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UNIQUENESS OF QUASI-ROOTS IN RIGHT-ANGLED ARTIN GROUPS

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Abstract. We introduce the notion of quasi-roots and study their uniqueness in right-angled Artin groups.

1. Introduction

In a group, if the equation $g^n = h$ holds, then g is called an *nth root* of h.

The uniqueness of roots in groups has been studied by many authors, e.g. [1]. Mal'cev proved that roots are unique (i.e., $g_1^n = g_2^n$ implies $g_1 = g_2$) in a torsion-free locally nilpotent group [17]. It is well-known that centralizers of nontrivial elements of a torsion-free hyperbolic group are necessarily infinite cyclic [3, 7], hence roots are unique. In the Artin groups of finite type $\mathbf{A}_n$, $\mathbf{B}_n = \mathbf{C}_n$ and affine type $\tilde{\mathbf{A}}_{n-1}$, $\tilde{\mathbf{C}}_{n-1}$, roots are unique up to conjugacy [8, 13, 16]. For the uniqueness of roots in general mapping class groups, see [2]. Pure braid groups and right-angled Artin groups are biorderable, hence roots are unique [6,12,18].

In this article, we introduce the notion of quasi-roots and study the uniqueness of quasi-roots in right-angled Artin groups.

1.1. Right-angled Artin groups

Before stating our results, we recall basic notions in right-angled Artin groups. See [4, 9, 19] for further details.

Throughout this article all graphs are finite and simple. For a graph Γ, let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively. For a graph Γ, the *right-angled Artin group* $A(\Gamma)$ on Γ is defined by the presentation

$$A(\Gamma) = \langle\, v \in V(\Gamma) \mid [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \in E(\Gamma)\,\rangle.$$

In this article, we use the opposite convention

$$G(\Gamma) = \langle\, v \in V(\Gamma) \mid [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \notin E(\Gamma)\,\rangle.$$

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In other words, $G(\Gamma) = A(\Gamma^c)$, where Γ^c denotes the complement graph of Γ.

Each element in $V(\Gamma) \cup V(\Gamma)^{-1}$ is called a *letter*. A *word* means a finite sequence of letters. For two words w_1 and w_2, the notation $w_1 \equiv w_2$ means that w_1 and w_2 coincide as sequences of letters. For example, if $w_1 \equiv v_1v_1v_2v_2^{-1}$ and $w_2 \equiv v_1v_1$, where $v_1, v_2 \in V(\Gamma)$, then $w_1 \not\equiv w_2$. A word w' is called a *subword* of a word w if $w \equiv w_1w'w_2$ for possibly empty words w_1 and w_2.

An element g in $G(\Gamma)$ can be expressed as a word w. The word w is called *reduced* if w is a shortest word among all words representing g. In this case, the length of w is called the *word length* of g, denoted by $|g|$.

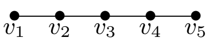
The *support* of $g \in G(\Gamma)$, denoted by $\operatorname{supp}(g)$, is defined as the set of all generators $v \in V(\Gamma)$ such that v or v^{-1} appears in a reduced word representing g. It is known that $\operatorname{supp}(g)$ is well-defined. For example, see [9, 10, 14, 15].

Let w be a (possibly non-reduced) word in $G(\Gamma)$. A subword $v^{\pm 1}w_1v^{\mp 1}$, $v \in V(\Gamma)$, of w is called a *cancellation* of v in w if each generator in $\operatorname{supp}(w_1)$ commutes with v. If, furthermore, no letter in w_1 is equal to v or v^{-1}, it is called an *innermost cancellation* of v in w. It is known that w is reduced if and only if w has no innermost cancellation [11, 14, 15].

Abusing notation, we shall sometimes regard a word as the group element represented by that word.

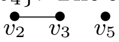
Definition 1.1 (disjointly commute)**.** We say that $g_1, g_2 \in G(\Gamma)$ *disjointly commute*, denoted by $g_1 \rightleftharpoons g_2$, if $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) = \emptyset$ and each $v_1 \in \operatorname{supp}(g_1)$ commutes with each $v_2 \in \operatorname{supp}(g_2)$ in $G(\Gamma)$, i.e., $\{v_1, v_2\} \notin E(\Gamma)$.

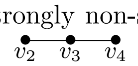
Definition 1.2 (strongly non-split)**.** An element $g \in G(\Gamma)$ is *non-split* if $\operatorname{supp}(g)$ spans a connected subgraph of Γ. Otherwise, it is *split*. We say that g is *strongly non-split* if g is non-split and there is no generator $v \in V(\Gamma)$ that disjointly commutes with g.

Example 1.3. Let $P_5 =$ 

be the path graph on five vertices $v_1, \ldots, v_5$. The right-angled Artin group $G(P_5)$ has a presentation

$$G(P_5) = \langle v_1, \ldots, v_5 \mid [v_i, v_j] = 1 \text{ if } |i-j| \geqslant 2 \rangle.$$

Let $g_1 = v_2v_3v_5^{-2}$, $g_2 = v_2v_3$ and $g_3 = v_2v_3v_4$. Then $\operatorname{supp}(g_1) = \{v_2, v_3, v_5\}$, $\operatorname{supp}(g_2) = \{v_2, v_3\}$ and $\operatorname{supp}(g_3) = \{v_2, v_3, v_4\}$. The element g_1 is split because $\operatorname{supp}(g_1)$ spans the disconnected subgraph 

of Γ. The element g_2 is non-split but it is not strongly non-split because v_5 disjontly commutes with g_2. The element g_3 is strongly non-split because $\operatorname{supp}(g) = \{v_2, v_3, v_4\}$ spans the connected subgraph 

of Γ and because no generator disjointly commutes with g.

Definition 1.4 (geodesic decomposition)**.** For elements $g, g_1, \ldots, g_k \in G(\Gamma)$, the decomposition $g = g_1 \cdots g_k$ is a *geodesic decomposition* of g if $|g| = |g_1| + \cdots + |g_k|$.

Definition 1.5 (cyclic conjugate)**.** For $g \in G(\Gamma)$ with a geodesic decomposition $g = g_1 g_2$, the element $g_2 g_1$ is a *cyclic conjugate* of g. For a word $w \equiv w_1 w_2$, the word $w_2 w_1$ is a *cyclic conjugate* of w.

