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REGULARITY RELATIVE TO A HEREDITARY TORSION THEORY FOR MODULES OVER A COMMUTATIVE RING

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Abstract. In this paper, we introduce and study regular rings relative to the hereditary torsion theory w (a special case of a well-centered torsion theory over a commutative ring), called w-regular rings. We focus mainly on the w-regularity for w-coherent rings and w-Noetherian rings. In particular, it is shown that the w-coherent w-regular domains are exactly the Prüfer v-multiplication domains and that an integral domain is w-Noetherian and w-regular if and only if it is a Krull domain. We also prove the w-analogue of the global version of the Serre–Auslander-Buchsbaum Theorem. Among other things, we show that every w-Noetherian w-regular ring is the direct sum of a finite number of Krull domains. Finally, we obtain that the global weak w-projective dimension of a w-Noetherian ring is 0, 1, or ∞.

1. Introduction

Throughout this paper, all rings are commutative with an identity element and all modules are unitary; in particular, R denotes such a ring.

1.1. Noetherian regularity

A famous theorem by Serre [29], Auslander and Buchsbaum [1] states that a Noetherian local ring is regular if and only if it has finite global dimension. In Matsumura's book [22], he says that the Serre–Auslander-Buchsbaum Theorem is one of the three top theorems in commutative ring theory, and that this grasps the essence of Noetherian regular local rings, and is also an important meeting-point of ideal theory and homological algebra. Also, Auslander and Buchsbaum [2] proved that a Noetherian regular local ring is a unique factorization domain (UFD). This result may be regarded as another important achievement of homological algebra.

From the Serre–Auslander-Buchsbaum Theorem one can deduce, at once, that localizations of Noetherian regular local rings at prime ideals are again

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regular. This fact leads one to define regularity for all Noetherian rings, not just local ones: A Noetherian ring is said to be regular (see [1,22]) if its localizations at all of its prime ideals are regular local rings. In [21], the following global version of the Serre–Auslander-Buchsbaum Theorem was established:

Theorem A ([21, Theorem 5.94])**.** *For any Noetherian ring R, the following statements are equivalent*:

(1) *R is a regular ring.*
(2) *$R_{\mathfrak{m}}$ is a regular local ring for all maximal ideals $\mathfrak{m}$ of R.*
(3) *Every maximal ideal of R has finite projective dimension.*
(4) *Every prime ideal of R has finite projective dimension.*
(5) *Every finitely generated R-module has finite projective dimension.*

Notice that, in general, a regular ring R need not have finite global dimension, even when R is an integral domain. For such an example, see [21, Example 5.96]. However, Theorem A may also suggest a good way to extend the definition of regular rings to the non-Noetherian setting. Following Glaz [12], we call a ring R *regular* if every finitely generated ideal of R has finite projective dimension.

1.2. Coherent regularity

The first generalization of the notion of regularity to coherent rings can probably be attributed to Bertin [3]. He proved that all coherent regular local rings are integrally closed domains. In [32], Vasconcelos showed that a coherent regular local ring is a greatest common divisor (GCD) domain. This result provides a generalization of the factoriality of Noetherian regular local rings. In general, Glaz [10] proved that if a polynomial ring over a coherent regular ring is a coherent ring, then it is a regular ring. For additional results on coherent regularity see [9, 11–13, 26]. In particular, we should also point out that a coherent regular domain must be a Prüfer v-multiplication domain (PvMD), see [35, Theorem 9.1.13]. It can be obtained immediately by a result ([32, Corollary 3.16]) in the Vasconcelos' proof of the GCD property of coherent regular local rings.

The notion of PvMDs comes from multiplicative ideal theory, and various ideal-theoretic properties of them have been considered by many authors. Although these rings were studied by Krull, more recent interest in them was sparked by Griffin's paper [16]. Examples of PvMDs are Prüfer domains, Krull domains, GCD domains, integrally closed coherent domains, etc.

1.3. Hereditary torsion theory w

The main purpose of this paper is to generalize regular rings to the hereditary torsion theory w setting. Next, we shall review some terminology related to the hereditary torsion theory w, see [35] for details. The reader should consult the books of Stenström [30] and Golan [15] for background in hereditary torsion theory.

Recall from [42] that an ideal J of R is called a *Glaz-Vasconcelos ideal* (a GV-*ideal* for short) if J is finitely generated and the natural homomorphism

$$\varphi : R \to J^* := \mathrm{Hom}_R(J, R)$$

is an isomorphism. It is clear that a finitely generated ideal J of R is a GV-ideal if and only if

$$\mathrm{Hom}_R(R/J, R) = 0 \text{ and } \mathrm{Ext}^1_R(R/J, R) = 0.$$

Notice that the set $\mathrm{GV}(R)$ of GV-ideals of R is a multiplicative system of ideals of R. Let M be an R-module. Define

$$\mathrm{tor_{GV}}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor_{GV}}(M)$ is a submodule of M. Now M is said to be GV-*torsion* (resp., GV-*torsionfree*) if $\mathrm{tor_{GV}}(M) = M$ (resp., $\mathrm{tor_{GV}}(M) = 0$). A GV-torsionfree module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for all $J \in \mathrm{GV}(R)$. Then flat modules and reflexive modules are both w-modules. For any GV-torsionfree module M,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M, where $E(M)$ denotes the injective envelope of M. It is obvious that a GV-torsionfree module M is a w-module if and only if $M_w = M$.

Let w-$\mathrm{Max}(R)$ denote the set of w-ideals of R maximal among proper integral w-ideals of R and we call $\mathfrak{m} \in w$-$\mathrm{Max}(R)$ a *maximal w-ideal* of R. Then every proper w-ideal is contained in a maximal w-ideal and every maximal w-ideal is a prime ideal. Let $f : M \to N$ be a homomorphism of R-modules. Then we say that f is a *w-monomorphism* (resp., *w-epimorphism*, *w-isomorphism*) if $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is a monomorphism (resp., an epimorphism, an isomorphism) over $R_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. Meanwhile, a sequence $A \to B \to C$ of R-modules is called a *w-exact sequence* if the sequence $A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m}$ is exact over $R_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. Thus, an R-module M is *w-finitely generated* (i.e., M contains a finitely generated submodule N with M/N GV-torsion) if and only if there is a w-exact sequence $F \to M \to 0$, where F is a finitely generated free module. Similarly, an R-module M is said to be of *w-finitely presented type* if there is a w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_1 and F_0 are finitely generated free modules; equivalently, provided that if $0 \to A \to F \to M \to 0$ is a w-exact sequence, where F is finitely generated free, then A is w-finitely generated.

