} (diag\ M)(diag\ M)^*.$$

Proof. Let $B = A$ in Theorem 3.2. Result follows by noting that diagonal entries of M are row sums of $A \circ A$. □

Following corollary is immediate from Corollary 3.3.

Corollary 3.4. *Let $M \in M_n(\mathcal{A})$ be a positive matrix such that all diagonal entries of M are one's. Then*

$$M \circ M \succeq \frac{1}{n} E_n.$$

4. C*-algebraic Novak's conjecture

It is well known that the *exponential map*

$$e : \mathcal{A} \ni x \mapsto e^x \coloneqq \sum_{n=0}^{\infty} \frac{x^n}{n!} \in \mathcal{A}$$

is a well defined map on a unital C*-algebra (more is true, it is well-defined on unital Banach algebras). Using this map and from the definition of trigonometric functions (for instance, see Chapter 8 in [44]) we define C*-algebraic sine and cosine functions as follows.

Definition 6. Let $\mathcal{A}$ be a unital C*-algebra. Define the *C*-algebraic sine* function by

$$\sin : \mathcal{A} \ni x \mapsto \sin x \coloneqq \frac{e^{ix} - e^{-ix}}{2i} \in \mathcal{A}.$$

Define the *C*-algebraic cosine* function by

$$\cos : \mathcal{A} \ni x \mapsto \cos x \coloneqq \frac{e^{ix} + e^{-ix}}{2} \in \mathcal{A}.$$

By a direct computation, we have the following result. The result also shows the similarity and differences of C*-algebraic trigonometric functions with usual trigonometric functions.

Theorem 4.1. *Let $\mathcal{A}$ be a unital C*-algebra. Then*

(i) $\sin(-x) = -\sin x, \forall x \in \mathcal{A}$.
(ii) $\cos(-x) = \cos x, \forall x \in \mathcal{A}$.
(iii) $\sin(x + y) = \sin x \cos y + \cos x \sin y, \forall x, y \in \mathcal{A}$ *such that* $xy = yx$.
(iv) $\cos(x + y) = \cos x \cos y - \sin x \sin y, \forall x, y \in \mathcal{A}$ *such that* $xy = yx$.
(v) $(\sin x)^* = \sin x^*, \forall x \in \mathcal{A}$.
(vi) $(\cos x)^* = \cos x^*, \forall x \in \mathcal{A}$.
(vii) $\sin^2 x + \cos^2 x = 1, \forall x \in \mathcal{A}$.

In the sequel, by $\mathcal{A}_{\text{sa}}$ we mean the set of all self-adjoint elements in the unital C*-algebra $\mathcal{A}$. Motivated from Novak's conjecture (Theorem 1.5), we formulate the following conjecture.

Conjecture 4.2 (C*-algebraic Novak's conjecture)**.** *Let $\mathcal{A}$ be a unital C*-algebra. Then the matrix*

$$\left[\prod_{l=1}^{d} \tfrac{1+\cos(x_{j,l}-x_{k,l})}{2} - \tfrac{1}{n}\right]_{1\leq j,k\leq n}$$

is positive for all $n, d \geq 2$ and all choices of $x_j = (x_{j,1}, \dots, x_{j,d}) \in \mathcal{A}_{sa}^d$, $\forall 1 \leq j \leq n$.

We solve a special case of Conjecture 4.2.

Theorem 4.3 (Commutative C*-algebraic Novak's conjecture)**.** *Let $\mathcal{A}$ be a commutative unital C*-algebra. Then the matrix*

$$\left[\prod_{l=1}^{d} \frac{1+\cos(x_{j,l}-x_{k,l})}{2} - \frac{1}{n}\right]_{1\leq j,k\leq n}$$

is positive for all $n, d \geq 2$ and all choices of $x_j = (x_{j,1}, \dots, x_{j,d}) \in \mathcal{A}_{sa}^d$, $\forall 1 \leq j \leq n$.

Proof. We first show that the matrix

$$A \coloneqq \left[\cos(z_j - z_k)\right]_{1\leq j,k\leq n}$$

is positive for all $n, d \geq 2$ and all choices of $z_1, \dots, z_n \in \mathcal{A}_{\text{sa}}$. First note that Theorem 4.1 says that the matrix A is self adjoint. An important theorem used by Vybíral in his proof of Novak's conjecture is the Bochner theorem [41]. Since *Bochner theorem* for C*-algebras is probably not known, we use Theorem 4.1 and make a direct computation which is inspired from computation done in [47]. Let $y = (y_1, \dots, y_n) \in \mathcal{A}_{\text{sa}}^d$. Then

$$\begin{aligned}
\langle Ay, y\rangle &= \sum_{j=1}^{n}\sum_{k=1}^{n}(\cos(z_j - z_k))y_j y_k^* \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}(\cos z_j \cos z_k + \sin z_j \sin z_k)y_j y_k^* \\
&= \left(\sum_{j=1}^{n}(\cos z_j)y_j\right)\left(\sum_{j=1}^{n}(\cos z_j)y_j\right)^* + \left(\sum_{j=1}^{n}(\sin z_j)y_j\right)\left(\sum_{j=1}^{n}(\sin z_j)y_j\right)^* \\
&\geq 0.
\end{aligned}$$

We define n by n matrices $M_1, \dots, M_d$ as follows.

$$M_l \coloneqq \left[\cos\left(\frac{x_{j,l}-x_{k,l}}{2}\right)\right]_{1\leq j,k\leq n}, \quad \forall 1 \leq l \leq d.$$

Theorem 2.2 then says that the matrix

$$M \coloneqq M_1 \circ \cdots \circ M_d = \left[\prod_{l=1}^{d} \cos\left(\frac{x_{j,l}-x_{k,l}}{2}\right)\right]_{1\leq j,k\leq n}$$

is positive. Since all diagonal entries of M are one's, we can apply Corollary 3.4 to get

$$\begin{aligned}
\left[\prod_{l=1}^{d} \frac{1+\cos(x_{j,l}-x_{k,l})}{2}\right]_{1\leq j,k\leq n} &= \left[\prod_{l=1}^{d} \cos^2\left(\frac{x_{j,l}-x_{k,l}}{2}\right)\right]_{1\leq j,k\leq n} \\
&= M \circ M \succeq \frac{1}{n}E_n,
\end{aligned}$$

i.e.,

$$\left[\prod_{l=1}^{d} \frac{1+\cos(x_{j,l}-x_{k,l})}{2} - \frac{1}{n}\right]_{1\leq j,k\leq n} \succeq 0.$$

□

Remark 4.4. Strategy of the Section 4 can be invoked to show that for some other classes of functions like cosine, C*-algebraic Novak's Conjecture can be formulated and solved.

We end the paper by asking an open problem similar to question asked by Vybíral in arXiv version (see https://arxiv.org/abs/1909.11726v1) of the paper [51].

Question 4.5. *Can the bound in Theorem 3.2 be improved for the C*-algebraic Schur product of positive matrices over* (*commutative*) *unital C*-algebras?*

Final sentence: Improved version of Theorem 1.2 is given by Dr. Apoorva Khare (see Theorem A in [28]) but it seems that the arguments used in the proof of Theorem A in [28] do not work for C*-algebras.

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