Definition 1.6 (cyclically reduced)**.** An element $g \in G(\Gamma)$ is *cyclically reduced* if g has the minimum word length in its conjugacy class.

Servatius [19, Proposition on p. 38] proved that for any $g \in G(\Gamma)$ there exists a geodesic decomposition $g = u^{-1}hu$ for $u, h \in G(\Gamma)$ such that h is cyclically reduced.

Definition 1.7 (primitive)**.** A nontrivial element $g \in G(\Gamma)$ is *primitive* if g is not a nontrivial power of another element, i.e., $g = u^n$ never holds for any $n \geqslant 2$ and $u \in G(\Gamma)$. Similarly, a nonempty word w is *primitive* if w is not a nontrivial power of another word.

1.2. Quasi-roots

We define quasi-roots as follows.

Definition 1.8. Let $0 \leqslant \lambda < \frac{1}{2}$ and $N \geqslant 2$. For $h \in G(\Gamma)$, an element $g \in G(\Gamma)$ is a (λ, N)-*quasi-root* of h if there exists a geodesic decomposition

$$h = ag^n b = a \cdot \underbrace{g \cdot g \cdots g}_{n} \cdot b \tag{1}$$

for some $n \geqslant N$ and $a, b \in G(\Gamma)$ with $|a|, |b| \leqslant \lambda |h|$. The above decomposition is called a (λ, N)-*quasi-root decomposition* of h.

We are interested in the uniqueness of quasi-roots: given an element $h \in G(\Gamma)$, we want to find conditions under which we can guarantee that a quasi-root of h is uniquely determined up to conjugacy.

Here are immediate observations concerning the quasi-root decomposition (1).

(i) If $|a| = |b| = 0$, then (1) becomes $h = g^n$, hence g is an nth root of h.

(ii) The condition $N \geqslant 2$ is natural because we cannot expect any kind of uniqueness of quasi-roots for $n = 1$ (and $|a| + |b| > 0$). Meanwhile, since $n \geqslant N \geqslant 2$ and $g^n = g \cdots g$ is geodesic, g is cyclically reduced (see Lemma 2.1).

(iii) The condition $\lambda < \frac{1}{2}$ is natural. If $|a| \geqslant \frac{1}{2}|h|$ or $|b| \geqslant \frac{1}{2}|h|$ is allowed, then g is not uniquely determined up to conjugacy. For example, let $h = v_1^n v_2^n$, where v_1 and v_2 are different generators. Then $h = v_1^n \cdot 1^n \cdot v_2^n = 1 \cdot v_1^n \cdot v_2^n = v_1^n \cdot v_2^n \cdot 1$, hence g is possibly one of 1, v_1 and v_2. However, the elements 1, v_1 and v_2 are not conjugate to each other.

The following is the main result of this article.

Theorem 1.9. *Let Γ be a finite connected graph, and let $0 \leqslant \lambda < \frac{1}{2}$ and $N \geqslant \frac{2|V(\Gamma)|+1}{1-2\lambda}$. If there are two (λ, N)-quasi-root decompositions*

$$h = a_1 g_1^{n_1} b_1 = a_2 g_2^{n_2} b_2$$

such that g_1 and g_2 are strongly non-split and primitive, then the quasi-roots g_1 and g_2 are conjugate such that $a_1g_1a_1^{-1} = a_2g_2a_2^{-1}$ and $b_1^{-1}g_1b_1 = b_2^{-1}g_2b_2$.

The constant N depends only on λ and $|V(\Gamma)|$. Hence we obtain the following.

Corollary 1.10. *Let Γ be a finite connected graph. For any $\lambda \in [0, \frac{1}{2})$, there exists $N \geqslant 3$ such that strongly non-split and primitive (λ, N)-quasi-roots are unique up to conjugacy.*

Concerning the main theorem, we remark the following.

(i) The strongly non-splitness of g_i is a necessary condition for the uniqueness of quasi-roots. For example, consider $G(P_5)$ in Example 1.3. Let $h = (v_2^m v_3^m v_5)^n$. Observe that $h = 1 \cdot (v_2^m v_3^m v_5)^n \cdot 1 = v_5^n \cdot (v_2^m v_3^m)^n \cdot 1 = 1 \cdot (v_2^m v_3^m)^n \cdot v_5^n$. Therefore, if m is large enough so that $(2m+1) \geqslant 1/\lambda$, then $n \leqslant \lambda(2m+1)n = \lambda|h|$. Hence both $v_2^m v_3^m v_5$ and $v_2^m v_3^m$ are (λ, n)-quasi-roots of h. However, $v_2^m v_3^m v_5$ and $v_2^m v_3^m$ are not conjugate. Notice that $v_2^m v_3^m v_5$ is split and that $v_2^m v_3^m$ is non-split but not strongly non-split.

(ii) The primitiveness of g_i is not a restriction. Without that condition, one may conclude that there exists an element u such that any (λ, N)-quasi-root is conjugate to a power of u.

Note that we cannot expect a naive generalization of the main theorem to mapping class groups. For example, consider the n-strand Artin braid group B_n. Let $\delta = \sigma_{n-1} \cdots \sigma_1$ and $\epsilon = \delta\sigma_1$, where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of B_n. Then $\delta^n = \epsilon^{n-1}$. For any $m \geqslant 1$, the element $h = \delta^{mn} = \epsilon^{m(n-1)}$ has two primitive (quasi-)roots δ and ϵ which are not conjugate to each other. In mapping class groups, pseudo-Anosov elements may behave like strongly non-split elements in right-angled Artin groups. Hence the following questions would be interesting.

Question 1.11. In mapping class groups, is it true that primitive pseudo-Anosov quasi-roots are unique up to conjugacy?

Question 1.12. In torsion-free hyperbolic groups, is it true that primitive quasi-roots are unique up to conjugacy?

The results in this article seem insufficient to build a finite-time algorithm for finding a quasi-root or deciding the existence of a quasi-root. Hence the following questions would be also interesting.

Question 1.13 (Quasi-root decision problem)**.** Given a graph Γ and $(\lambda, N, h) \in [0, 1/2) \times [2, \infty) \times G(\Gamma)$, decide whether or not h has a (λ, N)-quasi-root.

Question 1.14 (Quasi-root search problem)**.** Given a graph Γ and $(\lambda, N, h) \in [0, 1/2) \times [2, \infty) \times G(\Gamma)$, where h is assumed to have a (λ, N)-quasi-root, find a (λ, N)-quasi-root g of h together with a (λ, N)-quasi-root decomposition $h = ag^nb$.