In the integral domain case, w-modules were called *semi-divisorial modules* by Glaz and Vasconcelos in [14] and (in the ideal case) *F_∞-ideals* by Hedstrom and Houston in [17], which have been proved to be useful in the study of ideal theory and module theory. It is worthwhile to point out that "w" may be a natural bridge between homological algebra and multiplicative ideal theory. On one hand, from the torsion-theoretic point of view, the notion of w-modules coincides with that of w-closed (i.e., GV-torsionfree and GV-injective) modules, where the torsion theory w whose torsion modules are the GV-torsion modules and the torsionfree modules are the GV-torsionfree modules. In fact, one can see that the hereditary torsion theory w is a well-centered hereditary torsion theory (in the sense of [4]). On the other hand, in the ideal theory, "w" is better known as the so-called *w-operation*, which was introduced by Wang and McCasland [36] in order to define the notion of strong Mori domains. Since then, a lot of work has been done on the w-operation. Also, as we see in [43, p. 451]: "*Now what is so nice about the w-operation is that it is smoother than the t-operation in that $\cdots$. Smoothness of this sort can only mean one thing, that you can bring in a lot more homological algebra than with other star operations, if that is what you want.*" For a detailed study of star operations the reader may consult Gilmer's book [8].

1.4. (Weak) w-projective modules and weak w-projective dimension

It is known that the class of projective modules is one of the most important classes of modules in homological algebra. In [34], Wang and Kim generalized projective modules to the hereditary torsion theory w setting. Recall that an R-module M is said to be *w-projective* if $\mathrm{Ext}_R^1(L(M), N)$ is a GV-torsion module for any torsionfree w-module N, where $L(M) = (M/\mathrm{tor}_{\mathrm{GV}}(M))_w$. It is clear that both GV-torsion modules and projective modules are w-projective.

Recently, Wang and Zhou [40] have given a homological characterization of Krull domains in terms of w-projective modules. More precisely, it is shown that an integral domain R is a Krull domain if and only if every submodule of a finitely generated projective R-module is w-projective (see [40, Theorem 3.3]). However, they do not know whether the property that every submodule of an arbitrary projective module is w-projective can also characterize a Krull domain.

In a very recent paper [38], a new type of projective module is introduced and studied, which is called the weak w-projective module (see below). Then it is proved that an integral domain R is a Krull domain if and only if every submodule of a projective module is weak w-projective, if and only if every ideal of R is weak w-projective. Also, there is an example showing that not all submodules of a projective module over a Krull domain are w-projective.

Now, we recall the definition of weak w-projective modules. Following [38], we use $\mathcal{P}_w^{\dagger}$ to denote the class of GV-torsionfree R-modules N with the property that $\mathrm{Ext}_R^k(M, N) = 0$ for all w-projective R-modules M and for all integers $k \geq 1$. Then an R-module M is said to be a *weak w-projective module* if $\mathrm{Ext}_R^1(M, N) = 0$ for all $N \in \mathcal{P}_w^{\dagger}$. Clearly, every w-projective module is weak w-projective. However, in general, the converse is not true, even for a Krull domain (see [38, Example 4.5]).

Recall also that the weak w-projective dimension of modules and rings is defined as follows.

Definition. If M is an R-module, then w.w-pd$_R(M) \leqslant n$ (w.w-pd abbreviates *weak w-projective dimension*) if there exists a w-exact sequence of R-modules

$$(\star) \qquad 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is a weak w-projective module. The w-exact sequence $(\star)$ is called a *weak w-projective w-resolution* of length n of M. If no such finite w-resolution exists, then w.w-pd$_R(M) = \infty$; otherwise, define w.w-pd$_R(M) = n$ if n is the length of the shortest weak w-projective w-resolution of M.

Clearly, an R-module M is weak w-projective if and only if w.w-pd$_R(M) = 0$, and w.w-pd$_R(M) \leq$ pd$_R(M)$, where pd$_R(M)$ denotes the classical projective dimension of M.

Definition. The *global weak w-projective dimension* of a ring R is defined by

$$\text{gl.w.}w\text{-dim}(R) = \sup\{\text{w.}w\text{-pd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Thus, gl.w.w-dim$(R) = 0$ if and only if R is a semisimple ring ([38, Proposition 4.1]). Moreover, from the homological algebra point of view, Krull domains are exactly the integral domains of global weak w-projective dimension at most one.

Now, a natural way to generalize regular rings to the hereditary torsion theory w setting is as follows.

Definition. A ring R is called a *w-regular ring* if every finitely generated ideal of R has finite weak w-projective dimension.

In the present paper, we first discuss some basic properties of w-regular rings (see Section 2). Then, in Section 3, we focus on the w-regularity for w-coherent rings. It is shown that a w-coherent ring is w-regular if and only if its localizations at each of its maximal w-ideals are coherent regular local rings (see Theorem 3.5). Section 4 studies w-coherent w-regular domains. It turns out that they are exactly the PvMDs (see Proposition 4.1 and Corollary 4.6). In the last section, we investigate w-regularity for w-Noetherian rings. First, we observe that an integral domain is w-Noetherian and w-regular if and only if it is a Krull domain (see Corollary 5.1). Then the w-analogue of the global version of the Serre–Auslander-Buchsbaum Theorem is established in Theorem 5.4. Furthermore, it is proved that every w-Noetherian w-regular ring is the direct sum of a finite number of Krull domains (see Theorem 5.13). Finally, we obtain that the global weak w-projective dimension of a w-Noetherian ring is 0, 1, or ∞ (see Corollary 5.15).

Any undefined notions or notation are standard, as in [8, 15, 19, 27, 30, 35].

2. w-regular rings

The topic of this section is some simple properties of w-regular rings.

We first give some examples of w-regular rings. It is obvious that each ring of finite global weak w-projective dimension is a w-regular ring. In particular, a Krull domain is a w-regular ring. Moreover, all regular rings are w-regular rings. But the converse does not always hold. In [28], Samuel gave an example of a nonregular Noetherian UFD. Obviously, it is a Krull domain (and hence also a w-regular domain) but not regular.

When is a w-regular ring also regular? Recall from [35] that a ring R is called a *DW-ring* if every ideal of R is a w-ideal. Note that R is a DW-ring if and only if every maximal ideal of R is a w-ideal, if and only if $\mathrm{GV}(R) = \{R\}$ (see [35, Theorem 6.3.12]). Examples of DW-rings include zero-dimensional rings, rings of weak global dimension at most one, treed domains (in particular, one-dimensional domains), divisorial domains, etc. DW-domains were investigated first by Dobbs et al. [5, 6]; they were mentioned in their papers as *t-linkative domains*. In [24], Mimouni was the first to name t-linkative domains as DW-domains. In the ideal theory, the terminology of DW-domains reflects the feature that the d-operation and the w-operation on such a ring are identical. Thus, we have:

Proposition 2.1. *Every w-regular DW-ring is a regular ring.*

However, a d-dimensional Noetherian regular local ring ($d \geq 2$) is never a DW-ring. Indeed, if $(R, \mathfrak{m})$ is such a ring, then it is easy to see that $\mathfrak{m}$ is a GV-ideal of R, and hence R is not a DW-ring.

Next, we will show that localizations of w-regular rings at prime w-ideals are regular. But the notion of w-quasiprojective modules is needed. Recall from [31, p. 37] that an R-module M is said to be *w-quasiprojective* if $M_{\mathfrak{m}}$ is projective over $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in w\text{-Max}(R)$. Note that M is a w-quasiprojective module if and only if $M_{\mathfrak{p}}$ is projective over $R_{\mathfrak{p}}$ for all prime w-ideals of R. In [44], w-quasiprojective modules were called *w-almost projective modules*. It was shown in [38, Proposition 2.8] that every weak w-projective module is w-quasiprojective. To give an example of a w-quasiprojective module that is not weak w-projective, let us recall some concepts from star operation theory.

