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SHARP ORE-TYPE CONDITIONS FOR THE EXISTENCE OF AN EVEN $[4, b]$-FACTOR IN A GRAPH

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Abstract. Let a and b be positive even integers. An even $[a, b]$-factor of a graph G is a spanning subgraph H such that for every vertex $v \in V(G)$, $d_H(v)$ is even and $a \le d_H(v) \le b$. Let $\kappa(G)$ be the minimum size of a vertex set S such that $G - S$ is disconnected or one vertex, and let $\sigma_2(G) = \min_{uv \notin E(G)}(d(u)+d(v))$. In 2005, Matsuda proved an Ore-type condition for an n-vertex graph satisfying certain properties to guarantee the existence of an even $[2, b]$-factor. In this paper, we prove that for an even positive integer b with $b \ge 6$, if G is an n-vertex graph such that $n \ge b + 5$, $\kappa(G) \ge 4$, and $\sigma_2(G) \ge \frac{8n}{b+4}$, then G contains an even $[4, b]$-factor; each condition on n, $\kappa(G)$, and $\sigma_2(G)$ is sharp.

1. Introduction

Throughout the paper, a graph G is finite, simple, and undirected with the set of vertices $V(G)$ and the set of edges $E(G)$. Given a vertex $v \in V(G)$, the *degree* of v in G, written $d_G(v)$ (or $d(v)$ if G is clear from the context), is the number of edges in $E(G)$ incident to v. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Also, $\sigma_2(G) = \min_{uv \notin E(G)}(d_G(u) + d_G(v))$.

Given $S, T \subseteq V(G)$, $G[S]$ denotes the graph induced by S in a graph G, $G - S$ denotes the graph induced by $V(G) - S$ in a graph G, $[S, T]$ denotes the set of edges joining S and T, and $d_G(S) = \sum_{v \in S} d_G(v)$.

The *complement* of a graph G, denoted by $\overline{G}$, is a graph whose vertex set is $V(G)$ and two distinct vertices are adjacent if and only if they are not adjacent in G.

Let f and g be integer valued functions defined on $V(G)$ such that $g(v) \le f(v)$ for all $v \in V(G)$. A (g, f)*-factor* of a graph G is a spanning subgraph F of G such that $g(v) \le d_F(v) \le f(v)$ for all $v \in V(G)$. Let a and b be nonnegative

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integers such that $a \leq b$. Then an *$[a,b]$-factor* of G is a (g,f)-factor of G where $g(v)=a$ and $f(v)=b$ for all $v \in V(G)$. If $a=b$, then we call it an *a-factor*.

For even nonnegative integers a and b, an *even $[a,b]$-factor* of a graph is an $[a,b]$-factor such that $d_H(v)$ is even for all $v \in V(G)$.

A graph G is *k-edge-connected* if $S \subseteq E(G)$ with $|S| < k$, then $G - S$ is connected. The *edge-connectivity* of G, denoted $\kappa'(G)$, is the maximum k such that G is k-edge-connected. A graph G is *k-vertex-connected* if $|V(G)| \geq k+1$ and $S \subseteq V(G)$ with $|S| < k$, then $G - S$ is connected. The *vertex-connectivity* of G, denoted $\kappa(G)$, is the maximum k such that G is k-vertex-connected.

A *Hamiltonian cycle* in a graph is a cycle visiting all the vertices of the graph. Ore proved a sufficient condition related to $\sigma_2(G)$ (which is now called the Ore-type condition) for a graph G to have a Hamilton cycle.

Theorem 1.1 ([17]). *For $n \geq 3$, if G is an n-vertex graph with $\sigma_2(G) \geq n$, then G contains a Hamiltonian cycle.*

Note that a Hamiltonian cycle is a special kind of a 2-factor of G. Fleischner [3] proved that if $\kappa'(G) \geq 2$ and $\delta(G) \geq 3$, then G has an even factor. Kouider and Vestaargard extended his result and in fact explored sufficient conditions for a graph to have an even $[a,b]$-factor.

Theorem 1.2 ([8,9]). *Let a and b be even integers such that $2 \leq a < b$, and let G be an n-vertex 2-edge-connected graph. Then the following holds.*

(1) *If $a=2$, $n \geq 3$, and $\delta(G) \geq \max\{3, \frac{2n}{b+2}\}$, then G has an even $[2,b]$-factor.*
(2) *If $a \geq 4$, $n \geq \frac{(a+b)^2}{b}$, and $\delta(G) \geq \frac{an}{a+b} + \frac{a}{2}$, then G has an even $[a,b]$-factor.*
(3) *If $a \geq 4$, $n \geq \frac{(a+b)^2}{b}$, $\kappa'(G) \geq a + \min\{\sqrt{a}, \frac{b}{a}\}$, and $\delta(G) \geq \frac{an}{a+b}$, then G has an even $[a,b]$-factor.*
(4) *If $a \geq 4$, $n \geq \max\{\frac{(a+b)^2}{b}, \frac{3(a+b)}{2}\}$ and $\delta(G) \geq \frac{an}{a+b}$, then G has an even $[a,b]$-factor.*

Note that each statement contains a condition on the minimum degree of a graph. In 2005, Matsuda improved the sufficient condition for a graph G to have an even $[2,b]$-factor by giving a condition on $\sigma_2(G)$ instead of $\delta(G)$.

Theorem 1.3 ([14]). *Let $b \geq 2$ be an even integer. If G is an n-vertex 2-edge-connected graph such that $n \geq b+3$ and $\sigma_2(G) \geq \frac{4n}{b+2}$, then G has an even $[2,b]$-factor.*

In the same paper, he also proposed a conjecture on a generalization of the above theorem, which gives sufficient conditions for a graph to have an even $[a,b]$-factor.