2. SD-conical elements

Crisp, Godelle and Wiest introduced a normal form of elements and the notion of pyramidal elements in order to solve the conjugacy problem in right-angled Artin groups in linear-time [5]. We use their normal form and a variation of pyramidal elements, called SD-conical elements. In this section we define SD-conical elements and explore their properties.

The following equivalences are used implicitly in many works. We include a proof for readers' convenience.

Lemma 2.1. *The following are equivalent for $g \in G(\Gamma)$.*

(i) *g is cyclically reduced.*
(ii) *There is no geodesic decomposition as $g = u^{-1}hu$ for $u, h \in G(\Gamma)$ with $u \neq 1$.*
(iii) *$g^n = gg\cdots g$ is geodesic (i.e., $|g^n| = n|g|$) for all $n \geqslant 2$.*
(iv) *$g^n = gg\cdots g$ is geodesic (i.e., $|g^n| = n|g|$) for some $n \geqslant 2$.*
(v) *Every cyclic conjugate of g has the same word length as g, i.e., for any geodesic decomposition g_1g_2 of g, the decomposition g_2g_1 is also geodesic.*

Proof. "(i) $\Leftrightarrow$ (ii)" is known by Servatius [19, Proposition on p. 38]. The implications "(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii)" are obvious. Therefore it is enough to show "(ii) $\Rightarrow$ (iii)".

Assume that $g^n = gg\cdots g$ is not geodesic for some $n \geqslant 2$. Let w be a reduced word representing g. Then $w^n = ww\cdots w$ has an innermost cancellation. Since w is reduced, this cancellation occurs between two consecutive w's, i.e., there are words $w_1, \ldots, w_4$ and a letter $x \in V(\Gamma)^{\pm 1}$ such that

$$w \equiv w_1xw_2 \equiv w_3x^{-1}w_4, \qquad x \rightleftharpoons w_2, \qquad x^{-1} \rightleftharpoons w_3.$$

Since $x \rightleftharpoons w_2$, we have $x \notin \mathrm{supp}(w_2)$, hence the subword w_3x^{-1} must be a prefix of w_1. Therefore $w_1 \equiv w_3x^{-1}w_5$ for some word w_5. Now we have

$$w \equiv w_3x^{-1}w_5xw_2, \qquad x \rightleftharpoons w_2, \qquad x^{-1} \rightleftharpoons w_3.$$

Then g has a geodesic decomposition as $g = x^{-1}hx$, where $h = w_3w_5w_2$. This contradicts (ii). $\square$

Definition 2.2 (set of starting generators)**.** For $g \in G(\Gamma)$, the *set of starting generators*, denoted by $S(g)$, is the set of all generators $v \in V(\Gamma)$ such that either v or v^{-1} is the first letter of a reduced word representing g, i.e.,

$$S(g) = \{v \in V(\Gamma) \mid g = v^\epsilon h \text{ is geodesic for some } \epsilon = \pm 1 \text{ and } h \in G(\Gamma)\}.$$

Definition 2.3 (conical element and apex)**.** For $g \in G(\Gamma)$, if $S(g)$ consists of a single generator, say v_0, then we say that g is *conical* or *v_0-conical.* The generator v_0 is called the *apex* of g, denoted by $\mathrm{apex}(g)$.

Note that a conical element is necessarily non-split.

The following proposition is a quantitative version of Proposition 2.18 in [5]. Though it can be proved in almost the same way, we include a sketchy proof for better readability.

Proposition 2.4. *Let $g \in G(\Gamma)$ be non-split and cyclically reduced, and let $v_0 \in \operatorname{supp}(g)$. Then g is conjugate to a v_0-conical element $p \in G(\Gamma)$ by $a, b \in G(\Gamma)$ as*

$$g = apa^{-1} = b^{-1}pb,$$

where $g^k = ab$ holds and is geodesic for some $0 \leqslant k \leqslant |V(\Gamma)| - 1$. In particular, if $n \geqslant |V(\Gamma)|$, then g^n has a geodesic decomposition as $g^n = ap^{n-k}b$.

Sketchy proof. If g is v_0-conical, there is nothing to prove because we can take $k = 0$ and $a = b = 1$. So we assume that g is not v_0-conical.

We follow the argument in [5]. For any non-split element $h \in G(\Gamma)$ and for any $v \in \operatorname{supp}(h)$, there exists a unique geodesic decomposition as $h = tp$ such that p is v-conical and $v \notin \operatorname{supp}(t)$ [5, Lemma 2.16].

Starting from $g_0 = g$, we inductively do the following for $i = 0, 1, 2, \ldots$: (i) determine the geodesic decomposition $g_i = t_i \cdot p_i$, where p_i is v_0-conical and $v_0 \notin \operatorname{supp}(t_i)$; (ii) take the conjugation $g_{i+1} = t_i^{-1} g_i t_i = p_i t_i$.

Then g_j is v_0-conical for some $1 \leqslant j \leqslant |V(\Gamma)| - 1$. (In fact, the proof of [5, Proposition 2.18] says that one can take $j \leqslant \max\{\operatorname{dist}_{\Gamma_1}(v_0, v) \mid v \in \operatorname{supp}(g)\}$, where Γ_1 is the subgraph of Γ spanned by $\operatorname{supp}(g)$. Notice that $\max\{\operatorname{dist}_{\Gamma_1}(v_0, v) \mid v \in \operatorname{supp}(g)\} \leqslant |V(\Gamma)| - 1$.) Let k be the smallest integer such that g_k is v_0-conical. Then $1 \leqslant k \leqslant |V(\Gamma)| - 1$.

In the decomposition $g_k = t_k \cdot p_k$, we have $t_k = 1$ and $g_k = p_k$. Keeping track of the above procedure, we get

$$\begin{aligned}
g_0 &= g = t_0 \cdot p_0,\\
g_1 &= t_0^{-1} g_0 t_0 = p_0 t_0 = t_1 \cdot p_1,\\
g_2 &= t_1^{-1} g_1 t_1 = p_1 t_1 = t_2 \cdot p_2,\\
&\cdots\\
g_k &= t_{k-1}^{-1} g_{k-1} t_{k-1} = p_{k-1} t_{k-1} = t_k p_k = p_k.
\end{aligned}$$

Let $p = g_k = p_k$, $a = t_0 \cdots t_{k-1}$ and $b = p_{k-1} \cdots p_0$. Then the following hold

$$\begin{gathered}
|g| = |g_0| = \cdots = |g_{k-1}| = |g_k| = |p|,\\
|g_i^m| = m|g_i| \text{ for } 0 \leqslant i \leqslant k,\ m \geqslant 1
\end{gathered}$$

because g is cyclically reduced and each g_{i+1} is a cyclic conjugate of g_i, and hence also cyclically reduced for $0 \leqslant i \leqslant k-1$.

