J. Korean Math. Soc. **59** (2022), No. 4, pp. 685–697
https://doi.org/10.4134/JKMS.j210517
pISSN: 0304-9914 / eISSN: 2234-3008

ON 3^k-REGULAR CUBIC PARTITIONS

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Abstract. Recently, Gireesh, Shivashankar, and Naika [11] found some infinite classes of congruences for the 3- and the 9-regular cubic partitions modulo powers of 3. We extend their study to all the 3^k-regular cubic partitions. We also find new families of congruences.

1. Introduction

For complex numbers a and q such that $|q| < 1$, we define

$$(a;q)_\infty := \prod_{j=0}^{\infty} \left(1 - aq^j\right),$$
$$(a_1, a_2, \ldots, a_n; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty ,$$
$$f_n := (q^n;q^n)_\infty .$$

A partition of a non-negative integer n is a finite non-increasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The partition function $p(n)$ is defined as the number of partitions of n. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where by convention, $p(0) = 1$. The divisibility properties of $p(n)$ have been studied extensively since Ramanujan [16, 17, 19] found the following famous congruences for $p(n)$, for non-negative integers n,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Received August 27, 2021; Revised January 19, 2022; Accepted February 4, 2022.
2020 *Mathematics Subject Classification.* Primary 05A17, 11P83.
Key words and phrases. Congruence, m-regular cubic partition.
The first author was partially supported by Grant no. MTR/2018/000157 of Science & Engineering Research Board (SERB), DST, Government of India under the MATRICS scheme. The second author was partially supported by Council of Scientific & Industrial Research (CSIR), Government of India under CSIR-JRF scheme.

In a cubic partition, the parts that are multiples of 2 appear in two colors whereas the other parts appear in only one color. Here, we denote the number of cubic partitions of n by $a(n)$. The word 'cubic' is inspired by the involvement of Ramanujan's cubic continued fraction [18, p. 366] with $a(n)$. In 2010, H. C. Chan [6, 7] started investigating the congruence properties of $a(n)$. He found an infinite class of congruences for $a(n)$ modulo any power of 3. Further, H. C. Chan [8] proved that given a positive integer $j \geq 1$ and a prime $p \geq 5$, there are infinitely many congruences of the type $a(An + B) \equiv 0 \pmod{p^j}$. H. H. Chan and Toh [9] and Xiong [21] proved an infinite class of congruences for $a(n)$ modulo powers of 5, for which an elementary proof was also given by Hirschhorn [12]. The generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{f_1 f_2}.$$

For a positive integer $m > 1$, a partition of n is called an m-regular partition of n if there is no part that is divisible by m. If $b_m(n)$ counts the m-regular partitions of n, then the generating function of $b_m(n)$ is given by

$$\sum_{n=0}^{\infty} b_m(n)q^n = \frac{f_m}{f_1}.$$

The arithmetic properties of $b_m(n)$ have also been widely studied for many values of m. For example, see [1, 5, 10, 14, 15, 20] and the references cited there in.

Now, for non-negative integers $k \geq 1$, we denote the 3^k-regular cubic partition function by $a_{3^k}(n)$, which is generated by

$$\sum_{n=0}^{\infty} a_{3^k}(n)q^n = \frac{f_{3^k} f_{2\cdot 3^k}}{f_1 f_2}.$$

Recently, motivated by $a(n)$ and $b_m(n)$, Gireesh, Shivashankar, and Naika [11] studied $a_{3^k}(n)$ for $k = 1$ and 2. They found that for non-negative integers α, β, and n,

$$a_3\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2} - 1}{4}\right) \equiv 0 \pmod{3^{\alpha+1}}, \tag{1}$$

$$a_3\left(3^{2\alpha+2}n + \frac{7 \cdot 3^{2\alpha+1} - 1}{4}\right) \equiv 0 \pmod{3^{\alpha+2}}, \tag{2}$$

$$a_3\left(3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{4}\right) \equiv 0 \pmod{3^{\alpha+2}}, \tag{3}$$

$$a_9\left(3^{\beta+1}n + 3^{\beta+1} - 1\right) \equiv 0 \pmod{3^{\beta+1}}. \tag{4}$$

In this paper, we find infinite classes of congruences for $a_{3^k}(n)$ modulo powers of 3, some of which generalize (1)–(4). The following theorem is our result of the paper.