Conjecture 1.4 ([14]). *Let a and b be even integers such that $2 \leq a \leq b$. If G is an n-vertex 2-edge-connected graph such that $n \geq 2a + b + \frac{a^2-3a}{b} - 2$, $\delta(G) \geq a$, and $\sigma_2(G) \geq \frac{2an}{a+b}$, then G has an even $[a,b]$-factor.*

Theorem 1.3 says that Conjecture 1.4 is true when $a = 2$ if n is slightly bigger than $b+2-\frac{2}{b}$. Note that the lower bound on n in Theorem 1.3 is best possible (see Section 5 in [14]). In fact, if $n = b+2$, then we need 5, a slightly bigger than $4 = \frac{2\times 2(b+2)}{2+b}$ on σ_2 to have an even $[2, b]$-factor. Furthermore, Conjecture 1.4 is not true for $a \geq 4$ (see [2]). In fact, there are graphs with edge-connectivity or vertex-connectivity with $a-1$ satisfying the other conditions in Conjecture 1.4, which implies that we need high connectivity (at least a) to have an even $[a, b]$-factor in a graph ([2]). In this paper, we prove the conjecture for $a = 4$ by considering the vertex-connectivity at least 4 in the conjecture.

Theorem 1.5 (Main Theorem)**.** *Let $b \geq 6$ be an even integer. If G is an n-vertex graph such that* (i) $n \geq b+5$, (ii) $\kappa(G) \geq 4$, *and* (iii) $\sigma_2(G) \geq \frac{8n}{4+b}$, *then G has an even $[4, b]$-factor.*

Note that each condition in Theorem 1.5 is sharp (see Section 3). If we relax the condition on the minimum degree as well as the number of vertices, then $\sigma_2(G) \geq \frac{2an}{a+b}$ can be replaced by $\max\{d(u), d(v)\} \geq \frac{an}{a+b}$ for non-adjacent vertices u and v [10]. This implies that it is hard to prove the best lower bound for the number of vertices in an n-vertex graph to guarantee the existence of even $[a, b]$-factor. By checking the proof of Theorem 1.5, when n is close to $b+5$, we recognize that we need to handle separately from when n is big enough. Note that if we replace a by 4 on the lower bound for the number of vertices in Conjecture 1.4, then we have $n \geq b+6+\frac{4}{b}$, which is bigger than $b+5$ in Theorem 1.5.

To prove Theorem 1.5, we use Corollary 1.7, which is a simple application of Lovasz's parity (g, f)-factor theory.

Theorem 1.6 (Lovasz's parity (g, f)-factor theory [11])**.** *Let G be a graph and let g, f be two integer valued functions defined on $V(G)$ such that $0 \leq g(v) \leq f(v) \leq d_G(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. Then G has a (g, f)-factor F such that $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$ if and only if*

$$\sum_{v\in T}(d(v)-g(v)) + \sum_{u\in S} f(u) - |[S,T]| - q(S,T) \geq 0$$

for all disjoint subsets S and T of $V(G)$, where $q(S,T)$ is the number of components Q of $G-(S\cup T)$ such that

$$|[V(Q),T]| + \sum_{v\in V(Q)} f(v) \equiv 1 \pmod 2.$$

By applying Theorem 1.6 when $g(x) = a$ and $f(x) = b$, we obtain Corollary 1.7.

Corollary 1.7. *Let a and b be even integers with $2 \leq a \leq b$. A graph G has an even $[a, b]$-factor if and only if*

$$\delta(S,T) := q(S,T) - b|S| + a|T| - \sum_{v\in T} d_{G-S}(v) \leq 0$$

for all disjoint subsets S and T of $V(G)$, where $q(S,T)$ is the number of components Q of $G-(S\cup T)$ such that $|[V(Q),T]|$ is odd.

For convenience, let $\mathcal{Q}$ be the set of all the components Q of $G-(S\cup T)$ such that $|[V(Q),T]|$ is odd. For i an odd integer, let q_i be the number of components $Q\in\mathcal{Q}$ such that $|[V(Q),T]|=i$, and $q_{\geq i}$ be the number of components $Q'\in\mathcal{Q}$ such that $|[V(Q'),T]|\geq i$.

The parity lemma is also used in the proof of our main result.

Lemma 1.8 (Parity lemma)**.** *Let a and b be positive integers with the same parity. Then $\delta(S,T)$ has the same parity as a and b for any disjoint sets $S,T\subseteq V(G)$.*

For undefined terms, see West [18].

2. Proof of Theorem 1.5

In this section, we prove Theorem 1.5 with Claims 1, 2, 3, and 4. In particular, Claims 3 and 4, which are the essential tools to prove Theorem 1.5, results from utilizing Condition (iii).

Proof of Theorem 1.5. We prove the theorem by a contradiction. Assume to the contrary that G is a graph that satisfies the conditions of Theorem 1.5, and has no even $[4,b]$-factor. By Corollary 1.7 and Lemma 1.8, there exist disjoint $S,T\subseteq V(G)$ such that

$$\delta(S,T)=q(S,T)-b|S|+4|T|-\sum_{v\in T}d_{G-S}(v)\geq 2.$$

Among such sets S and T, we choose T so that $|T|$ is minimal.

Let $s=|S|$, $t=|T|$, and $p=-bs+4t$. For convenience, we often use q instead of $q(S,T)$ when there is no confusion.

Observation 1. If $s=0$, then $q=q_{\geq 5}$. If $1\leq s\leq 2$, then $q=q_{\geq 3}$. These follows from $\kappa(G)\geq 4$.

Observation 2. $\delta(G)\geq 4$. This follows from $\kappa(G)\geq 4$.

Observation 3. $\sigma_2(G)\geq 9$. This follows from $\sigma_2(G)\geq\frac{8n}{b+4}$ and $n\geq b+5$.

Claim 1. $p\geq 2$.

Proof. Note that $q\leq d_{G-S}(T)$. This implies $p=-bs+4t=\delta(S,T)-q+d_{G-S}(T)\geq 2$ as $\delta(S,T)\geq 2$. □

Observation 4. $t\geq 1$. This follows from Claim 1.

Claim 2. For any vertex $v\in T$, we have $d_{G[T]}(v)\leq 2$.