Using the identities $p_i t_i = t_{i+1} p_{i+1}$ for $0 \leqslant i \leqslant k-1$, we obtain

$$\begin{aligned}
g^k &= (t_0 p_0)^k = \underbrace{(t_0 p_0)(t_0 p_0) \cdots (t_0 p_0)}_{k} = (t_0 t_1 \cdots t_{k-1})(p_{k-1} \cdots p_1 p_0) = ab,\\
ga &= t_0 p_0 t_0 t_1 \cdots t_{k-1} = t_0 \cdots t_{k-1} t_k p_k = ap,
\end{aligned}$$

$$bg = p_{k-1} \cdots p_1 p_0 t_0 p_0 = t_k p_k p_{k-1} \cdots p_1 p_0 = pb,$$

namely $g^k = ab$ and $g = apa^{-1} = b^{-1}pb$.

On the other hand,

$$|a|+|b| \leqslant |t_0|+\cdots+|t_{k-1}|+|p_0|+\cdots+|p_{k-1}| = |g_0|+\cdots+|g_{k-1}| = k|g| = |g^k|,$$

where the first equality holds because each $g_i = t_i \cdot p_i$ is a geodesic decomposition. Since $|g^k| \leqslant |a| + |b|$ is obvious, we get $|a| + |b| = |g^k|$, namely $g^k = ab$ is a geodesic decomposition.

The identity $g^n = ap^{n-k}b$ follows from $g^k = ab$ and $g = b^{-1}pb$. Observe

$$|a| + |b| + |p^{n-k}| = k|g| + (n-k)|p| = n|g| = |g^n|,$$

hence $g^n = ap^{n-k}b$ is a geodesic decomposition. □

Definition 2.5 (v_0-conical conjugate)**.** Given a non-split and cyclically reduced element $g \in G(\Gamma)$ and a generator $v_0 \in \mathrm{supp}(g)$, the procedure in the proof of Proposition 2.4 uniquely determines a v_0-conical element p, called the v_0-*conical conjugate* of g.

The conical conjugate p in the above definition is also non-split and cyclically reduced because p is obtained from g by iterated cyclic conjugations and cyclic conjugations preserve non-splitness and cyclically reducedness. In particular, $|g| = |p|$.

Definition 2.6 (CGW-normal form)**.** Suppose $V(\Gamma)$ is endowed with a linear order.

(i) A reduced word $w \equiv v_{i_1}^{\epsilon_1} \cdots v_{i_k}^{\epsilon_k}$ on $V(\Gamma)^{\pm 1}$ representing an element $g \in G(\Gamma)$ is *initially normal* if either w is trivial or v_{i_1} is the largest element of $S(g)$.

(ii) A reduced word w is *normal* if all its suffixes are initially normal.

(iii) Any element $g \in G(\Gamma)$ has a unique normal representative word [5, Proposition 2.6]. We call this normal word the *CGW-normal form* of g, and denote it by $\sigma(g)$.

Note that all the subwords of a normal word are normal words.

One can understand $\sigma(g)$ as follows. Extend the linear order on $V(\Gamma)$ to $V(\Gamma)^{\pm 1}$ so that each generator v is the immediate predecessor of v^{-1}. (Namely, if the order on $V(\Gamma)$ is $v_1 < \cdots < v_r$, then the order on $V(\Gamma)^{\pm 1}$ is $v_1 < v_1^{-1} < v_2 < v_2^{-1} < \cdots < v_r < v_r^{-1}$.) Then $\sigma(g)$ is the largest in the lexicographic order among all the reduced words representing g.

Definition 2.7 (SD-conical)**.** Suppose that $V(\Gamma)$ is endowed with a linear order $<$.

(i) A conical element g is *pyramidal* if $\mathrm{apex}(g)$ is the smallest element of $\mathrm{supp}(g)$.

(ii) A conical element g is *SD-conical* if $v_0 = \mathrm{apex}(g)$ does not commute with any generator smaller than v_0, i.e., if $v < v_0$, then $\{v, v_0\} \in E(\Gamma)$. (From a graph theoretical point of view, the vertex v_0 dominates all the vertices $v \in V(\Gamma)$ smaller than v_0. SD stands for *smaller vertex dominating.*)

Example 2.8. Consider $G(P_5)$ in Example 1.3. Define a linear order on $V(P_5)$ by $v_1 < v_2 < v_3 < v_4 < v_5$. Let $g = v_2 v_4^{-1} v_3^{-1} v_5$. Then $S(g) = \{v_2, v_4\}$. The following are reduced words representing g.

$$w_1 \equiv v_2 v_4^{-1} v_3^{-1} v_5, \quad w_2 \equiv v_4^{-1} v_2 v_5 v_3^{-1}, \quad w_3 \equiv v_4^{-1} v_5 v_2 v_3^{-1}, \ldots.$$

The word w_1 is not initially normal because v_2 is not the largest element of $S(g)$. The word w_2 is initially normal, but it is not normal because the suffix $v_2 v_5 v_3^{-1}$ is not initially normal. The word w_3 is normal. Hence $\sigma(g) \equiv w_3 \equiv v_4^{-1} v_5 v_2 v_3^{-1}$.

The element $g = v_2 v_4^{-1} v_3^{-1} v_5 = v_4^{-1} v_5 v_2 v_3^{-1}$ is not conical because $S(g)$ is not a singleton. The conjugate $g_1 = (v_4^{-1} v_5)^{-1} g (v_4^{-1} v_5) = v_2 v_3^{-1} v_4^{-1} v_5$ is pyramidal because $S(g_1) = \{v_2\}$ and $v_2 = \operatorname{apex}(g_1)$ is the smallest element of $\operatorname{supp}(g_1) = \operatorname{supp}(g) = \{v_2, v_3, v_4, v_5\}$. Moreover, g_1 is SD-conical because v_1 is the only generator smaller than $v_2 = \operatorname{apex}(g_1)$ but it does not commute with v_2.