Theorem 1.1. *For non-negative integers $k \geq 1$, α, β, γ, $\delta \geq 1$, and n, we have*

$$a_{3^{2k-1}}\left(3^{2k+2\alpha-1}n + \tfrac{2\cdot 3^{2k+2\alpha}-3^{2k-1}+1}{8}\right) \equiv 0 \pmod{3^{2k+\alpha-1}}, \tag{5}$$

$$a_{3^{2k-1}}\left(3^{2k+2\alpha}n + \tfrac{14\cdot 3^{2k+2\alpha-1}-3^{2k-1}+1}{8}\right) \equiv 0 \pmod{3^{2k+\alpha}}, \tag{6}$$

$$a_{3^{2k-1}}\left(3^{2k+2\alpha}n + \tfrac{22\cdot 3^{2k+2\alpha-1}-3^{2k-1}+1}{8}\right) \equiv 0 \pmod{3^{2k+\alpha}}, \tag{7}$$

$$a_{3^{2k}}\left(3^{2k+\beta-1}n + 3^{2k+\beta-1} - \tfrac{3^{2k}-1}{8}\right) \equiv 0 \pmod{3^{2k+\beta-1}}, \tag{8}$$

$$a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}n + 5\cdot 2^{\gamma}3^{2k+\beta-1} - \tfrac{3^{2k}-1}{8}\right) \equiv 0 \pmod{3^{2k+\beta}}, \tag{9}$$

$$a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}\ell^{2\delta}n + \tfrac{2^{\gamma+3}3^{2k+\beta-1}\ell^{2\delta-1}r-3^{2k}+1}{8}\right) \equiv 0 \pmod{3^{2k+\beta}}, \tag{10}$$

where $\ell \equiv 5 \pmod 6$ is any prime and $r \in \{5, 11, 17, \ldots, 6\ell-1\} - \{6\lfloor \ell/6 \rfloor + 5\}$.

Clearly, (1)–(4) are the initial cases $k = 1$ of (5)–(8), respectively.

In Section 2, we state and prove the lemmas required to prove Theorem 1.1 and in Section 3, we prove Theorem 1.1. This section ends with the extraction operator H_m, which acts on a power series as

$$H_m\left(\sum_{n=0}^{\infty} A(n)q^n\right) = \sum_{n=0}^{\infty} A(mn)q^{mn}.$$

2. Required lemmas

In this section, we present a few lemmas, which are used in Section 3. The next two lemmas state the 2-dissection of f_3^3/f_1 and the n-dissection of f_1 for $n \equiv \pm 1 \pmod 6$. Note that for a power series $P(q)$ in q, an n-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{n-1} q^j P_j\left(q^n\right),$$

where P_j's are power series in q.

Lemma 2.1 ([13, (1.36)]; also see [3, (2.9)]). *We have*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{11}$$

Lemma 2.2 ([4, p. 274, Theorem 12.1]). *Let $n \geq 5$ be an integer such that $n \equiv \pm 1 \pmod 6$. Let $n = 6m + (-1)^i$ for some integer $i \geq 1$. Then, we have*

$$f_1 = f_{n^2}\cdot\left((-1)^m q^{\left(n^2-1\right)/24} + \sum_{j=1}^{(n-1)/2} (-1)^{j+m} q^{(j-m)\left(3j-3m-(-1)^i\right)/2}\right.$$

$$\frac{\left(q^{2jn}, q^{n^2-2jn}; q^{n^2}\right)_\infty}{\left(q^{jn}, q^{n^2-jn}; q^{n^2}\right)_\infty}\Bigg). \tag{12}$$

The following lemma is required to establish Lemma 2.4.

Lemma 2.3 ([7, Proposition 1])**.** *If* $\xi := \frac{f_1 f_2}{q f_9 f_{18}}$ *and* $\rho := \frac{f_3^4 f_6^4}{q^3 f_9^4 f_{18}^4}$, *then, for integers* $i \geq 1$, *we have*

$$H_3\left(\frac{1}{\xi^i}\right) = \sum_j^i \frac{m_{i,j}}{\rho^j}, \tag{13}$$

where $m_{i,j}$ *are given by the following matrix* M:

$$M := (m_{i,j})_{i,j\geq 1} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 3^3 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2^3 & 3^5 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2\cdot 3^2 & 2^2\cdot 3^4 & 3^7 & 0 & 0 & 0 & \dots \\ 0 & 5 & 2\cdot 3^3\cdot 5 & 3^6\cdot 5 & 3^9 & 0 & 0 & \dots \\ 0 & 1 & 2\cdot 3^2\cdot 7 & 3^6\cdot 5 & 2\cdot 3^9 & 3^{11} & 0 & \dots \\ 0 & 0 & 2\cdot 3\cdot 7 & 2^2\cdot 3^4\cdot 7 & 3^8\cdot 7 & 3^{10}\cdot 7 & 3^{13} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The entries of M *have the following properties*:

1. $m_{i+3j,j} = 0$ *and* $m_{i,i+j} = 0$ *for all* $i, j \geq 1$;
2. $m_{i,j} = 9m_{i-1,j-1} + 3m_{i-2,j-1} + m_{i-3,j-1}$ *for* $i \geq 4$ *and* $j \geq 2$.

The next lemma helps in proving Lemma 2.5.