Let D be an integral domain with quotient field Q and let $F(D)$ be the set of nonzero fractional ideals of D. Then a *star operation* is a function $A \mapsto A_*$ on $F(D)$ with the following properties:

If $A, B \in F(D)$ and $a \in Q \backslash \{0\}$, then

(1) $(a)_* = (a)$ and $(aA)_* = aA_*$.
(2) $A \subseteq A_*$ and if $A \subseteq B$, then $A_* \subseteq B_*$.
(3) $(A_*)_* = A_*$.

An ideal A is said to be a *$*$-ideal* if $A_* = A$. Define $A_d = A$, $A_v = (A^{-1})^{-1}$, and

$$A_t = \bigcup \{B_v \mid 0 \neq B \text{ is a finitely generated subideal of } A\},$$

where $A^{-1} = \{x \in Q \mid xA \subseteq D\}$. Then the functions $A \mapsto A$, $A \mapsto A_v$, and $A \mapsto A_t$ are classical examples of star operations, which are called the *d-operation*, *v-operation*, and *t-operation*, respectively. Also, the function $A \mapsto A_w$ is the so-called *w-operation*. In general, we have $A = A_d \subseteq A_w \subseteq A_t \subseteq A_v$, i.e.,

$$v\text{-ideals} \Rightarrow t\text{-ideals} \Rightarrow w\text{-ideals (semi-divisorial ideals)},$$

where v-ideals are better known as *divisorial ideals*.

In [18], Kang called an integral domain D a *t-Dedekind domain* if $D_{\mathfrak{m}}$ is a discrete valuation domain for each maximal t-ideal $\mathfrak{m}$ of D. Moreover, it was shown in [44, Proposition 2.12] that a domain is a t-almost Dedekind domain if and only if all its ideals are w-quasiprojective. Thus, if D is a t-almost Dedekind domain which is not a Krull domain (see, for example, [18, p. 167, Remark]), then D has a w-quasiprojective ideal that is not weak w-projective.

However, we have:

Proposition 2.2. *The following statements are equivalent for an R-module M.*

(1) *M is a w-quasiprojective module of w-finitely presented type.*
(2) *M is a w-finitely generated weak w-projective module.*
(3) *M is a w-finitely generated w-projective module.*

Proof. (1) $\Leftrightarrow$ (3) It follows from [34, Theorems 2.8 and 2.19].
(2) $\Leftrightarrow$ (3) See [38, Corollary 2.9]. □

By using the fact that every weak w-projective module is w-quasiprojective, it is easy to obtain the following proposition.

Proposition 2.3. *If R is a w-regular ring, then $R_{\mathfrak{p}}$ is a regular ring for all prime w-ideals $\mathfrak{p}$ of R.*

Next, we will see that a w-regular ring can be characterized by the property that all its w-finitely generated ideal have finite weak w-projective dimension.

Lemma 2.4. *Assume that $0 \to A \to B \to C \to 0$ is a w-exact sequence of R-modules. If any two modules have finite weak w-projective dimension, then so does the third one.*

Proof. By using [38, Lemma 2.1 and Proposition 3.1], this proof is the same as that given for the case of the classical projective dimension. □

As a consequence of the above lemma, we have the following:

Corollary 2.5. *Let $f : M_1 \to M$ be a w-isomorphism of R-modules. Then* $\text{w.}w\text{-pd}_R(M_1) < \infty$ *if and only if* $\text{w.}w\text{-pd}_R(M) < \infty$.

Proposition 2.6. *A ring R is w-regular if and only if every w-finitely generated ideal of R has finite weak w-projective dimension.*

Proof. Since every w-finitely generated ideal is w-isomorphic to one of its finitely generated subideals, the proof follows immediately from Corollary 2.5. □

3. w-regularity of w-coherent rings

In this section, we discuss mainly the w-regularity of w-coherent rings. First recall from [7] that an R-module M is said to be *w-finitely presented* if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$ with F finitely generated free and K w-finitely generated, and that a ring R is said to be *w-coherent* if every finitely generated ideal of R is w-finitely presented. It is clear that every w-finitely presented module must be finitely generated, and that every w-finitely presented module is of w-finitely presented type. But there is a module of w-finitely presented type that is not w-finitely presented (see [39, Example 2.3]). Moreover, it was proved in [39, Theorem 2.10] that a ring R is w-coherent if and only if every w-finitely generated ideal of R is of w-finitely presented type.

Proposition 3.1. *Let R be a w-coherent ring. Then the following statements are equivalent.*

(1) *R is a w-regular ring.*
(2) *Every w-finitely presented R-module has finite weak w-projective dimension.*
(3) *Every finitely presented R-module has finite weak w-projective dimension.*
(4) *Every R-module of w-finitely presented type has finite weak w-projective dimension.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a w-regular ring and let $M = (x_1, \dots, x_n)$ be a w-finitely presented R-module. Now we prove $\text{w.}w\text{-pd}_R(M) < \infty$ by induction on n. For the case $n = 1$, we have that $M \cong R/I$, where I is a w-finitely generated ideal of R. By Proposition 2.6, $\text{w.}w\text{-pd}_R(I) < \infty$, and so $\text{w.}w\text{-pd}_R(M) < \infty$. Thus, we have already seen that every w-finitely presented cyclic R-module has finite weak w-projective dimension. If $n > 1$, then write $M_1 = (x_1, \dots, x_{n-1})$ and there is an exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ with M/M_1 a cyclic R-module. By [7, Corollary 2.6], M/M_1 is w-finitely presented, whence $\text{w.}w\text{-pd}_R(M/M_1) < \infty$. On the other hand, since R is a w-coherent ring, [7, Theorem 3.3] implies that M_1 is also w-finitely presented. Therefore, by induction, $\text{w.}w\text{-pd}_R(M_1) < \infty$ and so $\text{w.}w\text{-pd}_R(M) < \infty$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) Suppose (3) holds and let M be an R-module of w-finitely presented type. Then it follows from [35, Theorem 6.4.15] that there exist a finitely presented R-module M_1 and a w-isomorphism $f : M_1 \to M$. Hence, (4) holds by Corollary 2.5.

(4) $\Rightarrow$ (1) This follows from the fact that every finitely generated ideal of a w-coherent ring is of w-finitely presented type. □

Recall from [20] that an R-module M is called *w-flat* if $M_\mathfrak{m}$ is $R_\mathfrak{m}$-flat for all $\mathfrak{m} \in w\text{-Max}(R)$. One can see that every weak w-projective module is w-flat. Following [37], for an R-module M, we denote by $w\text{-fd}_R(M)$ the *w-flat* *dimension* of M, and we use the notation w-w.gl.dim(R) to denote the *w-weak global dimension* of R. Then for an R-module M of w-finitely presented type over a w-coherent ring R, we have $w\text{-fd}_R(M) = \text{w}.w\text{-pd}_R(M)$ (see [38, Proposition 3.4]). Thus, by using this fact and Proposition 3.1, we can obtain the following corollary.