Proof. Let $v \in T$ and $T' = T - v$. By the choice of T and Corollary 1.7, we have $\delta(S, T') \leq 0$. Therefore, we have

$$2 \leq \delta(S, T) - \delta(S, T') \leq q(S, T) - q(S, T') + 4 - d_{G-S}(v).$$

Then we have $d_{G-S}(v) \leq q(S,T) - q(S,T') + 4 - 2 \leq |[\{v\}, V(G) - (S \cup T)]| + 2$. Consequently, we have $d_{G[T]}(v) = d_{G-S}(v) - |[\{v\}, V(G) - (S \cup T)]| \leq 2$. □

To prove Claim 3, we use Theorem 1.1.

Claim 3. If $t \geq 6$, then $d_G(T) \geq \frac{4n}{b+4} t$.

Proof. For any $u, v \in T$, we have

$$\begin{aligned} d_{\overline{G[T]}}(u) + d_{\overline{G[T]}}(v) &= \left(t - 1 - d_{G[T]}(u)\right) + \left(t - 1 - d_{G[T]}(v)\right) \\ &= 2(t-1) - \left(d_{G[T]}(u) + d_{G[T]}(v)\right). \end{aligned}$$

By Claim 2, we have $2(t-1) - \left(d_{G[T]}(u) + d_{G[T]}(v)\right) \geq 2t - 2 - 4 \geq t$ as $t \geq 6$. By Theorem 1.1, $\overline{G[T]}$ contains a Hamiltonian cycle, say $(v_1, \ldots, v_t)$. Without loss of generality, we may assume that $d_G(v_1) \geq d_G(v_i)$ for all $i \in [t]$. Note that v_i and v_{i+1} are not adjacent in $G[T]$ for all $i \in [t-1]$. Thus we have $d_G(v_1) \geq \frac{4n}{b+4}$ by Condition (iii).

If t is even, then by Condition (iii), we have

$$\begin{aligned} d_G(T) &= (d_G(v_1) + d_G(v_2)) + \cdots + (d_G(v_{t-1}) + d_G(v_t)) \\ &\geq \frac{8n}{4+b} \cdot \frac{t}{2} = \frac{4n}{4+b} t. \end{aligned}$$

If t is odd, then by Condition (iii), we have

$$\begin{aligned} d_G(T) &= d_G(v_1) + (d_G(v_2) + d_G(v_3)) + \cdots + (d_G(v_{t-1}) + d_G(v_t)) \\ &\geq \frac{4n}{4+b} + \frac{8n}{4+b} \cdot \frac{t-1}{2} = \frac{4n}{4+b} t. \end{aligned}$$

□

Claim 4. If $t \geq 4$, then $d_G(T) \geq 5t - 3$.

Proof. Note that $\delta(G) \geq 4$. If there are $v_1, v_2, v_3, v_4 \in T$ such that $d_G(v_i) = 4$ for all $i \in \{1, 2, 3, 4\}$, then all v_i's should be adjacent to each other since $\sigma_2(G) \geq 9$. This implies that $d_{G[T]}(v_i) \geq 3$ for all $i \in \{1, 2, 3, 4\}$, contradicting Claim 2.

Therefore, there are at most three vertices in T that have degree 4 in G, and we obtain $d_G(T) \geq 5(t-3) + 4 \cdot 3 = 5t - 3$. □

Claim 5. If $4 \leq t \leq b$ and $s \geq 1$, then $q \geq b - 1$.

Proof. By Claim 4, we get

$$\begin{aligned} q &\geq 2 + bs - 4t + d_G(T) - st \\ &\geq 2 + bs + (1-s)t - 3 \\ &\geq b - 1. \end{aligned}$$

□

Claim 6. $s \leq \frac{4(n-q)-p}{4+b}$.

Proof. Note that $t \leq n - s - q$. This gives us

$$s = \frac{4t-p}{b} \leq \frac{4(n-s-q)-p}{b},$$

which implies

$$s \leq \frac{4(n-q)-p}{4+b}.$$

□

Let $f(t) = \frac{-3bn}{b+4} + 3t + (t-b-1)\left(\frac{-4q-2}{b+4}\right)$.

Claim 7. If $t \geq b+1$, then $\delta(S,T) \leq f(t)$.

Proof. By Claim 1, Claim 3 and Claim 6, we have

$$\begin{aligned}\delta(S,T) &\leq q - bs + 4t - d_G(T) + st \\ &\leq (n-s-t) - bs + 4t - \frac{4n}{b+4}t + st \\ &\leq \left(1 - \frac{4t}{b+4}\right) n + 3t + (t-b-1)\left(\frac{4(n-q)-2}{b+4}\right) \\ &= \frac{-3bn}{b+4} + 3t + (t-b-1)\left(\frac{-4q-2}{b+4}\right) = f(t).\end{aligned}$$

□

Now, we consider four cases depending on the value of t to prove Theorem 1.5.

Case 1: $t \geq b+4$. With Claim 3 and Claim 6, we have

$$\begin{aligned}2 &\leq \delta(S,T) \\ &= q - bs + 4t - d_G(T) + |[S,T]| \\ &\leq q - bs + 4t - d_G(T) + st \\ &\leq q - bs + 4t - \frac{4n}{b+4}t + \frac{4(n-q)-p}{b+4}t \\ &\leq q - bs + 4t - 4q - p \\ &= -3q + (-bs + 4t - p) \\ &= -3q \leq 0,\end{aligned}$$

which is a contradiction.

Case 2: $t = b+3$.

Case 2-1: $n \geq b+8$. With Claim 7, as $b \geq 6$, we have

$$2 \leq \delta(S,T) \leq f(t) = f(b+3) = \frac{-3bn}{b+4} + 3(b+3) + \frac{-8q-4}{b+4} \leq -3 + \frac{44-8q}{b+4} < 2,$$

which is a contradiction.

Case 2-2: $n = b + 7$. With Claim 3, as $b \geq 6$, we have

$$d_G(T) \geq \frac{4n}{b+4}t = \frac{4(b+7)}{b+4}(b+3) = 4b + 24 - \frac{12}{b+4} \geq 4b + 22.8,$$

which implies

$$d_G(T) \geq 4b + 23.$$

Note that $s \leq n - t - q = 4 - q$ implies that

$$\begin{aligned} d_G(T) &\leq q - bs + 4t + st - 2 \\ &= q - bs + 4(b+3) + s(b+3) - 2 \\ &= q + 4b + 3s + 10 \\ &\leq 4b - 2q + 22. \end{aligned}$$

Combining above inequalities, we obtain $4b - 2q + 22 \geq 4b + 23$, a contradiction since $q \geq 0$.