Lemma 2.4 ([11, (32), (34), and (36)])**.** *For all integers* $i \geq 1$, *we have*

$$H_3\left(q^{i-1}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4i-1}\right) = \sum_{j=1}^{3i} m_{4i-1,i+j-1} q^{3j-3}\left(\frac{f_9 f_{18}}{f_3 f_6}\right)^{4j-3}, \tag{14}$$

$$H_3\left(q^{i-3}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4i-3}\right) = \sum_{j=1}^{3i-2} m_{4i-3,i+j-1} q^{3j-3}\left(\frac{f_9 f_{18}}{f_3 f_6}\right)^{4j-1}, \tag{15}$$

$$H_3\left(q^{i}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4i}\right) = \sum_{j=1}^{3i} m_{4i,i+j} q^{3j}\left(\frac{f_9 f_{18}}{f_3 f_6}\right)^{4j}. \tag{16}$$

Lemma 2.5. *For non-negative integers* $k \geq 1$, α, β, *and* n, *we have*

$$\sum_{n=0}^{\infty} a_{3^{2k-1}}\left(3^{2k+2\alpha-1}n + \frac{2\cdot 3^{2k+2\alpha} - 3^{2k-1} + 1}{8}\right) q^n$$

$$= \sum_{j=1}^{\infty} x_{2k-1,j}^{2\alpha+1} q^{j-1}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4j-1}, \tag{17}$$

$$\sum_{n=0}^{\infty} a_{3^{2k-1}}\left(3^{2k+2\alpha}n+\frac{2\cdot 3^{2k+2\alpha}-3^{2k-1}+1}{8}\right)q^n$$

$$=\sum_{j=1}^{\infty} x_{2k-1,j}^{2\alpha+2} q^{j-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4j-3}, \tag{18}$$

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(3^{2k+\beta-1}n+3^{2k+\beta-1}-\frac{3^{2k}-1}{8}\right)q^n$$

$$=\sum_{j=1}^{\infty} y_{2k-1,j}^{\beta+1} q^{j-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4j}, \tag{19}$$

where

$$x_1 := (x_{1,1}, x_{1,2}, x_{1,3}, \ldots) = (3,0,0,0,\ldots),$$

$$x_{r+1} := \begin{cases} x_r \cdot (m_{4i,i+j})_{i,j\geq 1}, & \textit{when } r \textit{ is odd,} \\ x_r \cdot (m_{4i+1,i+j})_{i,j\geq 1}, & \textit{when } r \textit{ is even,} \end{cases} \tag{20}$$

$$x_{2k-1} = x_1 \cdot \left(\left(\sum_{b=1}^{3i} m_{4i,i+b}\cdot m_{4b+1,j+b}\right)_{i,j\geq 1}\right)^{k-1}, \tag{21}$$

$$x_{2k-1}^1 := \left(x_{2k-1,1}^1, x_{2k-1,2}^1, x_{2k-1,3}^1, \ldots\right) := x_{2k-1},$$

$$x_{2k-1}^{\alpha+1} := \begin{cases} x_{2k-1}^{\alpha} \cdot (m_{4i-1,i+j-1})_{i,j\geq 1}, & \textit{when } \alpha\geq 1 \textit{ is odd,} \\ x_{2k-1}^{\alpha} \cdot (m_{4i-3,i+j-1})_{i,j\geq 1}, & \textit{when } \alpha\geq 1 \textit{ is even,} \end{cases} \tag{22}$$

$$y_{2k-1}^1 := x_{2k-1},$$

$$y_{2k-1}^{\beta+1} := y_{2k-1}^{\beta}\cdot (m_{4i,i+j})_{i,j\geq 1}, \quad \beta\geq 1.$$

Proof. Proofs of (17), (18), and (19) are similar. So here, we prove (17) and (18) only by induction. We first establish that for $r\geq 1$,

$$\sum_{n=0}^{\infty} a_{3^{2k-1}}\left(3^{2r-1}n+\frac{5\cdot 3^{2r-1}+1}{8}\right)q^n$$

$$=\sum_{j=1}^{\infty} x_{2r-1,j}q^{j-1}\frac{f_3^{4j-1}f_6^{4j-1}f_{3^{2k-2r}}f_{2\cdot 3^{2k-2r}}}{f_1^{4j}f_2^{4j}}, \tag{23}$$

which when $r=k$, proves (17) for $\alpha=0$.

We have

$$\sum_{n=0}^{\infty} a_{3^{2k-1}}(n)\, q^{n+1} = q\frac{f_{3^{2k-1}}f_{2\cdot 3^{2k-1}}}{f_1f_2} = \frac{f_{3^{2k-1}}f_{2\cdot 3^{2k-1}}}{f_9f_{18}}\frac{1}{\xi},$$

which by (13) results in

$$\sum_{n=0}^{\infty} a_{3^{2k-1}}\,(3n+2)\,q^n = 3\frac{f_3^3 f_6^3 f_{3^{2k-2}} f_{2\cdot 3^{2k-2}}}{f_1^4 f_2^4}.$$