Corollary 3.2. *Every w-coherent ring of finite w-weak global dimension is a w-regular ring.*

We next consider the localization of w-regular rings in connection with the classical regularity. For this, we need to prepare a little.

Lemma 3.3. *Let R be a w-coherent ring and let $f : M \to N$ be a homomorphism of R-modules of w-finitely presented type. Then both* $\ker(f)$ *and* $\text{coker}(f)$ *are of w-finitely presented type.*

Proof. By [35, Theorem 6.9.18], M and N are both w-coherent modules. Note that $\text{im}(f)$ is a w-finitely generated submodule of N. Then [35, Proposition 6.9.16] implies that both $\text{im}(f)$ and $\text{coker}(f)$ are w-coherent (and hence, they are of w-finitely presented type). Moreover, by [35, Theroem 6.4.14], $\ker(f)$ is a w-finitely generated submodule of M, and so it is of w-finitely presented type. □

A key lemma is the following:

Lemma 3.4. *Let R be a w-coherent ring and let M be an R-module of w-finitely presented type. Then there exists a maximal w-ideal $\mathfrak{m}_0$ of R such that*

$$\text{w}.w\text{-pd}_R(M) = \text{pd}_{R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}) \geq \text{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$$

for all $\mathfrak{m} \in w\text{-Max}(R)$.

Proof. First note, for each $\mathfrak{m} \in w\text{-Max}(R)$, that $\text{w}.w\text{-pd}_R(M) \geq \text{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ follows from the fact that every weak w-projective module is a w-quasiprojective module. To complete the proof, we need show that $\text{w}.w\text{-pd}_R(M) \leq \text{pd}_{R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0})$ for some $\mathfrak{m}_0 \in w\text{-Max}(R)$. For this, we write $d(\mathfrak{m}) = \text{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) < \infty$ for any $\mathfrak{m} \in w\text{-Max}(R)$. For each integer $n > 0$, by [38, Proposition 3.4], we have a w-exact sequence of R-modules

$$0 \to K_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where $P_0, \ldots, P_{n-1}$ are finitely generated projective and $K_n = \ker(P_{n-1} \to P_{n-2})$ is of w-finitely presented type. Hence, for each $\mathfrak{m} \in w\text{-Max}(R)$, $(K_{d(\mathfrak{m})})_{\mathfrak{m}}$ is finitely generate and free, say, of rank $r(\mathfrak{m})$. Thus, it is not difficult to see that there exists an R-homomorphism

$$\phi(\mathfrak{m}) : R^{r(\mathfrak{m})} \longrightarrow K_{d(\mathfrak{m})}$$

which, when localized at $\mathfrak{m}$, is an isomorphism over $R_{\mathfrak{m}}$. From Lemma 3.3, this just means that there are finitely generated submodules $K(\mathfrak{m})$ and $C(\mathfrak{m})$

of $\ker(\phi(\mathfrak{m}))$ and $\operatorname{coker}(\phi(\mathfrak{m}))$, respectively, and a suitable element $s_{\mathfrak{m}} \in R\backslash\mathfrak{m}$ such that

$$s_{\mathfrak{m}} \cdot K(\mathfrak{m}) = s_{\mathfrak{m}} \cdot C(\mathfrak{m}) = 0.$$

Let I be the ideal generated by all such $s_{\mathfrak{m}}$. Then I is not contained in any maximal w-ideal of R, and so $I_w = R$. Therefore, I must contain some GV-ideal J of R. Since J is finitely generated, there exist $\mathfrak{m}_1, \dots, \mathfrak{m}_k \in w\text{-Max}(R)$ such that $(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_k})_w = R$. Set

$$d(\mathfrak{m}_0) = \text{Max}\{d(\mathfrak{m}_1), \dots, d(\mathfrak{m}_k)\}.$$

Next, we claim that $K_{d(\mathfrak{m}_0)}$ is weak w-projective. If so, then

$$\text{w.}w\text{-pd}_R(M) \leq d(\mathfrak{m}_0) = \text{pd}_{R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}).$$

Now, for each $\mathfrak{m} \in w\text{-Max}(R)$, $(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_k})_w = R$ says that we can fix an index i $(1 \leq i \leq k)$ with $s_{\mathfrak{m}_i} \in R\backslash\mathfrak{m}$. Since $s_{\mathfrak{m}_i}$ kills both $K(\mathfrak{m}_i)$ and $C(\mathfrak{m}_i)$, it is easily seen that

$$\ker(\phi(\mathfrak{m}_i))_{\mathfrak{m}} = K(\mathfrak{m}_i)_{\mathfrak{m}} = 0 \text{ and } \operatorname{coker}(\phi(\mathfrak{m}_i))_{\mathfrak{m}} = C(\mathfrak{m}_i)_{\mathfrak{m}} = 0.$$

Hence, $\phi(\mathfrak{m}_i)$ localizes to an isomorphism at $\mathfrak{m}$, whence $(K_{d(\mathfrak{m}_i)})_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$. But note that $d(\mathfrak{m}_0) \geq d(\mathfrak{m}_i)$. Thus, it follows that the freeness of $(K_{d(\mathfrak{m}_i)})_{\mathfrak{m}}$ gives that of $(K_{d(\mathfrak{m}_0)})_{\mathfrak{m}}$. This shows, by Proposition 2.2, that $K_{d(\mathfrak{m}_0)}$ is weak w-projective, as desired. □

Proposition 3.5. *If R is a w-coherent ring, then $R_{\mathfrak{p}}$ is a coherent ring for all prime w-ideals $\mathfrak{p}$ of R.*

Proof. Let $\mathfrak{p}$ be a prime w-ideal of R and A a finitely generated ideal of $R_{\mathfrak{p}}$. Then $A = I_{\mathfrak{p}}$ for some finitely generated ideal I of R. Since R is w-coherent, I is w-finitely presented, whence A is a finitely presented ideal of $R_{\mathfrak{p}}$. Thus, $R_{\mathfrak{p}}$ is coherent. □

We can now prove the main result of this section.

Theorem 3.6. *The following statements are equivalent for a w-coherent ring R.*

(1) *R is a w-regular ring.*
(2) *$R_{\mathfrak{p}}$ is a coherent regular local ring for all prime w-ideals $\mathfrak{p}$ of R.*
(3) *$R_{\mathfrak{m}}$ is a coherent regular local ring for all $\mathfrak{m} \in w\text{-Max}(R)$.*

Proof. (1) ⇒ (2) This follows immediately from Propositions 3.5 and 2.3.

(2) ⇒ (3) It is trivial.

(3) ⇒ (1) Assume that (3) holds and let I be a finitely generated ideal of R. Then I is of w-finitely presented type, and so by Lemma 3.4, there exists a maximal w-ideal $\mathfrak{m}_0$ of R such that $\text{w.}w\text{-pd}_R(I) = \text{pd}_{R_{\mathfrak{m}_0}}(I_{\mathfrak{m}_0}) < \infty$. Therefore, R is w-regular. □

In [10], Glaz proved that if the polynomial ring $R[X]$ over a coherent regular ring R is coherent, then $R[X]$ is regular. To close this section, we give a w-analogue of this result.