Case 2-3: $n = b + 6$. With Claim 3, as $b \geq 6$, we have

$$d_G(T) \geq \frac{4n}{b+4}t = \frac{4(b+6)}{b+4}(b+3) = 4b + 20 - \frac{8}{b+4} \geq 4b + 19.2,$$

which implies

$$d_G(T) \geq 4b + 20.$$

Note that $s \leq n - t - q = 3 - q$ implies that

$$\begin{aligned} d_G(T) &\leq q - bs + 4t + st - 2 \\ &= q - bs + 4(b+3) + s(b+3) - 2 \\ &= q + 4b + 3s + 10 \\ &\leq 4b - 2q + 19. \end{aligned}$$

Combining above inequalities, we obtain $4b - 2q + 19 \geq 4b + 20$, a contradiction since $q \geq 0$.

Case 2-4: $n = b + 5$. With Claim 4, as $b \geq 6$, we have

$$d_G(T) \geq 5t - 3 = 5(b+3) - 3 = 5b + 12.$$

Note that $s \leq n - t - q = 2 - q$ implies that

$$\begin{aligned} d_G(T) &\leq q - bs + 4t + st - 2 \\ &= q - bs + 4(b+3) + s(b+3) - 2 \\ &= q + 4b + 3s + 10 \\ &\leq 4b - 2q + 16. \end{aligned}$$

Thus we obtain $4b - 2q + 16 \geq 5b + 12$, which is a contradiction since $b \geq 6$ and $q \geq 0$.

Case 3: $t = b + 2$.

Case 3-1: $n \geq b+7$. With Claim 7, as $b \geq 6$, we have

$$2 \leq \delta(S,T) \leq f(t) = f(b+2) = \frac{-3bn}{b+4} + 3(b+2) + \frac{-4q-2}{b+4} \leq -3 + \frac{34-4q}{b+4} < 2,$$

which is a contradiction.

Case 3-2: $n = b+6$. With Claim 3, as $b \geq 6$, we have

$$d_G(T) \geq \frac{4n}{b+4}t = \frac{4(b+6)}{b+4}(b+3) = 4b+16 - \frac{16}{b+4} \geq 4b+14.4,$$

which implies

$$d_G(T) \geq 4b+15.$$

Note that $s \leq n-t-q = 4-q$ implies that

$$\begin{aligned} d_G(T) &\leq q - bs + 4t + st - 2 \\ &= q - bs + 4(b+2) + s(b+2) - 2 \\ &= q + 4b + 2s + 6 \\ &\leq 4b - q + 14. \end{aligned}$$

Combining above inequalities, we obtain $4b-q+14 \geq 4b+15$, a contradiction since $q \geq 0$.

Case 3-3: $n = b+5$. With Claim 4, as $b \geq 6$, we have

$$d_G(T) \geq 5t - 3 = 5(b+2) - 3 = 5b+7.$$

Note that $s \leq n-t-q = 3-q$ implies that

$$\begin{aligned} d_G(T) &\leq q - bs + 4t + st - 2 \\ &= q - bs + 4(b+2) + s(b+2) - 2 \\ &= q + 4b + 2s + 6 \\ &\leq 4b - q + 12. \end{aligned}$$

Combining above inequalities, we obtain $4b-q+12 \geq 5b+7$, a contradiction since $b \geq 6$ and $q \geq 0$. □

Case 4: $t = b+1$.

Case 4-1: $n \geq b+6$. With Claim 7, as $b \geq 6$, we have

$$2 \leq \delta(S,T) \leq f(t) = f(b+1) = \frac{-3bn}{b+4} + 3(b+1) \leq -3 + \frac{24}{b+4} < 0,$$

which is a contradiction.

Case 4-2: $n = b+5$. With Claim 4, as $b \geq 6$, we have

$$d_G(T) \geq 5t - 3 = 5(b+1) - 3 = 5b+2.$$

Note that $s \leq n-t-q = 4-q$ implies that

$$\begin{aligned} d_G(T) &\leq q - bs + 4t + st - 2 \\ &= q - bs + 4(b+1) + s(b+1) - 2 \end{aligned}$$

$$= q + 4b + s + 2$$
$$\leq 4b + 6.$$

Combining above inequalities, we obtain $4b + 6 \geq 5b + 2$, a contradiction since $b \geq 6$.

By Cases from 1 to 4, we have $t \leq b$. Thus, by Claim 1, we have $s \leq \frac{4t-2}{b} < 4$.

Case 5: $4 \leq t \leq b$.

Case 5-1: $s = 0$. By Observation 1, we have $d_G(T) \geq 5q$ which implies

$$2 \leq q - bs + 4t - d_G(T) \leq -4q + 4t$$

so that $q \leq t - 1$.

Also, by Claim 4, we have

$$2 \leq q - bs + 4t - d_G(T) \leq q - t + 3$$

so that $q \geq t - 1$. Therefore, $q = t - 1$, and this implies

$$d_G(T) \leq q - bs + 4t - 2 = 5t - 3.$$

With this inequality and by Claim 4, we obtain $d_G(T) = 5t - 3$.

Let $W = \bigcup_{Q \in \mathcal{Q}} Q$. Since $|[V(W), T]| = \sum_{Q \in \mathcal{Q}} |[V(Q), T]| \geq 5q = 5t - 5$ and $d_G(T) = 5t - 3$, we have

$$d_{G[T]}(T) \leq d_G(T) - |[V(W), T]| \leq 2,$$

implying that $|E(G[T])| \leq 1$. Since $\sigma_2(G) \geq 9$, at most two vertices in T have degree 4, yielding

$$d_G(T) \geq 5(t - 2) + 4 \cdot 2 = 5t - 2,$$

contradicting $d_G(T) = 5t - 3$. □

Case 5-2: $s = 1$. By Observation 1, we have $d_{G-S}(T) \geq 3q_3 + 5(q - q_3) = 5q - 2q_3$. With this inequality and Claim 5, we have

$$2 \leq q - bs + 4t - d_{G-S}(T) \leq -4q + 2q_3 + 3b \leq 2q_3 - b + 4,$$

so that $q_3 \geq \frac{b-2}{2}$.