Thus, (23) is true for $r=1$. We now assume (23) to be true for some integer $r \geq 1$. Then, applying H_3 in (23) and (16), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_{3^{2k-1}} \left(3^{2r}n + \frac{7\cdot 3^{2r-1}+1}{8}\right) q^{3n+3} \\
&= \sum_{i=1}^{\infty} x_{2r-1,i} H_3 \left(q^i \frac{f_3^{4i-1} f_6^{4i-1} f_{3^{2k-2r}} f_{2\cdot 3^{2k-2r}}}{f_1^{4i} f_2^{4i}}\right) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{2r-1,i}\cdot m_{4i,i+j} q^{3j} \frac{f_9^{4j} f_{18}^{4j} f_{3^{2k-2r}} f_{2\cdot 3^{2k-2r}}}{f_3^{4j+1} f_6^{4j+1}};
\end{aligned}$$

equivalently by (20),

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_{3^{2k-1}} \left(3^{2r}n + \frac{7\cdot 3^{2r-1}+1}{8}\right) q^{n+2} \\
&= \sum_{i=1}^{\infty} x_{2r,i} q^{i+1} \frac{f_3^{4i} f_6^{4i} f_{3^{2k-2r-1}} f_{2\cdot 3^{2k-2r-1}}}{f_1^{4i+1} f_2^{4i+1}}.
\end{aligned}$$

Again, employing H_3 in the above identity and (15), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_{3^{2k-1}} \left(3^{2r+1}n + \frac{5\cdot 3^{2r+1}+1}{8}\right) q^{3n+3} \\
&= \sum_{i=1}^{\infty} x_{2r,i} H_3 \left(q^{i+1} \frac{f_3^{4i} f_6^{4i} f_{3^{2k-2r-1}} f_{2\cdot 3^{2k-2r-1}}}{f_1^{4i+1} f_2^{4i+1}}\right) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_{2r,i}\cdot m_{4i+1,i+j} q^{3j} \frac{f_9^{4j-1} f_{18}^{4j-1} f_{3^{2k-2r-1}} f_{2\cdot 3^{2k-2r-1}}}{f_3^{4j} f_6^{4j}};
\end{aligned}$$

equivalently by (20),

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_{3^{2k-1}} \left(3^{2r+1}n + \frac{5\cdot 3^{2r+1}+1}{8}\right) q^{n} \\
&= \sum_{j=1}^{\infty} x_{2r+1,j} q^{j-1} \frac{f_3^{4j-1} f_6^{4j-1} f_{3^{2k-2(r+1)}} f_{2\cdot 3^{2k-2(r+1)}}}{f_1^{4j} f_2^{4j}}.
\end{aligned}$$

Therefore, (23) is true for $r+1$ also. Hence, (23) is true.

Note that due to (20), it is evident that

$$x_{2k-1} = x_1 \cdot \left(\left(m_{4i,i+j}\right)_{i,j\geq 1} \cdot \left(m_{4i+1,i+j}\right)_{i,j\geq 1} \right)^{k-1},$$

which implies (21).

The proofs of (17) and (18) are similar to that of (23). So, we sketch the proofs of (17) and (18). Having proved (23), we see that (17) is true for $\alpha = 0$. Let (17) be true for some integer $\alpha \geq 0$. Then, the application of H_3 in (17), with the aid of (14) and (22), gives (18). Again, employing H_3 in (18), and then (15) and (22), we find that (17) true for $\alpha+1$ as well. Hence, by induction, (17) and (18) are true. □

Finally in this section, we find the 3-adic orders: $\nu_3\left(x_{2k-1,j}\right)$, $\nu_3\left(x_{2k-1,j}^{2\alpha+1}\right)$, and $\nu_3\left(y_{2k-1,j}^{\beta+1}\right)$ in the following lemma.

Lemma 2.6. *For non-negative integers $k \geq 1$, $j \geq 1$, and α, we have*

$$\nu_3\left(x_{2k-1,j}\right) \geq 2k-1+3(j-1), \tag{24}$$

$$\nu_3\left(x_{2k-1,j}^{2\alpha+1}\right) \geq 2k-1+\alpha+2(j-1), \tag{25}$$

$$\nu_3\left(y_{2k-1,j}^{\beta+1}\right) \geq 2k-1+\beta+2(j-1). \tag{26}$$

Proof. We recall from [11, (51)] that

$$\nu_3\left(m_{i,j}\right) \geq 3j-i-1. \tag{27}$$

Due to x_1 given in Lemma 2.5, (24) is true for $k=1$. Let us assume that (24) is true for some $k \geq 1$. Since $m_{4,2} = 18$, $m_{8,3} = 8$, $m_{12,4} = 1$, and $m_{4i,i+1} = 0$ for $i \geq 4$, we have

$$\begin{aligned} \nu_3\left(x_{2k,1}\right) &= \nu_3\left(\sum_{i=1}^{\infty} x_{2k-1,i} \cdot m_{4i,i+1}\right) \\ &\geq \min\left\{2k+1, 2k+2, 2k+5\right\} = 2k+1. \end{aligned} \tag{28}$$

Again, for $j \geq 2$, we have

$$\begin{aligned} \nu_3\left(x_{2k,j}\right) = \nu_3\left(\sum_{i=1}^{\infty} x_{2k-1,i} \cdot m_{4i,i+j}\right) &\geq \min_i\left\{\nu_3\left(x_{2k-1,i}\right) + \nu_3\left(m_{4i,i+j}\right)\right\} \\ \geq \min_i\{2k+3j+2i-5\} = 2k+3(j-1). \end{aligned} \tag{29}$$

From (28) and (29), we obtain

$$\nu_3\left(x_{2k,j}\right) \geq 2k+3(j-1)+\Delta_{j,1}, \tag{30}$$

where $\Delta_{j,1}$ is the Kronecker delta function.