Lemma 3.7. *Let R be a coherent regular ring for which $R[X]$ is a coherent ring. Then for each prime ideal P of $R[X]$, $R[X]_P$ is a coherent regular local ring.*

Proof. This follows from [12, Corollary 6.2.14 and Theorem 6.2.3]. □

Corollary 3.8. *Let R be a w-coherent w-regular ring for which $R[X]$ is a coherent ring. Then $R[X]$ is a w-regular ring.*

Proof. Let P be a prime w-ideal of $R[X]$. Then since $R[X]$ is a free R-module (and hence is a w-module over R), it follows from [41, Lemma 4] that $\mathfrak{p} := P \cap R$ is a prime w-ideal of R. Notice that $R_{\mathfrak{p}}$ is a coherent regular local ring and that $R_{\mathfrak{p}}[X] = R[X]_S$ is a coherent ring, where $S = R \backslash \mathfrak{p}$. Then by applying Lemma 3.7 to $R_{\mathfrak{p}}$ and P_S, we see that

$$R[X]_P \cong (R[X]_S)_{P_S}$$

is a coherent regular ring. Thus, Theorem 3.6 says that $R[X]$ is w-regular. □

4. w-coherent w-regular domains

Throughout this section, R will be an integral domain with quotient field Q. We begin with an important class of w-coherent w-regular domains.

Let $*$ be a star operation on R. Recall that a nonzero fractional ideal A of R is said to be $*$-*invertible* if $(AB)_* = R$ for some $B \in F(R)$ or equivalently $(AA^{-1})_* = R$. Then it is known that $A \in F(R)$ is t-invertible if and only if it is w-invertible, if and only if A is w-finitely generated and $A_{\mathfrak{m}}$ is a principal ideal of $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in w\text{-Max}(R)$. Recall also that a v-ideal A is said to be of *finite type* if there is a finitely generated fractional ideal B such that $A = B_v$. Then a domain R is called a *Prüfer v-multiplication domain* (PvMD) if for each nonzero finitely generated ideal I of R, $(II^{-1})_v = R$ and I^{-1} is of finite type. It is known that a domain R is a PvMD if and only if every nonzero finitely generated ideal of R is w-invertible (or t-invertible).

It is shown in [35, Corollary 7.5.9] that every PvMD is a w-coherent domain. Also, note that a domain R is a PvMD if and only if its w-weak global dimension is at most one (see [37, Theorem 3.5]). Thus, as a consequence of Corollary 3.2, we state the following:

Proposition 4.1. *Every PvMD is a w-coherent w-regular domain.*

As mentioned in the introduction, every coherent regular domain is a PvMD. Thus it seems natural to ask whether the converse of Proposition 4.1 is also true. In the rest of this section, we will give a positive answer to this question.

Recall from [12] that the *small finitistic projective dimension* of a ring R is defined as:

$$\text{fP.dim}R = \sup\left\{ \text{pd}_R(M) \;\middle|\; \begin{array}{l} M \text{ an } R\text{-module with } \text{pd}_R(M) < \infty \text{ and} \\ M \text{ admits a finite projective resolution} \end{array} \right\}.$$

It is shown in [11, Corollary 3.3.17] that for a local ring R, $\text{fP.dim}(R) = 0$ if and only if R satisfies the property that every proper finitely generated ideal of R admits a nonzero annihilator. Thus, if $\mathfrak{p}$ is a minimal prime ideal over $\text{Ann}_R(x)$ for some $x \in R\backslash\{0\}$, then $\text{fP.dim}(R_\mathfrak{p}) = 0$.

Proposition 4.2. *Let N be a torsionfree R-module with a finite weak w-projective w-resolution, i.e., there exists a w-exact sequence*

$$0 \to P^n \to \cdots \to P^1 \to P^0 \to N \to 0,$$

where all $P^0, P^1, \ldots, P^n$ are w-finitely generated and weak w-projective R-modules. Then for any $\mathfrak{p} \in \text{Ass}_R(Q/R)$, $N_\mathfrak{p}$ is free over $R_\mathfrak{p}$.

Proof. Let $\mathfrak{p} \in \text{Ass}_R(Q/R)$. Then $\mathfrak{p}$ is minimal over $(Ra \underset{R}{:} b)$ for some $a, b \in R\backslash\{0\}$. Since $a \in (Ra \underset{R}{:} b) \subseteq \mathfrak{p}$, $\frac{a}{1}$ is not a unit in $R_\mathfrak{p}$. Notice that $\mathfrak{p}$ is a prime w-ideal (see [25, Lemma 3.1(2)]). Then

$$0 \to P^n_\mathfrak{p} \to \cdots \to P^1_\mathfrak{p} \to P^0_\mathfrak{p} \to N_\mathfrak{p} \to 0$$

is a finite free resolution of $N_\mathfrak{p}$ over $R_\mathfrak{p}$. Since N is torsionfree, $\frac{a}{1}$ is a non-zero-divisor of $N_\mathfrak{p}$. Therefore,

$$0 \to P^n_\mathfrak{p}/\tfrac{a}{1}P^n_\mathfrak{p} \to \cdots \to P^1_\mathfrak{p}/\tfrac{a}{1}P^1_\mathfrak{p} \to P^0_\mathfrak{p}/\tfrac{a}{1}P^0_\mathfrak{p} \to N_\mathfrak{p}/\tfrac{a}{1}N_\mathfrak{p} \to 0$$

is a finite free resolution of $N_\mathfrak{p}/\frac{a}{1}N_\mathfrak{p}$ over $R_\mathfrak{p}/\frac{a}{1}R_\mathfrak{p}$. Set $\bar{R} = R/Ra$, $\bar{N} = N/aN$, and $\bar{\mathfrak{p}} = \mathfrak{p}/Ra$. Then $N_\mathfrak{p}/\frac{a}{1}N_\mathfrak{p} \cong \bar{N}_{\bar{\mathfrak{p}}}$ and $R_\mathfrak{p}/\frac{a}{1}R_\mathfrak{p} \cong \bar{R}_{\bar{\mathfrak{p}}}$, whence $\bar{N}_{\bar{\mathfrak{p}}}$ has a finite free resolution over $\bar{R}_{\bar{\mathfrak{p}}}$. Since $\text{Ann}_{\bar{R}}(\bar{b}) = (Ra \underset{R}{:} b)/Ra$, $\bar{\mathfrak{p}}$ is minimal over $\text{Ann}_{\bar{R}}(\bar{b})$ (where $\bar{b}$ denotes the coset $\bar{b} = b + Ra$), and so $\text{fP.dim}(\bar{R}_{\bar{\mathfrak{p}}})$=0. Thus it follows that $N_\mathfrak{p}/\frac{a}{1}N_\mathfrak{p}$ is free over $R_\mathfrak{p}/\frac{a}{1}R_\mathfrak{p}$. Hence, by [35, Theorem 3.9.15], $N_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module. □

