Honam Mathematical J. **44** (2022), No. 2, pp. 148–164
https://doi.org/10.5831/HMJ.2022.44.2.148

HESITANT FUZZY p-IDEALS AND QUASI-ASSOCIATIVE IDEALS IN BCI-ALGEBRAS

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Abstract. The main purpose of this paper is to apply the notion of hesitant fuzzy sets to an algebraic structure, so called a BCI-algebra. The primary goal of the study is to define hesitant fuzzy p-ideals and hesitant fuzzy quasi-associative ideals in BCI-algebras, and to investigate their properties and relations.

1. Introduction

In real world problems, dealing with uncertainty is always a challenging problem, and a part of long-term research challenge was to find a tool to deal with uncertainty. Fuzzy sets along with their extensions, for example, Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets, fuzzy multisets and interval-valued fuzzy sets etc., played a significant role of tools able to deal with uncertainty in different type of problems. As another generalization of fuzzy sets, hesitant fuzzy sets whose membership functions represented by a set of possible values are a new effective and useful tool to express human's hesitancy in daily life. Torra [8] originally introduced this concept as an extension of Zadeh's fuzzy sets, and it has attracted very quickly the attention of several researchers that have discussed its applications. It is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Hesitant fuzzy set theory is mainly used in decision making problem, etc. (see [7, 10, 11, 12, 14]). Gu et al. [1] investigated the evaluation model for risk investment with hesitant fuzzy information. Xu and Xia [13, 14] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can

Received June 1, 2020. Revised March 7, 2022. Accepted March 8, 2022.
2010 Mathematics Subject Classification. 06F35, 03G25, 03E72.
Key words and phrases. hesitant fuzzy ideal, hesitant fuzzy p-ideal, hesitant fuzzy quasi-associative ideal.
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be obtained, and defined the distance and correlation measures for hesitant fuzzy information and discussed their properties in detail. They investigated the connections of the aforementioned distance measures and further developed a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Xia et al. [15] developed several series of aggregation operators for hesitant fuzzy information with the aid of quasi-arithmetic means. In [16], Yu et al. explored aggregation methods for prioritized hesitant fuzzy elements and their application on personnel evaluation. Zhang [19] developed a wide range of hesitant fuzzy power aggregation operators for hesitant fuzzy information. The study of hesitant fuzzy set theory in algebraic structures is now start, and Jun (together with his research colleagues) applied hesitant fuzzy sets to residuated lattices, MTL-algebras and BCK/BCI-algebras (see [3, 4, 6]).

In order to provide algebraic tools for dealing with uncertainty based on hesitant fuzzy sets, the aim of this paper is first to consider applications of hesitant fuzzy sets in ideal theory, so called p-ideals and quasi-associative ideals in BCI-algebras, and to discuss their properties, relations, characterizations and establishing extension properties for a hesitant fuzzy p-ideal and a hesitant fuzzy quasi-associative ideal. We provide a characterization of a p-semisimple BCI-algebra by using the notion of hesitant fuzzy p-ideals.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra. A BCI-algebra X is said to be associative (see [2]) if it satisfies:

$$(\forall x, y, z \in X)\left((x * y) * z = x * (y * z)\right). \tag{1}$$

Any BCK/BCI-algebra X satisfies the following conditions:

(2) $(\forall x \in X)\,(x * 0 = x)\,,$

(3) $(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,,$

(4) $(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,,$

(5) $(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

(6) $(\forall x, y, z \in X)\,(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,,$

(7) $(\forall x, y \in X)\,(0 * (0 * (x * y)) = (0 * y) * (0 * x))\,,$

(8) $(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)\,.$

A BCI-algebra X is said to be p-semisimple (see [2]) if $0 * (0 * x) = x$ for all $x \in X$.

Every p-semisimple BCI-algebra X satisfies:

(9) $(\forall x, y, z \in X)\,((x * z) * (y * z) = x * y)\,.$

A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an ideal of X if it satisfies:

(10) $0 \in A,$

(11) $(\forall x \in X)\,(x * y \in A,\; y \in A \;\Rightarrow\; x \in A)\,.$

A subset A of a BCI-algebra X is called a p-ideal of X (see [18]) if it satisfies (10) and

(12) $(\forall x, y, z \in X)\,((x * z) * (y * z) \in A,\; y \in A \;\Rightarrow\; x \in A)\,.$

Note that an ideal A of a BCI-algebra X is a p-ideal of X if and only if the following assertion is valid:

(13) $(\forall x, y, z \in X)\,((x * z) * (y * z) \in A \;\Rightarrow\; x * y \in A)\,.$

A subset Q of a BCI-algebra X is called a quasi-associative ideal of X (see [17]) if it satisfies (10) and

(14) $(\forall x, y, z \in X)\,(x * (y * z) \in Q,\; y \in Q \;\Rightarrow\; x * z \in Q)\,.$

Note that an ideal Q of a BCI-algebra X is a quasi-associative ideal of X if and only if the following assertion is valid:

(15) $(\forall x, y \in X)\,(x * (0 * y) \in Q \;\Rightarrow\; x * y \in Q)\,.$

We refer the reader to the books [2, 5] for further information regarding BCK/BCI-algebras.