Now, for $j \geq 1$, by (30), we have

$$\begin{aligned} \nu_3\left(x_{2k+1,j}\right) &\geq \min_i\left\{\nu_3\left(x_{2k,i}\right) + \nu_3\left(m_{4i+1,i+j}\right)\right\} \\ &\geq \min_i\{2k+3j+2i+\Delta_{i,1}-5\} = 2k+1+3(j-1), \end{aligned}$$

which assures that (24) is true for $k+1$ as well. Thus, by induction, (24) is true.

Proofs of (25) and (26) are similar. We only prove (25) using (24) and induction. From $x^1_{2k-1,j} := x_{2k-1,j}$ given in Lemma 2.5 and (24), (25) is true for $\alpha = 0$. We let (25) be true for some $\alpha \geq 0$. Since $m_{3,1} = 1$ and $m_{4i-1,i} = 0$ for $i \geq 2$, we find that

$$\nu_3\left(x^{2\alpha+2}_{2k-1,1}\right) \geq \nu_3\left(x^{2\alpha+1}_{2k-1,1}\right) \geq 2k-1+\alpha \tag{31}$$

and for $j \geq 2$,

$$\nu_3\left(x^{2\alpha+2}_{2k-1,j}\right) \geq \min_i\left\{\nu_3\left(x^{2\alpha+1}_{2k-1,i}\right) + \nu_3\left(m_{4i-1,i+j-1}\right)\right\},$$

which by (27) gives

$$\begin{aligned}\nu_3\left(x^{2\alpha+2}_{2k-1,j}\right) &\geq \min_i\{2k-1+\alpha+3j+i-5\} = 2k-1+\alpha+3j-4\\ &\geq 2k-1+\alpha+2(j-1).\end{aligned} \tag{32}$$

Again, since $m_{1,1} = 3$, $m_{5,2} = 5$, $m_{9,3} = 1$, and $m_{4i-3,i} = 0$ for $i \geq 4$, on account of (31) and (32), we obtain

$$\nu_3\left(x^{2\alpha+3}_{2k-1,1}\right) \geq \min_{i\leq 3}\left\{\nu_3\left(x^{2\alpha+1}_{2k-1,1}\right)\cdot m_{4i-3,i}\right\} \geq 2k+\alpha \tag{33}$$

and for $j \geq 2$,

$$\nu_3\left(x^{2\alpha+3}_{2k-1,j}\right) \geq \min_i\left\{\nu_3\left(x^{2\alpha+2}_{2k-1,i}\right) + \nu_3\left(m_{4i-3,i+j-1}\right)\right\},$$

which by (27) gives

$$\begin{aligned}\nu_3\left(x^{2\alpha+3}_{2k-1,j}\right) &\geq \min_i\{2k-1+\alpha+3j+i-3\} = 2k-1+\alpha+3j-2\\ &\geq 2k+\alpha+2(j-1).\end{aligned} \tag{34}$$

Thus, (33) and (34) imply that (25) is true for $\alpha+1$. Therefore, (25) is true. □

3. Proof of Theorem 1.1

Congruence (5) follows evidently from (17) and (25). Since by (25), $\nu_3\left(x^{2\alpha+1}_{2k-1,j}\right) \geq 2k+\alpha$ for $j \geq 2$, (17) gives

$$\begin{aligned}&\sum_{n=0}^{\infty} a_{3^{2k}}\left(3^{2k+2\alpha-1}n + \frac{2\cdot 3^{2k+2\alpha} - 3^{2k-1}+1}{8}\right)q^n\\ &\equiv x^{2\alpha+1}_{2k-1,1}\frac{f_3^3 f_6^3}{f_1^3 f_2^3} \equiv x^{2\alpha+1}_{2k-1,1}\frac{f_3^3 f_6^3}{f_3 f_6}\\ &\equiv x^{2\alpha+1}_{2k-1,1} f_3^2 f_6^2 \pmod{3^{2k+\alpha}}.\end{aligned} \tag{35}$$

Extractions of terms involving the forms q^{3n+1} and q^{3n+2} from (35) give (6) and (7), respectively.