Let $\{Q_1, \ldots, Q_{q_3}\} \subseteq \mathcal{Q}$ such that $|[V(Q_i), T]| = 3$ for all $i \in \{1, \ldots, q_3\}$. Note that for $i \in \{1, \ldots, q_3\}$ and $u \in V(Q_i)$, we have $d_G(u) \leq (|V(Q_i)| - 1) + 1 + 3$, and the equality is achieved only if $V(Q_i) = \{u\}$ as $\kappa(G) \geq 4$ and $|[V(Q_i), T]| = 3$. In this case, $d_G(u) \leq 4$.

Let $u \in V(Q_1)$ and $v \in V(Q_2)$. Then we have

$$\begin{aligned}\frac{8n}{b+4} &\leq d_G(u) + d_G(v)\\ &\leq (|V(Q_1)| - 1 + 1 + 3) + (|V(Q_2)| - 1 + 1 + 2)\\ &\leq |V(Q_1)| + |V(Q_2)| + 5\end{aligned}$$

so that $|V(Q_1)| + |V(Q_2)| \geq \frac{8n}{b+4} - 5$ and $|V(Q_2)| \geq \frac{4n}{b+4} - \frac{5}{2}$. Therefore

$$\begin{aligned}\sum_{i=1}^{q_3} |V(Q_i)| &\geq |V(Q_1)| + |V(Q_2)| + (q_3 - 2)|V(Q_2)| \\ &\geq \left(\frac{8n}{b+4} - 5\right) + \left(\frac{4n}{b+4} - \frac{5}{2}\right)(q_3 - 2) = \left(\frac{4n}{b+4} - \frac{5}{2}\right) q_3.\end{aligned}$$

Since $d_{G-S}(T) \geq 3q_3 + 5q_{\geq 5}$, we have

$$\begin{aligned}2 &\leq q - bs + 4t - d_{G-S}(T) \\ &\leq (q_3 + q_{\geq 5}) - b + 4t - (3q_3 + 5q_{\geq 5}) \\ &\leq -4q_{\geq 5} - 2q_3 - b + 4t,\end{aligned}$$

which yields $t \geq \frac{4q_{\geq 5} + 2q_3 + b + 2}{4}$.

If $b \geq 8$, then combining all the above results, and noting that $q_{\geq 5} = q - q_3 \geq (b-1) - q_3$, we have

$$\begin{aligned}n &\geq s + t + \sum_{i=1}^{q_3} |V(Q_i)| + q_{\geq 5} \\ &\geq 1 + \left(\frac{4q_{\geq 5} + 2q_3 + b + 2}{4}\right) + \left(\frac{4n}{b+4} - \frac{5}{2}\right) q_3 + q_{\geq 5} \\ &\geq \left(\frac{4n}{b+4} - 2\right) q_3 + 2(b - 1 - q_3) + \frac{b+6}{4} \\ &\geq \left(\frac{4n}{b+4} - 4\right)\left(\frac{b-2}{2}\right) + 2(b-1) + \frac{b+6}{4} \\ &= n + \frac{(b-8)n}{b+4} + \frac{b+14}{4} > n,\end{aligned}$$

which is a contradiction. Therefore, $b = 6$. Since $d_{G-S}(T) \geq 3q$, we have

$$2 \leq q - bs + 4t - d_{G-S}(T) \leq -2q - 6 + 4t,$$

resulting $q \leq 2t - 4$. This implies

$$d_G(T) \leq q - bs + 4t + st - 2 = q + 5t - 8 \leq 7t - 12.$$

Case 5-2-1: $n \geq b + 7$. By the same argument as in Observation 3, we have $\sigma_2(G) \geq 11$. Thus, by the argument as in Claim 4, there can be at most three vertices in T that have degree at most 5 in G. If T has a vertex of degree 4 in G, then

$$d_G(T) \geq 4 \cdot 3 + 7(t - 3) = 7t - 9,$$

and otherwise

$$d_G(T) \geq 5 \cdot 3 + 6(t - 3) = 6t - 3.$$

Since $t \leq b = 6$, we have $d_G(T) \geq 7t - 9$.

Therefore, we have $7t - 9 \leq d_G(T) \leq 7t - 12$, which is a contradiction.

Case 5-2-2: $n = b + 6$. By the same argument as in Observation 3, we have $\sigma_2(G) \geq 10$. Thus, by the argument as in Claim 4, there can be at most three vertices in T that have degree at most 4 in G. If T has a vertex of degree 4 in G, then

$$d_G(T) \geq 4 \cdot 3 + 6(t - 3) = 6t - 6,$$

and otherwise

$$d_G(T) \geq 5t.$$

Since $t \leq b = 6$, we have $d_G(T) \geq 6t - 6$.

Therefore, we have $6t - 6 \leq d_G(T) \leq 7t - 12$, which implies that $t \geq 6$. Thus we have $6 \leq t \leq b = 6$ so that $t = 6$ and $d_G(T) = 30$.

Now, notice that

$$\begin{aligned} 2 &\leq q - bs + 4t - d_G(T) + st \\ &= q - 6 + 24 - 30 + 6 = q - 6 \end{aligned}$$

implies that $q \geq 8$, and this implies $12 = b + 6 = n \geq s + t + q \geq 15$, which is a contradiction.

Case 5-2-3: $n = b + 5$. By Claim 4, we have $5t - 3 \leq d_G(T) \leq 7t - 12$, which implies that $t \geq 5$. Also, note that by Claim 5, $q \geq b - 1 = 5$. Thus,

$$11 = b + 5 = n \geq s + t + q \geq 11$$

so that the equality holds for each inequality above, and $t = q = 5$. In particular, every component in $\mathcal{Q}$ has exactly one vertex. Let $S = \{s_1\}$, $T = \{t_1, t_2, t_3, t_4, t_5\}$, and $Q_i = \{q_i\}$ for all $i \in \{1, 2, 3, 4, 5\}$ where $\mathcal{Q} = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$. Note that

$$d_{G-S}(T) \leq q - bs + 4t - 2 = 5 - 6 + 20 - 2 = 17$$

and

$$d_G(T) \geq 5t - 3 = 22$$

implies that $|[S, T]| = d_G(T) - d_{G-S}(T) \geq 22 - 17 = 5$. Since $s = 1$ and $t = 5$, the inequality is actually an equality. Thus we know that $d_G(T) = 22$, $d_{G-S}(T) = 17$, and s_1 is adjacent to t_i for all $i \in \{1, 2, 3, 4, 5\}$.