Definition 2.1 ([8, 9])**.** *Let E be a reference set. A hesitant fuzzy set on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:*

$$H_E := \{(e, h_E(e)) \mid e \in E\}$$

where $h_E : E \to \mathscr{P}([0,1])$.

Definition 2.2. *Let X be a BCK/BCI-algebra. Given a non-empty subset A of X, a hesitant fuzzy set*

$$H_X := \{(x, h_X(x)) \mid x \in X\}$$

on X satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{16}$$

is called a hesitant fuzzy set related to A (briefly, A-hesitant fuzzy set) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

Definition 2.3 ([3])**.** *Let X be a BCK/BCI-algebra. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a hesitant fuzzy subalgebra of X related to A (briefly, A-hesitant fuzzy subalgebra of X) if it satisfies the following condition:*

$$(\forall x, y \in A)\,(h_A(x * y) \supseteq h_A(x) \cap h_A(y))\,. \tag{17}$$

An A-hesitant fuzzy subalgebra of X with $A = X$ is called a hesitant fuzzy subalgebra of X.

Definition 2.4 ([3])**.** *Let X be a BCK/BCI-algebra. Given a non-empty subset (subalgebra as much as possible) A of X, an A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy ideal of X related to A (briefly, A-hesitant fuzzy ideal of X) if it satisfies:*

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(y) \subseteq h_A(x) \subseteq h_A(0))\,. \tag{18}$$

An A-hesitant fuzzy ideal of X with $A = X$ is called a hesitant fuzzy ideal of X.

3. Hesitant fuzzy p-ideals

In what follows, we take a BCI-algebra X as a reference set and let A be a non-empty subset (subalgebra as much as possible) of X unless otherwise specified.

Definition 3.1. *An A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy p-ideal of X related to A (briefly, A-hesitant fuzzy p-ideal of X) if it satisfies:*

(19) $$(\forall x \in A)\,(h_A(x) \subseteq h_A(0)),$$

(20) $$(\forall x, y, z \in A)\,(h_A((x * z) * (y * z)) \cap h_A(y) \subseteq h_A(x)).$$

An A-hesitant fuzzy p-ideal of X with $A = X$ is called a hesitant fuzzy p-ideal of X.

Example 3.2. *Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table.*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0, 1]), (a, (\tfrac{1}{4}, \tfrac{3}{4})), (b, (\tfrac{3}{8}, \tfrac{5}{8})), (c, (\tfrac{3}{8}, \tfrac{5}{8}))\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy p-ideal of X.

Proposition 3.3. *Every A-hesitant fuzzy p-ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies:*

(21) $$(\forall x \in A)\,(h_A(0 * (0 * x)) \subseteq h_A(x)).$$

Proof. If we put $z := x$ and $y := 0$ in (20), then

$$h_A(x) \supseteq h_A((x*x)*(0*x)) \cap h_A(0) = h_A(0*(0*x)) \cap h_A(0) = h_A(0*(0*x))$$

for all $x \in A$ by (III) and (19). □

Corollary 3.4. *Every hesitant fuzzy ideal $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X satisfies:*

(22) $$(\forall x \in X)\,(h_X(0 * (0 * x)) \subseteq h_X(x)).$$

Theorem 3.5. *Every A-hesitant fuzzy p-ideal of X is an A-hesitant fuzzy ideal of X.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy p-ideal of X. Since $x * 0 = x$ for all $x \in X$, it follows from (20) that

$$h_A(x) \supseteq h_A((x * 0) * (y * 0)) \cap h_A(y) = h_A(x * y) \cap h_A(y)$$

for all $x, y \in A$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X. □

The converse of Theorem 3.5 is not true in general as seen in the following example.

Example 3.6. *Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.*

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

For $A = X$, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_A = \left\{(0, [0,1]), (1, (\tfrac{1}{4}, \tfrac{3}{4})), (a, (\tfrac{3}{8}, \tfrac{5}{8})), (b, (\tfrac{3}{8}, \tfrac{5}{8})), (c, (\tfrac{3}{8}, \tfrac{5}{8}))\right\}.$$

Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, but it is not a hesitant fuzzy p-ideal of X since $h_A(1) = (\frac{1}{4}, \frac{3}{4}) \subsetneq [0, 1] = h_A((1 * a) * (0 * a)) \cap h_A(0)$.

Proposition 3.7. *Every A-hesitant fuzzy p-ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies:*

$$(\forall x, y, z \in A)\,(h_A(x * y) \subseteq h_A((x * z) * (y * z))). \tag{23}$$

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy p-ideal of X. Then it is an A-hesitant fuzzy ideal of X by Theorem 3.5. Hence

$$\begin{aligned} h_A((x * z) * (y * z)) &\supseteq h_A(((x * z) * (y * z)) * (x * y)) \cap h_A(x * y) \\ &= h_A(0) \cap h_A(x * y) = h_A(x * y) \end{aligned}$$

for all $x, y, z \in A$. □

We provide conditions for an A-hesitant fuzzy ideal to be an A-hesitant fuzzy p-ideal.