From (19) and (26), (8) follows immediately. Now, since by (26), $\nu_3\left(y_{2k-1,j}^{\beta+1}\right) \geq 2k+\beta$ for $j \geq 2$, (19) gives

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(3^{2k+\beta-1}n + 3^{2k+\beta-1} - \frac{3^{2k}-1}{8}\right)q^n$$
$$\equiv y_{2k-1,1}^{\beta+1}\frac{f_3^3 f_6^3}{f_1 f_2} \pmod{3^{2k+\beta}}. \tag{36}$$

Using (11) in (36), and then extracting the terms that involve q^{2n+1}, we find that

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2\cdot 3^{2k+\beta-1} + 2\cdot 3^{2k+\beta-1} - \frac{3^{2k}-1}{8}\right)q^n$$
$$\equiv y_{2k-1,1}^{\beta+1}\frac{f_3^3 f_6^3}{f_1 f_2} \pmod{3^{2k+\beta}}. \tag{37}$$

Therefore, by induction, (36) and (37) imply that for integers $\gamma \geq 0$,

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma}3^{2k+\beta-1}n + 2^{\gamma}3^{2k+\beta-1} - \frac{3^{2k}-1}{8}\right)q^n$$
$$\equiv y_{2k-1,1}^{\beta+1}\frac{f_3^3 f_6^3}{f_1 f_2} \pmod{3^{2k+\beta}}.$$

Again, employing (11) in the above identity, and then extracting the terms that involve q^{2n}, we obtain

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta-1}n + 2^{\gamma}3^{2k+\beta-1} - \frac{3^{2k}-1}{8}\right)q^n$$
$$\equiv y_{2k-1,1}^{\beta+1}\frac{f_2^3 f_3^5}{f_1^3 f_6} \equiv y_{2k-1,1}^{\beta+1} f_3^4 \pmod{3^{2k+\beta}}. \tag{38}$$

Extracting the terms that involve q^{3n+2} from (38), we deduce (9).

Meanwhile, to prove (10), we establish the following identity first when $\ell = 6m-1$ is a prime.

$$H_\ell\left(q^{-\left(\ell^2-6\ell\lfloor \ell/6\rfloor-1\right)/6} f_1 f_3\right) = q^{\ell\lfloor \ell/6\rfloor} f_{\ell^2} f_{3\ell^2}. \tag{39}$$

Using the ℓ-dissections of f_1 and f_3 given by (12), we have

$$f_1 f_3 = f_{\ell^2} f_{3\ell^2}\cdot\left(q^{\left(\ell^2-1\right)/6} + \sum_{j=1}^{(\ell-1)/2}(-1)^j q^{(j-m)(3j-3m+1)/2+(\ell^2-1)/8}\right.$$
$$\times\frac{\left(q^{2j\ell}, q^{\ell^2-2j\ell}; q^{\ell^2}\right)_\infty}{\left(q^{j\ell}, q^{\ell^2-j\ell}; q^{\ell^2}\right)_\infty} + \sum_{k=1}^{(\ell-1)/2}(-1)^k q^{3(k-m)(3k-3m+1)/2+(\ell^2-1)/24}$$

$$\times \frac{\left(q^{6k\ell}, q^{3\ell^2-6k\ell}; q^{3\ell^2}\right)_\infty}{\left(q^{3k\ell}, q^{3\ell^2-3k\ell}; q^{3\ell^2}\right)_\infty} + \sum_{k=1}^{(\ell-1)/2} \left(\sum_{j=1}^{(\ell-1)/2} (-1)^{j+k} \right.$$

$$\times q^{(j-m)(3j-3m+1)/2+3(k-m)(3k-3m+1)/2} \frac{\left(q^{2j\ell}, q^{\ell^2-2j\ell}; q^{\ell^2}\right)_\infty}{\left(q^{j\ell}, q^{\ell^2-j\ell}; q^{\ell^2}\right)_\infty}$$

$$\left. \left. \times \frac{\left(q^{6k\ell}, q^{3\ell^2-6k\ell}; q^{3\ell^2}\right)_\infty}{\left(q^{3k\ell}, q^{3\ell^2-3k\ell}; q^{3\ell^2}\right)_\infty} \right) \right). \tag{40}$$

To prove (39), we require the following inequalities. For all integers $1 \leq j, k \leq (\ell-1)/2$,

$$\frac{(j-m)(3j-3m+1)}{2} \neq \frac{\ell^2-1}{24}, \tag{41}$$

$$\frac{(j-m)(3j-3m+1)}{2} + \frac{3(k-m)(3k-3m+1)}{2} \neq \frac{\ell^2-1}{6}. \tag{42}$$