Corollary 4.3. *Let R be a w-coherent domain and let N be a w-finitely generated torsionfree R-module with finite weak w-projective dimension. Then for any $\mathfrak{p} \in \text{Ass}_R(Q/R)$, $N_\mathfrak{p}$ is free over $R_\mathfrak{p}$.*

Proof. By hypothesis, N is of w-finitely presented type (cf. [35, Theorem 6.9.20]). Hence, it follows from [38, Proposition 3.4] that N has a finite weak w-projective w-resolution. Thus, the proof is a consequence of Proposition 4.2. □

Lemma 4.4. *The following statements are equivalent for a domain R.*

(1) *R is a w-coherent domain.*
(2) *Every finitely generated torsionfree R-module is w-finitely presented.*

(3) *If N is a* GV*-torsionfree w-finitely presented R-module, then* $\mathrm{Hom}_R(M, N)$ *is w-finitely generated for each w-finitely presented R-module M.*
(4) *M^* is w-finitely generated for each w-finitely presented R-module M.*

Proof. This proof is the same as that given for [25, Proposition 4.2]. □

Proposition 4.5. *Let R be a w-coherent domain and let I be a nonzero finitely generated ideal of R with finite weak w-projective dimension. Then*:
(1) *II^{-1} contains a* GV*-ideal of R.*
(2) *I is a w-invertible ideal.*

Proof. (1) Let $\mathfrak{p} \in \mathrm{Ass}_R(Q/R)$. Then note, by Lemma 4.4, that $I^{-1} \cong I^*$ is w-finitely generated, and so it contains a finitely generated fractional subideal A with $A_\mathfrak{p} = (I^{-1})_\mathfrak{p}$. Set $J = IA$. Then J is a finitely generated subideal of II^{-1}. By Corollary 4.3, $I_\mathfrak{p}$ is a principal ideal of $R_\mathfrak{p}$. So it follows that

$$J_\mathfrak{p} = I_\mathfrak{p} A_\mathfrak{p} = I_\mathfrak{p}(I^{-1})_\mathfrak{p} = I_\mathfrak{p}(I_\mathfrak{p})^{-1} = R_\mathfrak{p},$$

that is, $J \nsubseteq \mathfrak{p}$. Thus, $J^{-1} = R$, i.e., $J \in \mathrm{GV}(R)$.

(2) By (1), there exists $J \in \mathrm{GV}(R)$ with $J \subseteq II^{-1}$. Therefore,

$$R = J_w \subseteq (II^{-1})_w \subseteq R,$$

whence $(II^{-1})_w = R$, that is, I is a w-invertible ideal. □

As a consequence of Proposition 4.5, we can obtain the promised result.

Corollary 4.6. *If R is a w-coherent w-regular domain, then it is a PvMD. In particular, every w-coherent domain of finite w-weak global dimension is a PvMD.*

Thus we have shown that a domain is a PvMD if and only if it is w-coherent and w-regular, and that the w-weak global dimension of a w-coherent w-regular domain is at most one.

Corollary 4.7. *Let R be a w-coherent w-regular ring with zero divisors. If R is a connected ring, then R is a PvMD.*

Proof. By Corollary 4.6, it is sufficient to show that R is a domain. For this, let a be a nonzero element of R and set $I = \mathrm{Ann}_R(a)$. Then I is a w-finitely generated ideal of R as R is w-coherent. Also, note that $R_\mathfrak{m}$ is a domain for all $\mathfrak{m} \in w$-$\mathrm{Max}(R)$. Thus, by [33, Lemma 4.12], I is generated by an idempotent element e. But $a \neq 0$, so we must have $e = 0$, i.e., $I = 0$. Hence R is a domain. □

Remark 4.8. It is known that a coherent regular ring does not necessarily have finite weak global dimension (see [12, p. 202]). Although in the connected ring case, the w-weak global dimension of a w-coherent w-regular ring is at most one, we do not know if any w-coherent w-regular ring has finite w-weak global dimension, in particular, we do not know if it has w-weak global dimension at most one.

5. w-regularity of w-Noetherian rings

This section is devoted to a discussion of the w-regularity of w-Noetherian rings. Recall that a ring R is called *w-Noetherian* if each ideal of R is w-finitely generated, or equivalently, R satisfies the ascending chain condition on integral w-ideals. In the integral domain case, a w-Noetherian ring is better known as a *strong Mori domain* (an SM domain).

First, we give an important class of w-Noetherian w-regular rings. Since a domain is a Krull domain if and only if it is both an SM domain and a PvMD, we have the following:

Proposition 5.1. *A domain is a Krull domain if and only if it is a w-regular SM domain.*

Next, we will prove the w-analogue of the global version of the Serre–Auslander-Buchsbaum Theorem.

Lemma 5.2. *Let R be a w-Noetherian ring and M a w-finitely generated R-module. Then there exists a maximal w-ideal $\mathfrak{m}_0$ of R such that*

$$\text{w.}w\text{-pd}_R(M) = \text{pd}_{R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}) \geq \text{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$$

for all $\mathfrak{m} \in w\text{-Max}(R)$.

Proof. It follows from Lemma 3.4 and the fact that every w-finitely generated module over a w-Noetherian ring is of w-finitely presented type (cf. [35, Theorem 6.8.5]). □

For any ring R, the global dimension of R is denoted by gl.dim(R).

Proposition 5.3. *Let R be a w-Noetherian ring. Then for any $\mathfrak{m} \in w\text{-Max}(R)$, we have*

$$\text{gl.dim}(R_{\mathfrak{m}}) = \text{w.}w\text{-pd}_R(R/\mathfrak{m})$$

and

$$\text{gl.w.}w\text{-dim}(R) = \sup\{\text{gl.dim}(R_{\mathfrak{m}}) \mid \mathfrak{m} \in w\text{-Max}(R)\}.$$

Proof. For any $\mathfrak{m} \in w\text{-Max}(R)$, apply Lemma 5.2 to $M = R/\mathfrak{m}$. Since $M_{\mathfrak{m}} \cong R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ and $M_{\mathfrak{m}'} = 0$ for other maximal w-ideals $\mathfrak{m}'$, we see that

$$\text{w.}w\text{-pd}_R(R/\mathfrak{m}) = \text{pd}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}) = \text{gl.dim}(R_{\mathfrak{m}}).$$

The second equality follows easily from Lemma 5.2 and the fact that any $R_{\mathfrak{m}}$-module is a localization of an R-module. □

Theorem 5.4. *The following statements are equivalent for a w-Noetherian ring R.*

(1) *R is a w-regular ring.*
(2) *$R_{\mathfrak{p}}$ is a Noetherian regular local ring for all prime w-ideals $\mathfrak{p}$ of R.*
(3) *$R_{\mathfrak{m}}$ is a Noetherian regular local ring for all $\mathfrak{m} \in w\text{-Max}(R)$.*
(4) *$\text{w.}w\text{-pd}_R(\mathfrak{m}) < \infty$ for all $\mathfrak{m} \in w\text{-Max}(R)$.*

(5) $\mathrm{w}.w\text{-}\mathrm{pd}_R(\mathfrak{p}) < \infty$ *for all prime* w*-ideals* $\mathfrak{p}$ *of* R.