In the same way, one can see that $h_1 = v_4 v_5$ is pyramidal but not SD-conical, and that $h_2 = v_2 v_1 v_3 v_4$ is SD-conical but not pyramidal.

The notions of normal form, pyramidal and SD-conical are defined via a linear order on $V(\Gamma)$. Whenever such terms are used, it is assumed that $V(\Gamma)$ is endowed with some linear order even though it is not specified.

Lemma 2.9. *Let $g_1 g_2$ be geodesic for $g_1, g_2 \in G(\Gamma)$.*

(i) *If g_2 is SD-conical, then $\sigma(g_1 g_2) \equiv \sigma(g_1)\sigma(g_2)$.*
(ii) *If g_1 is SD-conical and strongly non-split, then $g_1 g_2$ is also SD-conical and strongly non-split.*

Proof. In the proof of Proposition 2.20 in [5], the following is observed:
If $v_1^{\epsilon_1}, v_2^{\epsilon_2}$ are letters and w is a reduced word such that the words $v_1^{-\epsilon_1} w$ and $w v_2^{\epsilon_2}$ are reduced but the word $v_1^{-\epsilon_1} w v_2^{\epsilon_2}$ is not, then $v_1^{\epsilon_1} = v_2^{\epsilon_2}$ and $v_1 \rightleftharpoons w$.

Using this, we get the following.

Claim. For $v \in V(\Gamma)$, if $v \in S(g_1 g_2)$ and $v \notin S(g_1)$, then $v \in S(g_2)$ and $v \rightleftharpoons g_1$.

Proof of Claim. Let w_1 and w_2 be reduced words representing g_1 and g_2, respectively. Since $v \in S(g_1 g_2)$ and $v \notin S(g_1)$, we can write $w_2 \equiv w_2' v_2^{\epsilon_2} w_2''$ such that $v \notin S(w_1 w_2')$ and $v \in S(w_1 w_2' v_2^{\epsilon_2})$, i.e., $v^{-\epsilon} w_1 w_2'$ is reduced but $v^{-\epsilon} w_1 w_2' v_2^{\epsilon_2}$ is not. Since $g_1 g_2$ is geodesic, $w_1 w_2' v_2^{\epsilon_2}$ is reduced. From the above observation, $v \rightleftharpoons w_1 w_2'$ and $v^{\epsilon} = v_2^{\epsilon_2}$. Hence $v \rightleftharpoons g_1$ and $v \in S(g_2)$. □

(i) Our proof is similar to the one in [5, Proposition 2.20], where it is shown that if p is pyramidal and cyclically reduced, then $\sigma(p)\sigma(p)$ is normal, hence $\sigma(p^2) \equiv \sigma(p)\sigma(p)$.

Assume $\sigma(g_1g_2) \not\equiv \sigma(g_1)\sigma(g_2)$, i.e., $\sigma(g_1)\sigma(g_2)$ is not normal. Then there exists a suffix of $\sigma(g_1)\sigma(g_2)$ that is not initially normal. Since $\sigma(g_1)$ and $\sigma(g_2)$ are normal, we can write

$$\sigma(g_1) \equiv w_1' v_1^{\epsilon_1} w_1, \qquad \sigma(g_2) \equiv w_2 v_2^{\epsilon_2} w_2'$$

such that $v_1^{\epsilon_1} w_1 w_2$ is initially normal but $v_1^{\epsilon_1} w_1 w_2 v_2^{\epsilon_2}$ is not. Therefore there exists $v_0^{\epsilon_0}$ such that

$$v_0 > v_1, \qquad v_0 \notin S(v_1^{\epsilon_1} w_1 w_2), \qquad v_0 \in S(v_1^{\epsilon_1} w_1 w_2 v_2^{\epsilon_2}).$$

By the above claim, $v_0 \rightleftharpoons v_1^{\epsilon_1} w_1 w_2$ and $v_0 \in S(v_2^{\epsilon_2}) = \{v_2\}$. Therefore,

$$v_0 \rightleftharpoons v_1, \qquad v_0 \rightleftharpoons w_1, \qquad v_0 \rightleftharpoons w_2, \qquad v_0 = v_2.$$

Assume w_2 is the empty word. Since g_2 is SD-conical, we have $v_0 = v_2 = \operatorname{apex}(g_2)$ and v_0 does not commute with any generator smaller than v_0. This is a contradiction because $v_0 \rightleftharpoons v_1$ and $v_0 > v_1$. Therefore w_2 is not the empty word, hence $S(w_2)$ is not empty.

Let $v_2' \in S(w_2)$. Since $v_2 = v_0 \rightleftharpoons w_2$, both v_2' and v_0 belong to $S(w_2 v_2^{\epsilon_2})$. Moreover, $v_2' \neq v_0$ because $v_2' \in S(w_2)$ and $v_0 \rightleftharpoons w_2$ imply $v_2' \in \operatorname{supp}(w_2)$ and $v_0 \notin \operatorname{supp}(w_2)$, respectively. Since $S(w_2 v_2^{\epsilon_2}) \subset S(g_2)$, we have $|S(g_2)| \geqslant 2$, which is a contradiction because g_2 is conical.

(ii) We first show that g_1g_2 is conical. Assume $|S(g_1g_2)| \geqslant 2$. Let $v_1 = \operatorname{apex}(g_1)$. Then $v_1 \in S(g_1) \subset S(g_1g_2)$. Because $|S(g_1g_2)| \geqslant 2$, there exists $v_2 \in S(g_1g_2) \setminus \{v_1\}$. Since g_1 is conical and $v_2 \neq v_1$, we have $v_2 \notin S(g_1)$. By the claim, we have $v_2 \rightleftharpoons g_1$. This contradicts that g_1 is strongly non-split. Therefore g_1g_2 is conical, and hence non-split.

Since $S(g_1) \subset S(g_1g_2)$ and $|S(g_1g_2)| = 1$, we have $S(g_1g_2) = S(g_1)$, hence $\operatorname{apex}(g_1g_2) = \operatorname{apex}(g_1)$. Because g_1 is SD-conical, $\operatorname{apex}(g_1g_2) = \operatorname{apex}(g_1)$ does not commute with any generator smaller than itself. Therefore g_1g_2 is SD-conical.