Simplifying the negation of (41) with $\ell = 6m-1$, we find that $\ell = 3j$, which is a contradiction. Therefore, (41) is true. Again, the negation of (42) gives

$$\ell = 3\frac{j^2+3k^2}{j+3k}.$$

Now, if $3 \nmid j+3k$, then $\ell = 3(j^2+3k^2)/(j+3k)$ is not a prime, which is a contradiction. If $3 \mid j+3k$, then $j = 3t$ for some integer t. Then, we have $\ell = 3(3t^2+k^2)/(t+k)$. Again, if $3 \nmid t+k$, $\ell = 3(3t^2+k^2)/(t+k)$ is not a prime, which is a contradiction. So, $3 \mid t+k$. Let $t+k = 3s$ for some integer s so that we have

$$\ell = \frac{3t^2+(3s-t)^2}{s} = \frac{4t^2}{s} - 6t + 9s.$$

Therefore, $s \mid 4t^2$, otherwise, we have a contradiction. We consider any prime p such that $p \mid s$. Since, $s = (t+k)/3 = j/9 + k/3 < \ell - 1$, $p \neq \ell$. Now, let $t = p^\alpha \cdot r$ and $s = p^\beta \cdot u$ for some integers $\alpha \geq 0, \beta \geq 1$ with $p \nmid r, s$. Then, we have

$$\begin{aligned} \nu_p(\ell) &= 0, \\ \nu_p(\ell) &= \nu_p\left(\frac{4p^{2\alpha}\cdot r^2 - 6p^{\alpha+\beta}\cdot r\cdot u + 9p^{2\beta}\cdot u^2}{p^\beta \cdot u}\right) \\ &\geq 2\cdot \min\{\alpha,\beta\} - \beta. \end{aligned}$$

The above two imply that $2\cdot\min\{\alpha,\beta\} \leq \beta$ which further gives that $\min\{\alpha,\beta\} = \alpha$. Therefore, we have $2\alpha \leq \beta$. But when p is odd, $s \mid 4t^2$ gives $2\alpha \geq \beta$. Again, when $p = 2$, $s \mid 4t^2$ gives $\beta \leq 2\alpha + 2$. So, we must have that $s \in \{t^2, 2t^2, 4t^2\}$. If $s = t^2, 4t^2$, we have $\ell \equiv 1 \pmod 3$, which is a contradiction to $\ell = 6m-1$. Finally, if $s = 2t^2$, we have $\ell = 2(1-3t+9t^2)$, which is again a contradiction. Therefore, (42) is true.

Thus, multiplying $q^{-\left(\ell^2-6\ell\lfloor\ell/6\rfloor-1\right)/6}$ on both sides of (40), then applying H_ℓ, and with the aid of (41) and (42), we arrive at (39).

Now, we come back to the proof of (10). For that, it is enough to show that

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}\ell^{2\delta-1}n+\frac{5\cdot 2^{\gamma+3}3^{2k+\beta-1}\ell^{2\delta-1}-3^{2k}+1}{8}\right)q^n$$

$$\equiv y_{2k-1,1}^{\beta+1}q^{\lfloor\ell/6\rfloor}f_\ell f_{3\ell} \pmod{3^{2k+\beta}}. \tag{43}$$

Because, extracting the terms involving $q^{\ell n+\epsilon}$ for $\epsilon\in\{0,1,2,\ldots,\ell-1\}-\{\lfloor\ell/6\rfloor\}$ with $r := 6\epsilon+5$, we arrive at (10). From (38), we have

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}n+2^{\gamma}3^{2k+\beta-1}-\frac{3^{2k}-1}{8}\right)q^n$$

$$\equiv y_{2k-1,1}^{\beta+1}f_1^4 \equiv y_{2k-1,1}^{\beta+1}f_1f_3 \pmod{3^{2k+\beta}}. \tag{44}$$

Employing (39) in (44) and the fact that $\ell-6\lfloor\ell/6\rfloor=5$, we find that

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}\ell n+\frac{5\cdot 2^{\gamma+3}3^{2k+\beta-1}\ell-3^{2k}+1}{8}\right)q^n$$

$$\equiv y_{2k-1,1}^{\beta+1}q^{\lfloor\ell/6\rfloor}f_\ell f_{3\ell} \pmod{3^{2k+\beta}}.$$

Thus, (43) is true for $\delta=1$. If we assume (43) to be true for some $\delta\geq 1$, then extracting the terms involving $q^{\ell n+\lfloor\ell/6\rfloor}$ along with the fact that $6\lfloor\ell/6\rfloor+5=\ell$, we have

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}\ell^{2\delta}n+\frac{2^{\gamma+3}3^{2k+\beta-1}\ell^{2\delta}-3^{2k}+1}{8}\right)q^n$$

$$\equiv y_{2k-1,1}^{\beta+1}f_1f_3 \pmod{3^{2k+\beta}}.$$

Finally, using (39) in the above identity, we obtain

$$\sum_{n=0}^{\infty} a_{3^{2k}}\left(2^{\gamma+1}3^{2k+\beta}\ell^{2\delta+1}n+\frac{5\cdot 2^{\gamma+3}3^{2k+\beta-1}\ell^{2\delta+1}-3^{2k}+1}{8}\right)q^n$$

$$\equiv y_{2k-1,1}^{\beta+1}q^{\lfloor\ell/6\rfloor}f_\ell f_{3\ell} \pmod{3^{2k+\beta}},$$

which shows that (43) is true for $\delta+1$ as well. Thus, by induction, (43) is true. □

4. Concluding remark

Let $a_{3^k;3}(n)$ denote the 3^k-regular partitions in three colors. In [2], the authors offered the following conjecture.