(6) $\mathrm{w}.w\text{-}\mathrm{pd}_R(M) < \infty$ *for all* w*-finitely generated* R*-modules* M.

(7) $\mathrm{w}.w\text{-}\mathrm{pd}_R(M) < \infty$ *for all finitely generated* R*-modules* M.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Theorem 3.6.

(6) $\Rightarrow$ (5) and (5) $\Rightarrow$ (4) are both clear.

(4) $\Rightarrow$ (3) Let $\mathfrak{m} \in w\text{-}\mathrm{Max}(R)$. Then the exactness of $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$ and (4) imply that $\mathrm{w}.w\text{-}\mathrm{pd}_R(R/\mathfrak{m}) < \infty$, and hence $\mathrm{gl.dim}(R_\mathfrak{m}) < \infty$ by Proposition 5.3. Therefore, $R_\mathfrak{m}$ is a Noetherian regular local ring.

(3) $\Rightarrow$ (6) It is a direct consequence of Lemma 5.2.

(6) $\Rightarrow$ (7) It is trivial.

(7) $\Rightarrow$ (6) It follows from Corollary 2.5. □

Recall from [35] that the *w-Krull dimension* of R is defined as

$$w\text{-}\dim(R) = \sup\{\mathrm{ht}(\mathfrak{m}) \mid \mathfrak{m} \in w\text{-}\mathrm{Max}(R)\}.$$

Corollary 5.5. *If R is a w-Noetherian w-regular ring, then*

$$\mathrm{gl.w.}w\text{-}\dim(R) = w\text{-}\dim(R).$$

Proof. This follows immediately from Proposition 5.3, Theorem 5.4 and the fact that the global dimension of a Noetherian regular local ring is equal to its Krull dimension. □

Our next two results are on the invariance of the w-regularity of w-Noetherian rings under localizations and polynomial extensions.

Proposition 5.6. *Let R be a w-Noetherian w-regular ring and S a multiplicatively closed set of R. Then R_S is also w-Noetherian and w-regular.*

Proof. Note, by [35, Theorem 6.8.31(3)], that R_S is w-Noetherian. Let P be a prime w-ideal of R_S. Then it is easy to see from [35, Theorem 6.8.31(2)] that there exists a prime w-ideal $\mathfrak{p}$ of R with $\mathfrak{p}_S = P$ and $\mathfrak{p} \cap S = \emptyset$. Thus, $(R_S)_P = (R_S)_{\mathfrak{p}_S} \cong R_\mathfrak{p}$ is a Noetherian regular local ring, and so R_S is w-regular. □

Proposition 5.7. *If R is a w-Noetherian w-regular ring, then so is $R[X]$.*

Proof. First, the fact that $R[X]$ is w-Noetherian follows from [42, Theorem 4.9]. Thus, by using Lemma 3.7, the rest of this proof is the same as that given for Corollary 3.8. □

A connection between w-Noetherian w-regular rings and Noetherian regular rings can be established by the notion of w-Nagata rings. Let us recall the definition of them from [34]. For any $\alpha \in R[X]$, we denote by $c(\alpha)$ the subideal of R generated by the coefficients of α and is called the *content* of α. Set

$$S_w := \{f \in R[X] \mid c(f)_w = R\} = \{f \in R[X] \mid c(f) \in \mathrm{GV}(R)\}.$$

Then it is easy to see that S_w is a multiplicatively closed set of $R[X]$. Write

$$R\{X\} := R[X]_{S_w}.$$

Then $R\{X\}$ is called a *w-Nagata ring*. Notice that the set of maximal ideals of $R\{X\}$ is

$$\{\mathfrak{m}\{X\} := \mathfrak{m}[X]_{S_w} = R\{X\} \bigotimes_R \mathfrak{m}\},$$

where $\mathfrak{m}$ takes all maximal w-ideals of R.

Proposition 5.8. *A ring R is w-Noetherian w-regular if and only if $R\{X\}$ is a Noetherian regular ring.*

Proof. By [35, Theorem 6.8.8], a ring R is w-Noetherian if and only if $R\{X\}$ is a Noetherian ring.

Assume that R is a w-Noetherian w-regular ring and let $\mathfrak{M}$ be a maximal ideal of $R\{X\}$. Then $\mathfrak{M} = \mathfrak{m}\{X\}$ for some $\mathfrak{m} \in w$-Max(R). Since $R_\mathfrak{m}$ is a Noetherian regular local ring, it follows from [11, Proposition 1] that

$$R\{X\}_\mathfrak{M} = R\{X\}_{\mathfrak{m}\{X\}} \cong R_\mathfrak{m}(X)$$

is a regular local ring too. Therefore, $R\{X\}$ is a regular ring by [21, Theorem 5.94].

Conversely, assume that $R\{X\}$ is a Noetherian regular ring and let $\mathfrak{m} \in w$-Max(R). Then $\mathfrak{m}\{X\}$ is a maximal ideal of $R\{X\}$, and so

$$R_\mathfrak{m}(X) \cong R\{X\}_{\mathfrak{m}\{X\}}$$

is a regular local ring. Thus, also by [11, Proposition 1], $R_\mathfrak{m}$ is regular too. Hence, by Theorem 5.4, R is w-regular. □

We know that a Noetherian regular ring does not necessarily have finite global dimension. So it is natural to ask whether the global weak w-projective dimension of a w-Noetherian w-regular ring is finite. We have seen that w-Noetherian w-regular domains are exactly the Krull domains, and so a w-Noetherian w-regular domain must have global weak w-projective at most one. In the rest of this section, we will show that this is true for arbitrary w-Noetherian w-regular rings. This result will be obtained by proving that any w-Noetherian w-regular ring is the direct sum of a finite number of Krull domains.

Lemma 5.9. *Let I, J and K be ideals of R with $(I+J)_w = R$ and $(I+K)_w = R$. Then $(I + (J \cap K))_w = R$.*

Proof. Choose $A_1, A_2 \in \mathrm{GV}(R)$ with $A_1 \subseteq I + J$ and $A_2 \subseteq I + K$. Then

$$A_1A_2 \subseteq (I + J)(I + K) \subseteq I + JK \subseteq I + (J \cap K).$$

Since $A_1A_2 \in \mathrm{GV}(R)$, we obtain $(I + (J \cap K))_w = R$. □

Lemma 5.10. *Let I and J be w-ideals of R with $(I + J)_w = R$ and $IJ = 0$. Then $R = I \bigoplus J$.*

Proof. For any maximal w-ideal $\mathfrak{m}$ of R, $(I+J)_w = R$ implies that $I_\mathfrak{m} + J_\mathfrak{m} = R_\mathfrak{m}$, and so

$$(IJ)_\mathfrak{m} = I_\mathfrak{m} J_\mathfrak{m} = I_\mathfrak{m} \cap J_\mathfrak{m} = (I \cap J)_\mathfrak{m}.$$

Note that both $I \cap J$ and $I \bigoplus J$ are w-ideals of R. Hence, $I \cap J = (IJ)_w = 0$, whence

$$R = (I+J)_w = (I \bigoplus J)_w = I \bigoplus J. \qquad \square$$

Proposition 5.11. *The following statements are equivalent for R.*

(1) *Every principal ideal of R is w-flat.*
(2) *$R_\mathfrak{p}$ is a domain for all prime w-ideals $\mathfrak{p}$ of R.*
(3) *$R_\mathfrak{m}$ is a domain for all $\mathfrak{m} \in w\text{-Max}(R)$.*
(4) *R is reduced and every maximal w-ideal $\mathfrak{m}$ of R contains a unique minimal prime ideal $\mathfrak{q}$ of R. In this case,*

$$\mathfrak{q} = \{r \in R \mid ur = 0 \text{ for some } u \in R \backslash \mathfrak{m}\}$$

and $R_\mathfrak{q}$ is the field of quotients of $R_\mathfrak{m}$.