Note that $q = q_{\geq 3} = 5$ by Observation 1. Since $d_{G-S}(T) = 17$, we have either (i) $q_3 = 4$ and $q_5 = 1$ or (ii) $q_3 = 5$. Thus in each case, there are at most one edge in $G[T]$.

Without loss of generality, let t_1 be nonadjacent to t_i for all $i \in \{2, 3, 4, 5\}$. Since t_2 is not adjacent to t_1, without loss of generality, $q_1, q_2, q_3 \in N_G(t_2) \cap N_G(t_1)$ since $\kappa(G) \geq 4$. Without loss of generality, let $|[Q_1, T]| = |[Q_2, T]| = 3$. Without loss of generality, let $t_3 \in N_G(q_1)$ and $t_4, t_5 \notin N_G(q_1)$. Since t_3 is not adjacent to t_1, without loss of generality, $q_4 \in N_G(t_3) \cap N_G(t_1)$ since $\kappa(G) \geq 4$.

If t_4 and t_5 are not adjacent, then at least two among $\{q_2, q_3, q_4\}$ are in $N_G(t_4) \cap N_G(t_5)$ since $\kappa(G) \geq 4$, which results in at least two among $\{Q_2, Q_3, Q_4\}$ satisfies $|[Q_i, T]| \geq 4$, which is a contradiction to $q_5 \leq 1$.

Thus t_4 and t_5 are adjacent, and thus $q = q_3$. Then since t_4 is not adjacent to t_1, at least two among $\{q_2, q_3, q_4\}$ are in $N_G(t_4)$ since $\kappa(G) \geq 4$. Analogously, at least two among $\{q_2, q_3, q_4\}$ are in $N_G(t_5)$. Thus at least one among $\{q_2, q_3, q_4\}$ is in $N_G(t_4) \cap N_G(t_5)$. This means at least one among $\{Q_2, Q_3, Q_4\}$ satisfies $|[Q_i, T]| \geq 4$, which is a contradiction to $q_5 = 0$.

Case 5-3: $s = 2$. By Observation 1, we have $d_{G-S}(T) \geq 3q_3 + 5(q - q_3) = 5q - 2q_3$. With this inequality and Claim 5, we have

$$2 \leq q - bs + 4t - d_{G-S}(T) \leq -4q + 2q_3 - 2b + 4t \leq 2q_3 - 2b + 4,$$

so that $q_3 \geq b - 1$. With this and $d_{G-S}(T) \geq 3q_3 + 5q_{\geq 5}$, we have

$$\begin{aligned} 2 &\leq q - bs + 4t - d_{G-S}(T) \\ &\leq (q_3 + q_{\geq 5}) - 2b + 4t - (3q_3 + 5q_{\geq 5}) \\ &= -4q_{\geq 5} - 2q_3 - 2b + 4t \\ &\leq -4q_{\geq 5} - 2(b-1) - 2b + 4t \\ &= -4q_{\geq 5} - 4b + 4t + 2 \end{aligned}$$

which yields $t \geq q_{\geq 5} + b$. Since $t \leq b$, we get $t = b$ and $q_{\geq 5} = 0$.

Case 5-3-1: $n \geq b + 7$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$. Then $|[V(Q_i), T]| = 3$ for all $i \in \{1, \ldots, q\}$ by Observation 1 and $q_{\geq 5} = 0$. Assume $|V(Q_1)| \leq \cdots \leq |V(Q_q)|$. Let $u \in V(Q_1)$ and $v \in V(Q_2)$. Then we have

$$\begin{aligned} \frac{8n}{b+4} &\leq d_G(u) + d_G(v) \\ &\leq ((|V(Q_1)| - 1) + 2 + 3) + ((|V(Q_2)| - 1) + 2 + 3) \\ &\leq |V(Q_1)| + |V(Q_2)| + 8 \end{aligned}$$

so that $|V(Q_1)| + |V(Q_2)| \geq \frac{8n}{b+4} - 8$ and $|V(Q_2)| \geq \frac{4n}{b+4} - 4$. Therefore

$$\begin{aligned} \sum_{i=1}^{q_3} |V(Q_i)| &\geq |V(Q_1)| + |V(Q_2)| + (q-2)|V(Q_2)| \\ &\geq \left(\frac{8n}{b+4} - 8\right) + \left(\frac{4n}{b+4} - 4\right)(q-2) = \left(\frac{4n}{b+4} - 4\right) q. \end{aligned}$$

Thus noting $n \geq b + 7$, we get

$$\begin{aligned} n &\geq s + t + \sum_{i=1}^{q} |V(Q_i)| \\ &\geq 2 + b + \left(\frac{4n}{b+4} - 4\right) q \\ &\geq 2 + b + \left(\frac{4n}{b+4} - 4\right)(b-1) \\ &= n + \frac{(3b-8)n - 3b^2 - 6b + 24}{b+4} \end{aligned}$$

$$\geq n + \frac{7b-32}{b+4} > n,$$

which is a contradiction since $b \geq 6$.

Case 5-3-2: $b+5 \leq n \leq b+6$. We get

$$b+6 \geq n \geq s+t+q \geq 2+b+(b-1)$$

which is a contradiction since $b \geq 6$.

Case 5-4: $s = 3$. Since $d_{G-S}(T) \geq q_1 + 3(q-q_1) = 3q - 2q_1$, and by Claim 5, we have

$$\begin{aligned} 2 &\leq q - bs + 4t - d_{G-S}(T) \\ &\leq -2q + 2q_1 - 3b + 4t \\ &\leq -2(b-1) + 2q_1 - 3b + 4t \\ &\leq 2q_1 - b + 2, \end{aligned}$$

so that $q_1 \geq \frac{1}{2}b$.