Conjecture 4.1. *For non-negative integers $k\geq 1$, α, and n, we have*

$$a_{3^{2k};3}\left(3^{2k+\alpha-1}\cdot n+3^{2k+\alpha-1}-\frac{3^{2k}-1}{8}\right)\equiv 0 \pmod{3^{3k+2\alpha-1}},$$

$$a_{3^{2k-1};3}\left(3^{2k+2\alpha-1}\cdot n+\frac{2\cdot 3^{2k+2\alpha}-3^{2k-1}+1}{8}\right)\equiv 0 \pmod{3^{3k+2\alpha-1}}.$$

Note that the method of this paper can be used to prove Conjecture 4.1. We omit the details.

Acknowledgement. The authors are thankful to the anonymous reviewer for his/her helpful comments on the paper.

References

[1] Z. Ahmed and N. D. Baruah, *New congruences for ℓ-regular partitions for $\ell \in \{5,6,7,49\}$*, Ramanujan J. **40** (2016), no. 3, 649–668. https://doi.org/10.1007/s11139-015-9752-2

[2] N. D. Baruah and H. Das, *Generating functions and congruences for 9-regular and 27-regular partitions in 3 colours*, Hardy-Ramanujan J. **44** (2021), 101–115.

[3] N. D. Baruah and K. Nath, *Infinite families of arithmetic identities and congruences for bipartitions with 3-cores*, J. Number Theory **149** (2015), 92–104. https://doi.org/10.1016/j.jnt.2014.10.010

[4] B. C. Berndt, *Ramanujan's notebooks. Part III*, Springer-Verlag, New York, 1991. https://doi.org/10.1007/978-1-4612-0965-2

[5] R. Carlson and J. J. Webb, *Infinite families of infinite families of congruences for k-regular partitions*, Ramanujan J. **33** (2014), no. 3, 329–337. https://doi.org/10.1007/s11139-013-9523-x

[6] H.-C. Chan, *Ramanujan's cubic continued fraction and an analog of his "most beautiful identity"*, Int. J. Number Theory **6** (2010), no. 3, 673–680. https://doi.org/10.1142/S1793042110003150

[7] H.-C. Chan, *Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function*, Int. J. Number Theory **6** (2010), no. 4, 819–834. https://doi.org/10.1142/S1793042110003241

[8] H.-C. Chan, *Distribution of a certain partition function modulo powers of primes*, Acta Math. Sin. (Engl. Ser.) **27** (2011), no. 4, 625–634. https://doi.org/10.1007/s10114-011-8620-2

[9] H. H. Chan and P. C. Toh, *New analogues of Ramanujan's partition identities*, J. Number Theory **130** (2010), no. 9, 1898–1913. https://doi.org/10.1016/j.jnt.2010.02.017

[10] B. Dandurand and D. Penniston, *ℓ-divisibility of ℓ-regular partition functions*, Ramanujan J. **19** (2009), no. 1, 63–70. https://doi.org/10.1007/s11139-007-9042-8

[11] D. S. Gireesh, C. Shivashankar, and M. S. M. Naika, *On 3- and 9-regular cubic partitions*, J. Integer Seq. **23** (2020), no. 7, Art. 20.7.2, 12 pp.

[12] M. D. Hirschhorn, *Cubic partitions modulo powers of 5*, Ramanujan J. **51** (2020), no. 1, 67–84. https://doi.org/10.1007/s11139-018-0074-z

[13] M. Hirschhorn, F. Garvan, and J. Borwein, *Cubic analogues of the Jacobian theta function $\theta(z,q)$*, Canad. J. Math. **45** (1993), no. 4, 673–694. https://doi.org/10.4153/CJM-1993-038-2

[14] Q.-H. Hou, L. H. Sun, and L. Zhang, *Quadratic forms and congruences for ℓ-regular partitions modulo 3, 5 and 7*, Adv. in Appl. Math. **70** (2015), 32–44. https://doi.org/10.1016/j.aam.2015.06.005

[15] B. L. S. Lin, *Arithmetic of the 7-regular bipartition function modulo 3*, Ramanujan J. **37** (2015), no. 3, 469–478. https://doi.org/10.1007/s11139-013-9542-7

[16] S. Ramanujan, *Congruence properties of partitions*, Proc. London Math. Soc. **18** (1920), 19.

[17] S. Ramanujan, *Congruence properties of partitions*, Math. Z. **19** (1921), no. 1–2, 147–153.
[18] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.
[19] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Proc. Cambridge Philos. Soc. **19** (1919), 207–210.
[20] J. J. Webb, *Arithmetic of the 13-regular partition function modulo* 3, Ramanujan J. **25** (2011), no. 1, 49–56. https://doi.org/10.1007/s11139-010-9227-4
[21] X. Xiong, *The number of cubic partitions modulo powers of* 5, Sci. Sin. Math. **41** (2011), no. 1, 1–15.

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