The element g_1g_2 is strongly non-split because $\operatorname{supp}(g_1g_2) \supset \operatorname{supp}(g_1)$ and g_1 is strongly non-split. ☐

Proposition 2.10. *For $n \geqslant 1$, if ag^nb is geodesic and g is strongly non-split and SD-conical, then* $\sigma(ag^nb) \equiv \sigma(a)\sigma(g)^{n-1}\sigma(gb) \equiv \sigma(a)\underbrace{\sigma(g)\cdots\sigma(g)}_{n-1}\sigma(gb)$.

Proof. Because g is strongly non-split and SD-conical, by Lemma 2.9(ii), $g^kb = g\cdot g^{k-1}b$ is strongly non-split and SD-conical for all $1 \leqslant k \leqslant n$. By Lemma 2.9(i), $\sigma(ag^nb) \equiv \sigma(a)\sigma(g^nb)$ and $\sigma(g^kb) \equiv \sigma(g\cdot g^{k-1}b) \equiv \sigma(g)\sigma(g^{k-1}b)$ for $2 \leqslant k \leqslant n$. By an induction on k, we can show $\sigma(g^kb) \equiv \underbrace{\sigma(g)\cdots\sigma(g)}_{k-1}\sigma(gb)$ for all $1 \leqslant k \leqslant n$. Therefore

$$\sigma(ag^nb) \equiv \sigma(a)\sigma(g^nb) \equiv \sigma(a)\underbrace{\sigma(g)\cdots\sigma(g)}_{n-1}\sigma(gb).$$

☐

Lemma 2.11. *Let $g_1, g_2 \in G(\Gamma)$ be strongly non-split. Then there exists a linear order on $V(\Gamma)$ together with vertices $v_1 \in \operatorname{supp}(g_1)$ and $v_2 \in \operatorname{supp}(g_2)$ such that both the v_1-conical conjugate of g_1 and the v_2-conical conjugate of g_2 are strongly non-split and SD-conical.*

Proof. First, suppose $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) \neq \emptyset$. Choose any $v_0 \in \operatorname{supp}(g_1) \cap \operatorname{supp}(g_2)$. Give a linear order on $V(\Gamma)$ such that v_0 is the smallest element of $V(\Gamma)$. Let p_1 and p_2 be the v_0-conical conjugates of g_1 and g_2, respectively. Then $\operatorname{apex}(p_1) = \operatorname{apex}(p_2) = v_0$ is the smallest element of $V(\Gamma)$, hence p_1 and p_2 are SD-conical.

Now suppose $\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2) = \emptyset$. Since g_1 is strongly non-split, there exists $(v_1, v_2) \in \operatorname{supp}(g_1) \times \operatorname{supp}(g_2)$ such that $\{v_1, v_2\} \in E(\Gamma)$. Give a linear order on $V(\Gamma)$ such that $v_1 = \min V(\Gamma)$ and $v_2 = \min (V(\Gamma) - \{v_1\})$. For $i = 1, 2$, let p_i be the v_i-conical conjugate of g_i. Then p_1 is SD-conical because $\operatorname{apex}(p_1) = v_1$ is the smallest element of $V(\Gamma)$. And p_2 is SD-conical because $\operatorname{apex}(p_2) = v_2$ is smaller than any generator other than v_1 and $\{v_1, v_2\} \in E(\Gamma)$.

In both cases above, p_1 and p_2 are strongly non-split because $\operatorname{supp}(p_i) = \operatorname{supp}(g_i)$ for $i = 1, 2$. □

3. Uniqueness of quasi-roots

In this section, we prove Theorem 1.9 in three steps.

3.1. Step 1: Quasi-roots of words

Let X be a set of symbols. In this subsection, a letter means an element of $X^{\pm 1}$ and a word means a word on $X^{\pm 1}$. In Proposition 3.3, we will prove the uniqueness of primitive quasi-roots up to conjugacy for words.

Lemma 3.1. *Let $\{x_n\}_{n \geqslant 1}$ and $\{y_n\}_{n \geqslant 1}$ be sequences of letters of period $p \geqslant 1$ and $q \geqslant 1$, respectively, such that the first $p + q$ terms coincide, i.e.,*

$$\begin{aligned} &x_{n+p} = x_n && \text{for } n \geqslant 1,\\ &y_{n+q} = y_n && \text{for } n \geqslant 1,\\ &x_n = y_n && \text{for } 1 \leqslant n \leqslant p+q. \end{aligned}$$

Then $x_n = y_n$ for all $n \geqslant 1$ and the sequence $\{x_n\}_{n \geqslant 1}$ has period $\gcd(p, q)$.

Proof. For $1 \leqslant n \leqslant p$, we have

$$x_n = y_n = y_{n+q} = x_{n+q}.$$

Because $\{x_n\}_{n \geqslant 1}$ has period p, the identity $x_n = x_{n+q}$ holds for all $n \geqslant 1$. Therefore $\{x_n\}_{n \geqslant 1}$ has period q, hence it has period $\gcd(p, q)$. By the same argument $\{y_n\}$ has period $\gcd(p, q)$. Because $\{x_n\}$ and $\{y_n\}$ are sequences of period $\gcd(p, q)$ whose first $p + q$ terms are identical, we have $x_n = y_n$ for all $n \geqslant 1$. □

Corollary 3.2. *Let $w_1 \equiv x_1 \cdots x_p$ and $w_2 \equiv y_1 \cdots y_q$ be two primitive words of lengths p and q, respectively. Suppose that there exist powers $w_1^{m_1}$ and $w_2^{m_2}$*

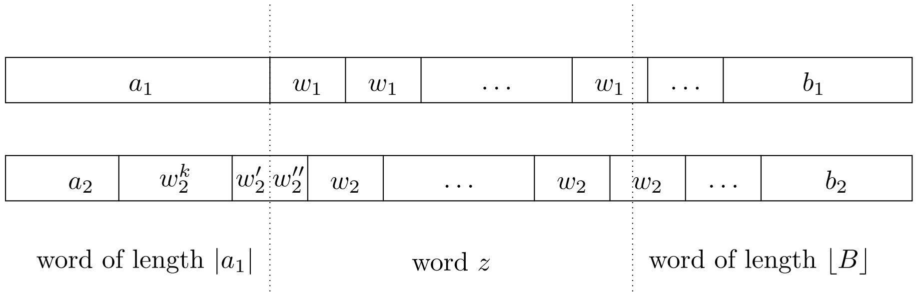


FIGURE 1. The word $w \equiv a_1 w_1^{m_1} b_1 \equiv a_2 w_2^{m_2} b_2$

such that w_1^2 *is a prefix of* $w_2^{m_2}$ *and* w_2^2 *is a prefix of* $w_1^{m_1}$. *Then* $p = q$ *and* $w_1 \equiv w_2$.