Let $\{Q_1, \ldots, Q_{q_1}\} \subseteq \mathcal{Q}$ such that $|[V(Q_i), T]| = 1$ for all $i \in \{1, \ldots, q_1\}$. Assume $|V(Q_1)| \leq \cdots \leq |V(Q_{q_1})|$. Let $u \in V(Q_1)$ and $v \in V(Q_2)$. Then we have

$$\begin{aligned} \frac{8n}{b+4} &\leq d_G(u) + d_G(v) \\ &\leq (|V(Q_1)| - 1 + 3 + 1) + (|V(Q_2)| - 1 + 3 + 1) \\ &\leq |V(Q_1)| + |V(Q_2)| + 6 \end{aligned}$$

so that $|V(Q_1)| + |V(Q_2)| \geq \frac{8n}{b+4} - 6$ and $|V(Q_2)| \geq \frac{4n}{b+4} - 3$. Thus we have

$$\begin{aligned} \sum_{i=1}^{q_3} |V(Q_i)| &\geq |V(Q_1)| + |V(Q_2)| + (q_1 - 2)|V(Q_2)| \\ &\geq \left(\frac{8n}{b+4} - 6\right) + \left(\frac{4n}{b+4} - 3\right)(q_1 - 2) = \left(\frac{4n}{b+4} - 3\right) q_1. \end{aligned}$$

Since $d_{G-S}(T) \geq q_1 + 3q_{\geq 3}$, we have

$$\begin{aligned} 2 &\leq q - bs + 4t - d_{G-S}(T) \\ &\leq (q_1 + q_{\geq 3}) - 3b + 4t - (q_1 + 3q_{\geq 3}) \\ &= -2q_{\geq 3} - 3b + 4t, \end{aligned}$$

which yields $t \geq \frac{2+3b+2q_{\geq 3}}{4}$.

Combining all the above results, and noting $n \geq b+5$ and $b \geq 6$, we get

$$\begin{aligned} n &\geq s + t + \sum_{i=1}^{q_1} |V(Q_i)| + q_{\geq 3} \\ &\geq 3 + \left(\frac{2+3b+2q_{\geq 3}}{4}\right) + \left(\frac{4n}{4+b} - 3\right) q_1 + q_{\geq 3} \end{aligned}$$

$$\geq \left(\frac{4n}{b+4}-3\right)\frac{b}{2}+\frac{3}{2}q_{\geq 3}+\frac{3b+14}{4}$$
$$= n+\frac{(b-4)n}{b+4}+\frac{-3b+14}{4}+\frac{3}{2}q_{\geq 3}$$
$$\geq n+\frac{b^2+6b-24}{4(b+4)}+\frac{3}{2}q_{\geq 3}$$
$$= n+\left(\frac{1}{4}b+\frac{1}{2}\right)-\frac{32}{4(b+4)}+\frac{3}{2}q_{\geq 3}$$
$$\geq n+2-\frac{32}{40}+\frac{3}{2}q_{\geq 3} > n,$$

which is a contradiction.

Case 6: $1 \leq t \leq 3$. Note that $s \leq \frac{4t-2}{b} \leq \frac{5}{3}$ by Claim 1 and $b \geq 6$ so that $s \leq 1$. By Observation 2, $d_G(T) \geq 4t$, and this implies $q \geq bs - 4t + d_G(T) - st + 2 \geq 2$.

If $1 \leq t \leq 2$ or $s = 0$, then it is a contradiction to $\kappa(G) \geq 4$ since $q \geq 2$. Thus $t = 3$ and $s = 1$, and we have $q \geq bs - 4t + d_G(T) - st + 2 \geq 5$.

By Observation 1, we have $d_{G-S}(T) \geq 3q$. Since

$$d_{G-S}(T) \leq q - bs + 4t - 2 \leq q + 4,$$

we have $q \leq 2$, which is a contradiction. □

3. Sharpness of Theorem 1.5

In this section, we provide some examples demonstrating that the Conditions (i), (ii), and (iii) in Theorem 1.5 are sharp. The examples are from [2] and [14]. For the completeness of the paper, we present them in $a = 4$ format in the examples below.

Example 3.1 shows that the lower bound for n is the best possible in Theorem 1.5

Example 3.1 ([14])**.** Let $b \geq 6$, and let $G = K_2 \vee C_{b+2}$, where K_n and C_n are the complete graph and cycle on n vertices, respectively and for disjoint gerahs G_1 and G_2, $G_1 \vee G_2$ is the graph obtained from the union of G_1 and G_2 by adding all possible edges between G_1 and G_2. Then note that $\kappa(G) = 4$, $|V(G)| = b + 4$, and $\sigma_2(G) = 8(= \frac{8|V(G)|}{b+4})$. However, G does not contain an even $[4, b]$-factor. □

Matsuda ([14]) considered the graph $K_{a-2+(a^2-3a)/b} \vee C_{a+b-1}$ for $a^2 - 3a \equiv 0 \pmod{b}$ to show that the lower bound for n in Conjecture 1.4 is best possible. Note that Example 3.1 does not satisfy $a^2 - 3a \equiv 0 \pmod{b}$.

Example 3.2 shows that the lower bound for the vertex-connectivity is best possible in Theorem 1.5.