Proof. The proof is similar to that of [12, Theorem 4.2.2]. $\square$

The proposition below collects some properties of the w torsion theories of finite products of rings.

Proposition 5.12. *Let $R = R_1 \times R_2$ be a product decomposition of rings. Then*:

(1) *Let $J = J_1 \times J_2$ be an ideal of R. Then $J \in \mathrm{GV}(R)$ if and only if $J_i \in \mathrm{GV}(R_i)$ for $i = 1, 2$.*
(2) *Let $I = I_1 \times I_2$ be an ideal of R. Then*:
   (a) *I is a w-finitely generated ideal of R if and only if I_i is w-finitely generated of R_i for $i = 1, 2$.*
   (b) *I is a w-ideal of R if and only if I_i is a w-ideal of R_i for $i = 1, 2$.*
(3) *R is w-Noetherian if and only if so are R_1 and R_2.*
(4) $w\text{-Max}(R) = \{\mathfrak{m}_1 \times R_1 \mid \mathfrak{m}_1 \in w\text{-Max}(R_1)\} \bigcup \{R_1 \times \mathfrak{m}_2 \mid \mathfrak{m}_2 \in w\text{-Max}(R_2)\}$.
(5) *Let $M = M_1 \times M_2$ be an R-module. Then*

$$w\text{-fd}_R(M) = \sup\{w\text{-fd}_{R_1}(M_1), w\text{-fd}_{R_2}(M_2)\}.$$

(6) $w\text{-w.gl.dim}(R) = \sup\{w\text{-w.gl.dim}(R_1), w\text{-w.gl.dim}(R_2)\}$.
(7) *R is w-Noetherian and w-regular if and only if so are R_1 and R_2.*

Proof. (1) See [42, Proposition 1.2(5)].

(2) By using (1), this proof is straightforward.

(3) This follows immediately from (2)(a).

(4) Let $\mathfrak{m} = \mathfrak{m}_1 \times \mathfrak{m}_2 \in w\text{-Max}(R)$. Then

$$\mathfrak{m} = \mathfrak{m}_1 \times \mathfrak{m}_2 \subseteq R_1 \times \mathfrak{m}_2 \subseteq R = R_1 \times R_2.$$

But by (2)(b), $R_1 \times \mathfrak{m}_2$ is a w-ideal of R. Hence, we have either $\mathfrak{m}_1 = R_1$ or $\mathfrak{m}_2 = R_2$. On the other hand, if $\mathfrak{m}_i \in w\text{-Max}(R_i)$ for $i = 1, 2$, then it follows easily from (2)(b) that both $\mathfrak{m}_1 \times R_2$ and $R_1 \times \mathfrak{m}_2$ are maximal w-ideals of R.

(5) First, note that if S_1 is a multiplicatively closed set of R_1, then $S := S_1 \times R_2$ is a mulitplicatively closed set of R and $M_S \cong (M_1)_{S_1}$. Therefore, it follows from [37, Proposition 2.4] and (4) that

$$\begin{aligned} w\text{-fd}_R(M) &= \sup\{\text{fd}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in w\text{-Max}(R)\} \\ &= \sup\{\text{fd}_{(R_i)_{\mathfrak{m}_i}}((M_i)_{\mathfrak{m}_i}) \mid \mathfrak{m}_i \in w\text{-Max}(R_i),\ i = 1, 2\} \\ &= \sup\{w\text{-fd}_{R_1}(M_1), w\text{-fd}_{R_2}(M_2)\}. \end{aligned}$$

(6) It follows immediately from (5).

(7) This follows at once from (3), [38, Proposition 3.4(2)], and (5). □

Theorem 5.13. *Every w-Noetherian w-regular ring is the direct sum of a finite number of Krull domains.*

Proof. Let R be a w-Noetherian w-regular ring. By [35, Corollary 6.8.22], R has only a finite number of minimal prime ideals, say $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Now, we prove this result by induction on n.

If $n = 1$, then the nilradical $nil(R)$ of R is $\mathfrak{p}_1$. But Proposition 5.11 implies that R is reduced, that is, $nil(R) = \mathfrak{p}_1 = 0$. Thus, R is a domain, and so it is a Krull domain by Proposition 5.1.

If $n > 1$, then we have $(\mathfrak{p}_i + \mathfrak{p}_j)_w = R$ for $i \neq j$. Otherwise, there exists some $\mathfrak{m}_0 \in w\text{-Max}(R)$ such that

$$\mathfrak{p}_i, \mathfrak{p}_j \subseteq \mathfrak{p}_i + \mathfrak{p}_j \subseteq (\mathfrak{p}_i + \mathfrak{p}_j)_w \subseteq \mathfrak{m}_0,$$

which contradicts the fact that every maximal w-ideal of R contains a unique minimal prime ideal (see Proposition 5.11). By repeated use of Lemma 5.9 we obtain $(\mathfrak{p}_1 + I)_w = R$, where $I = \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$. Also, note that $\mathfrak{p}_1 I \subseteq \mathfrak{p}_1 \cap I = nil(R) = 0$. Therefore, by Lemma 5.10, $R = \mathfrak{p}_1 \bigoplus I$. So it follows easily from Proposition 5.12 that $I \cong R/\mathfrak{p}_1$ is a Krull domain, and that $\mathfrak{p}_1 \cong R/I$ is a w-Noetherian w-regular ring with $n-1$ minimal prime ideals. By induction, $\mathfrak{p}_1$ as a ring is the direct sum of a finite number of Krull domains. Consequently, this gives the desired result. □

Corollary 5.14. *Let R be a w-Noetherian ring. Then the following statements are equivalent.*

(1) *R is a w-regular ring.*
(2) $\text{gl.w.}w\text{-dim}(R) \leq 1$.
(3) $\text{gl.w.}w\text{-dim}(R) < \infty$.

Proof. This follows immediately from Theorem 5.13, [38, Proposition 3.5(2)], Proposition 5.12(6) and the fact that every Krull domain has the global weak w-projective dimension at most one (see [38, Theorem 4.3]). □

In [23, Example 3.15], Mimouni gave an example of an SM domain R which is not a Krull domain. In any such example, we must have $\text{gl.w.}w\text{-dim}(R) = \infty$ by Corollary 5.14.

Corollary 5.15. *The global weak w-projective dimension of a w-Noetherian ring is* 0, 1, *or* ∞.

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