Proof. Consider the infinite words

$$w_1^* \equiv w_1 w_1 \cdots \equiv x_1 \cdots x_p x_1 \cdots x_p x_1 \cdots x_p \cdots ,$$
$$w_2^* \equiv w_2 w_2 \cdots \equiv y_1 \cdots y_q y_1 \cdots y_q y_1 \cdots y_q \cdots .$$

From the hypothesis, the first $p + q$ letters of w_1^* and w_2^* coincide.

Let $d = \gcd(p, q)$. By Lemma 3.1, w_1^* and w_2^* coincide and both have period d. Namely, $u = x_1 \cdots x_d$ is a common root of w_1 and w_2, i.e., $w_1 \equiv u^a$ and $w_2 \equiv u^b$, where $a = p/d$ and $b = q/d$. Since w_1 and w_2 are primitive, we have $a = b = 1$, hence $p = q$ and $w_1 \equiv w_2$. □

Proposition 3.3. *Let* w *be a word with the following two decompositions*

$$w \equiv a_1 w_1^{m_1} b_1 \equiv a_2 w_2^{m_2} b_2,$$

where $m_i \geqslant 2$ *and* w_i, a_i, b_i *are words for* $i = 1, 2$. *Suppose that* w_1 *and* w_2 *are primitive and that there exist non-negative constants* A *and* B *such that*

$$|a_i| \leqslant A, \quad |b_i| \leqslant B, \quad |w| - (A + B) \geqslant 2|w_i|$$

for $i = 1, 2$. *Then* w_1 *and* w_2 *are cyclically conjugate. Furthermore,* $a_1 w_1 a_1^{-1} = a_2 w_2 a_2^{-1}$ *and* $b_1^{-1} w_1 b_1 = b_2^{-1} w_2 b_2$ *when the words are regarded as elements of a free group.*

Proof. Without loss of generality, we may assume $|a_1| \geqslant |a_2|$. Then we have a decomposition $w_2 \equiv w_2' w_2''$ such that $a_1 \equiv a_2 w_2^k w_2'$ for some $k \geqslant 0$. (Notice that $|a_1| + |b_2| + 2|w_2| \leqslant A + B + 2|w_2| \leqslant |w| = |a_2| + |b_2| + m_2|w_2|$ and hence $|a_1| \leqslant |a_2| + (m_2 - 2)|w_2|$.) See Figure 1. Therefore

$$w \equiv a_2 w_2^{m_2} b_2 \equiv a_2 w_2^k w_2' (w_2'' w_2')^{m_2 - k - 1} w_2'' b_2 \equiv a_1 (w_2'' w_2')^{m_2 - k - 1} w_2'' b_2.$$

Let z be the subword of w obtained by removing the first $|a_1|$ letters and the last $\lfloor B \rfloor$ letters from w. From the hypothesis that $|w| - (A + B) \geqslant 2|w_i|$, we have $|z| \geqslant 2|w_i|$ for $i = 1, 2$. Notice that z is a prefix of both $w_1^{m_1}$ and $(w_2''w_2')^{m_2-k-1}w_2''$.

Since z is a prefix of $w_1^{m_1}$, it is of the form

$$z \equiv w_1 \cdots w_1 w_1' \equiv w_1^{n_1} w_1'$$

for some $n_1 \geqslant 2$ and some prefix w_1' of w_1. On the other hand, since z is prefix of $(w_2''w_2')^{m_2-k-1}w_2''$, it is of the form

$$z \equiv (w_2''w_2')^{n_2} w_2'''$$

for some $n_2 \geqslant 2$ and some prefix w_2''' of $w_2''w_2'$.

Notice that $w_2''w_2'$ is primitive because it is a cyclic conjugate of the primitive word w_2. The words w_1 and $w_2''w_2'$ now satisfy the hypothesis of Corollary 3.2. Therefore $w_1 \equiv w_2''w_2'$, hence w_1 and w_2 are cyclically conjugate.

When the words are regarded as elements of a free group,

$$\begin{aligned} a_1 w_1 a_1^{-1} &= (a_2 w_2^k w_2')(w_2''w_2')(a_2 w_2^k w_2')^{-1} = a_2 w_2^k w_2' w_2'' w_2^{-k} a_2^{-1} \\ &= a_2 w_2^k w_2 w_2^{-k} a_2^{-1} = a_2 w_2 a_2^{-1}. \end{aligned}$$

Using the same argument, we obtain $b_1^{-1} w_1 b_1 = b_2^{-1} w_2 b_2$. □

3.2. Step 2: SD-conical quasi-roots

Proposition 3.4. *Let* Γ *be a connected graph, and let*

$$h = a_1 p_1^{n_1} b_1 = a_2 p_2^{n_2} b_2$$

be two geodesic decompositions of $h \in G(\Gamma)$*, where* $n_i \geqslant 3$ *and* $a_i, b_i, p_i \in G(\Gamma)$ *for* $i = 1, 2$*. Suppose that* p_1 *and* p_2 *are strongly non-split, SD-conical and primitive and that there exist non-negative constants* A *and* B *such that*

$$|a_i| \leqslant A, \quad |b_i| \leqslant B, \quad |h| - (A + B) \geqslant 3|p_i|$$

for $i = 1, 2$*. Then the CGW-normal forms* $\sigma(p_1)$ *and* $\sigma(p_2)$ *are cyclically conjugate. Furthermore,* $a_1 p_1 a_1^{-1} = a_2 p_2 a_2^{-1}$ *and* $b_1^{-1} p_1 b_1 = b_2^{-1} p_2 b_2$*.*

Proof. Without loss of generality, we may assume $|p_1| \geqslant |p_2|$.