Example 3.2 ([2])**.** Let b and t be even integers such that $b \geq 28$ and $-22 + \frac{5}{4}b \geq t \geq -21 + b + \frac{4}{b}$ (≥ 6). Let L_0 be the trivial graph with $V(L_0) =$

$\{y_1, y_2, y_3\}$. For $1 \le i \le 4$, let L_i be a copy of K_6 with $V(L_i) = \{x_{i1}, \dots, x_{i6}\}$. Let L_5 be a copy of K_t with $V(L_5) = \{x_{51}, \dots, x_{5t}\}$. Suppose that L is the graph obtained from $L_0, \dots, L_5$ by adding edges between y_j and x_{ij} for all $i \in \{1, \dots, 5\}$ and for all $j \in \{1, 2, 3\}$ (see Figure 1). Then note that $|V(L)| \ge b+7$, $\kappa(L) = 3$, and $\sigma_2(L) = 10(\ge \frac{8|V(L)|}{b+4})$. However, L does not contain an even $[4, b]$-factor. □

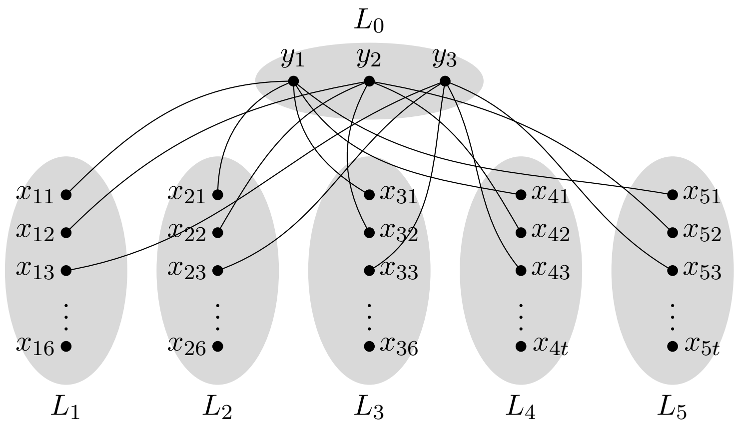

FIGURE 1. The graph L in Example 3.2

Example 3.3 shows the necessity of increasing an edge connectivity in the Condition of Conjecture 1.4.

Example 3.3 ([2])**.** Let b and t be even integers such that $b \ge 12$ and $t \ge 2 + \frac{b}{2} + \frac{2}{b}$. For $i \in \{1, 2\}$, let H_i be a copy of K_t with $V(H_i) = \{x_{i1}, \dots, x_{it}\}$. Let H_3 be a copy of K_2 with $V(H_3) = \{y, z\}$. Suppose that H is the graph obtained from H_1, H_2, and H_3 by adding edges between z and x_{11}, x_{22}, x_{23}, and between y and x_{21}, x_{12}, x_{13} (see Figure 2). Then $|V(H)| \ge b + 7$, $\kappa'(H) = 3$, and $\sigma_2(H) = 2 + \frac{|V(H)|}{2}(\ge \frac{8|V(H)|}{4+b})$ for $b \ge 12$. However, H does not contain an even $[4, b]$-factor.

Example 3.4 shows that the lower bound of $\sigma_2(G)$ is best possible in Theorem 1.5.

Example 3.4 ([14])**.** Let b and t be even integers such that $b \ge 6$ and $t \ge 2$. Suppose that F is a copy of the complete bipartite graph with partite sets A and B such that $|A| = 4t$ and $|B| = bt + 1$. Then $|V(F)| = (b + 4)t + 1 > b + 7$ for $t \ge 2$, $\kappa(F) = 4t > 4$, and $\sigma_2(F) = 8t < \frac{8|V(F)|}{b+4}$ for $b \ge 6$. However, F does not contain an even $[4, b]$-factor.

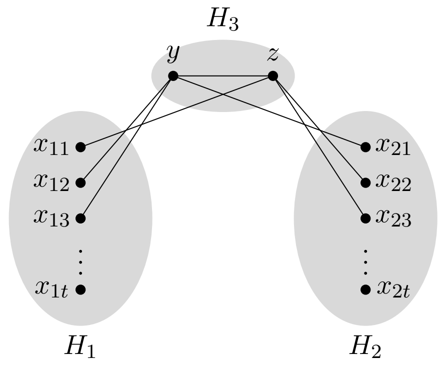

FIGURE 2. The graph H in Example 3.3

4. Concluding remarks

By Example 3.3, we know that the 2-edge-connectivity condition in Conjecture 1.4 is not enough. The example actually tells us that even 3-edge-connectivity condition is not enough.

However, it is still interesting for us to check whether Conjecture 1.4 is true if we replace a 2-edge-connected condition into a 4-edge-connected condition. If we can prove this, then it improves our result as 4-vertex-connectivity implies 4-edge-connectivity.

Question 4.1. Let $b \geq 6$ be an even integer. If G is an n-vertex graph such that (i) $n \geq b+5$, (ii) $\kappa'(G) \geq 4$, and (iii) $\sigma_2(G) \geq \frac{8n}{4+b}$, then does G have an even $[4, b]$-factor?

Moreover, it will be very interesting to check whether the result in this paper can be extended into general a as follows.

Question 4.2. Let a and b be even integers such that $4 \leq a < b$. If G is an n-vertex graph such that (i) $n \geq 2a + b + \frac{a^2-3a}{b} - 2$, (ii) $\kappa(G) \geq a$, and (iii) $\sigma_2(G) \geq \frac{2an}{a+b}$, then does G have an even $[a, b]$-factor?

Naturally, we can ask a similar question with an edge-connectivity condition instead of vertex-connectivity condition.

Question 4.3. Let a and b be even integers such that $4 \leq a < b$. If G is an n-vertex graph such that (i) $n \geq 2a + b + \frac{a^2-3a}{b} - 2$, (ii) $\kappa'(G) \geq a$, and (iii) $\sigma_2(G) \geq \frac{2an}{a+b}$, then does G have an even $[a, b]$-factor?

In 2020, with Sungeun Kim, Jihwan Park, and Hyo Ree, the third author [7] found an eigenvalue condition for a regular graph to guarantee the existence of an even $[1, b]$-factor, improving the result of Lu, Wu, and Yang [13]. Very recently, the third author [15] proved eigenvalue conditions for a regular graph to have an even $[a, b]$-factor or odd $[a, b]$-factor, extending the previous results $[1, 4, 5, 12]$. In a subsequent paper [6], eigenvalue conditions for certain eigenvalues in an h-edge-connected graph with given minimum degree to have a parity factor have been proven. With the method in the subsequent paper, we can show that the eigenvalue conditions are sharp by adjusting the parameter t of a graph in [16] (see Section 4) and the number of bullets $B_{r,t}$ or $B'_{r,t}$ (see the definitions in [16]) depending on the parity of r and t.

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