Since $a_1 p_1^{n_1} b_1$ and $a_2 p_2^{n_2} b_2$ are geodesic decompositions and since p_1 and p_2 are strongly non-split and SD-conical, the following holds by Proposition 2.10.

$$\sigma(h) \equiv \sigma(a_1)\sigma(p_1)^{n_1-1}\sigma(p_1 b_1) \equiv \sigma(a_2)\sigma(p_2)^{n_2-1}\sigma(p_2 b_2). \tag{2}$$

Let $A' = A$ and $B' = B + |p_1|$. Then

$$|h| - (A' + B') = |h| - (A + B) - |p_1| \geqslant 3|p_1| - |p_1| = 2|p_1| \geqslant 2|p_2|.$$

Applying Proposition 3.3 to (2), we get the desired result. □

3.3. Step 3: Proof of Theorem 1.9

Proposition 3.5. *Let Γ be a connected graph, and let*

$$h = a_1 g_1^{n_1} b_1 = a_2 g_2^{n_2} b_2 \tag{3}$$

be two geodesic decompositions of $h \in G(\Gamma)$, where $n_i \geqslant 1$ and $a_i, b_i, g_i \in G(\Gamma)$ for $i = 1, 2$. Suppose that g_1 and g_2 are strongly non-split and primitive and that there exist non-negative constants A and B such that

$$|a_i| \leqslant A, \quad |b_i| \leqslant B, \quad |h| - (A + B) \geqslant (2|V(\Gamma)| + 1)|g_i|$$

for $i = 1, 2$. Then g_1 and g_2 are conjugate such that $a_1 g_1 a_1^{-1} = a_2 g_2 a_2^{-1}$ and $b_1^{-1} g_1 b_1 = b_2^{-1} g_2 b_2$.

Proof. Let $V = |V(\Gamma)|$. Since $n_i|g_i| = |g_i^{n_i}| = |h| - (|a_i| + |b_i|) \geqslant |h| - (A+B) \geqslant (2V + 1)|g_i|$, we have $n_i \geqslant 2V + 1 \geqslant 2$. Since $a_i g_i^{n_i} b_i$ is geodesic, $g_i^{n_i}$ is also geodesic, hence g_i is cyclically reduced by Lemma 2.1.

By Lemma 2.11, we can choose a linear order on $V(\Gamma)$ together with vertices $v_1 \in \mathrm{supp}(g_1)$ and $v_2 \in \mathrm{supp}(g_2)$ such that the v_i-conical conjugate p_i of g_i is strongly non-split and SD-conical for $i = 1, 2$. Note that $|g_i| = |p_i|$.

By Proposition 2.4, we have the following geodesic decompositions

$$g_1^{n_1} = c_1 p_1^{n_1 - k_1} d_1, \quad g_2^{n_2} = c_2 p_2^{n_2 - k_2} d_2 \tag{4}$$

for some $0 \leqslant k_1, k_2 \leqslant V - 1$, where the elements $c_i, d_i \in G(\Gamma)$ are such that $g_i^{k_i} = c_i d_i$ is geodesic and $g_i = c_i p_i c_i^{-1} = d_i^{-1} p_i d_i$ for $i = 1, 2$. Since $g_i^{k_i} = c_i d_i$ is geodesic, we have

$$|c_i|, \ |d_i| \leqslant |g_i^{k_i}| \leqslant k_i |g_i| \leqslant (V - 1)|g_i|$$

for $i = 1, 2$. Combining (3) and (4), we have the following two decompositions of h:

$$h = a_1 c_1 p_1^{n_1 - k_1} d_1 b_1 = a_2 c_2 p_2^{n_2 - k_2} d_2 b_2. \tag{5}$$

They are geodesic decompositions because both $h = a_i g_i^{n_i} b_i$ and $g_i^{n_i} = c_i p_i^{n_i - k_i} d_i$ are geodesic.

Let $r = \max\{|g_1|, |g_2|\}$, $A' = A + (V - 1)r$ and $B' = B + (V - 1)r$. Then for $i = 1, 2$

$$|a_i c_i| \leqslant |a_i| + |c_i| \leqslant A + (V - 1)|g_i| \leqslant A',$$
$$|d_i b_i| \leqslant |d_i| + |b_i| \leqslant B + (V - 1)|g_i| \leqslant B',$$

$$\begin{aligned} |h| - (A' + B') &= |h| - (A + B) - 2(V - 1)r \\ &\geqslant (2V + 1)r - 2(V - 1)r = 3r \\ &\geqslant 3|g_i| = 3|p_i|. \end{aligned}$$

Since p_i is conjugate to the primitive element g_i, it is also primitive. Since $n_i \geqslant 2V+1$ and $k_i \leqslant V-1$, one has $n_i - k_i \geqslant V+2 \geqslant 3$. Applying Proposition 3.4 to (5), we get

$$a_1c_1p_1c_1^{-1}a_1^{-1} = a_2c_2p_2c_2^{-1}a_2^{-1}, \qquad b_1^{-1}d_1^{-1}p_1d_1b_1 = b_2^{-1}d_2^{-1}p_2d_2b_2.$$

From the identity $g_i = c_ip_ic_i^{-1} = d_i^{-1}p_id_i$, we obtain

$$a_1g_1a_1^{-1} = a_2g_2a_2^{-1}, \qquad b_1^{-1}g_1b_1 = b_2^{-1}g_2b_2. \qquad \square$$

Proof of Theorem 1.9. We are given $0 \leqslant \lambda < \frac{1}{2}$, $N \geqslant \frac{2|V(\Gamma)|+1}{1-2\lambda}$ and two geodesic decompositions of h

$$h = a_1g_1^{n_1}b_1 = a_2g_2^{n_2}b_2, \tag{6}$$

where $n_i \geqslant N$, $a_i, b_i, g_i \in G(\Gamma)$, $|a_i|, |b_i| \leqslant \lambda|h|$, and g_i is strongly non-split and primitive for $i = 1, 2$.

Since $|h| \geqslant |g_i^{n_i}| = n_i|g_i| \geqslant N|g_i|$, one has $|g_i| \leqslant |h|/N$ for $i = 1, 2$. Let $A = B = \lambda|h|$. Then $|a_i| \leqslant A$, $|b_i| \leqslant B$ and

$$|h| - (A+B) = (1-2\lambda)|h| \geqslant (2|V(\Gamma)|+1)\frac{|h|}{N} \geqslant (2|V(\Gamma)|+1)|g_i|$$

for $i = 1, 2$. Applying Proposition 3.5, we get the desired results. $\square$

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