Theorem 3.8. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X such that*

$$(\forall x, y, z \in A)\,(h_A(x * y) \supseteq h_A((x * z) * (y * z))). \tag{24}$$

Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy p-ideal of X.

Proof. If the condition (24) is valid, then

$$h_A(x) \supseteq h_A(x * y) \cap h_A(y) \supseteq h_A((x * z) * (y * z)) \cap h_A(y)$$

for all $x, y, z \in A$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy p-ideal of X. □

Corollary 3.9. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy ideal of X such that*

$$(\forall x, y, z \in X)\,(h_X(x * y) \supseteq h_X((x * z) * (y * z))).$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy p-ideal of X.

Lemma 3.10. *Every A-hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies the following condition:*

$$(\forall x \in A)\,(h_A(x) \subseteq h_A(0 * (0 * x))).$$

Proof. For every $x \in A$, we have

$$h_A(x) = h_A(0) \cap h_A(x) = h_A((0 * (0 * x)) * x) \cap h_A(x) \subseteq h_A(0 * (0 * x))$$

which is the desired result. □

Theorem 3.11. *If an A-hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies the condition (21), then it is an A-hesitant fuzzy p-ideal of X.*

Proof. Let $x, y, z \in A$. Using Lemma 3.10, (6), (7) and (21), we have

$$\begin{aligned} h_A((x * z) * (y * z)) &\subseteq h_A(0 * (0 * ((x * z) * (y * z)))) \\ &= h_A((0 * y) * (0 * x)) \\ &= h_A(0 * (0 * (x * y))) \\ &\subseteq h_A(x * y). \end{aligned}$$

It follows from Theorem 3.8 that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy p-ideal of X. □

Corollary 3.12. *If a hesitant fuzzy ideal $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X satisfies the condition (22), then it is a hesitant fuzzy p-ideal of X.*

Theorem 3.13. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy p-ideal of X, then the set*

$$I := \{x \in X \mid h_X(x) = h_X(0)\}$$

is a p-ideal of X.

Proof. Obviously $0 \in I$. Let $x, y, z \in X$ be such that $(x*z)*(y*z) \in I$ and $y \in I$. Then

$$h_X(x) \supseteq h_X((x * z) * (y * z)) \cap h_X(y) = h_X(0),$$

and so $h_X(x) = h_X(0)$, that is, $x \in I$. Therefore I is a p-ideal of X. □

For any subset I of X, let $H_X^I = \big\{\big(x, h_X^I(x)\big) \mid x \in X\big\}$ be a hesitant fuzzy set on X defined by

$$h_X^I(x) = \begin{cases} [0,1] & \text{if } x \in I, \\ \{1\} & \text{otherwise.} \end{cases}$$

Lemma 3.14. *For any subset I of X, the following are equivalent:*

(1) *I is an ideal (resp. p-ideal) of X.*
(2) *The hesitant fuzzy set $H_X^I = \big\{(x, h_X^I(x)) \mid x \in X\big\}$ on X is a hesitant fuzzy ideal (resp. hesitant fuzzy p-ideal) of X.*

Proof. The proof is straightforward. □

Theorem 3.15. *A BCI-algebra X is p-semisimple if and only if every hesitant fuzzy ideal of X is a hesitant fuzzy p-ideal of X.*

Proof. Assume that X is a p-semisimple BCI-algebra and let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy ideal of X. Then

$$h_X(x) \supseteq h_X(x * y) \cap h_X(y) = h_X((x * z) * (y * z)) \cap h_X(y)$$

by using (9). Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy p-ideal of X.

Conversely, suppose that every hesitant fuzzy ideal of X is a hesitant fuzzy p-ideal of X. Since the hesitant fuzzy set

$$H_X^{\{0\}} = \left\{\left(x, h_X^{\{0\}}(x)\right) \mid x \in X\right\}$$

on X is a hesitant fuzzy ideal of X, it is also a hesitant fuzzy p-ideal of X. It follows from Lemma 3.14 that $\{0\}$ is a p-ideal of X. For any $x \in X$, we have

$$\begin{aligned}((x * (0 * (0 * x))) * x) * (0 * x) &= ((x * x) * (0 * (0 * x))) * (0 * x) \\ &= (0 * (0 * (0 * x))) * (0 * x) \\ &= (0 * (0 * x)) * (0 * (0 * x)) = 0 \in \{0\}\end{aligned}$$

by using (4) and (III), which implies from (12) that $x*(0*(0*x)) \in \{0\}$. Hence $x * (0 * (0 * x)) = 0$, that is, $x \leq 0 * (0 * x)$. Since $0 * (0 * x) \leq x$, we get $0 * (0 * x) = x$. Therefore X is a p-semisimple BCI-algebra. □

Theorem 3.16. (Extension property for hesitant fuzzy p-ideals) *Let*

$$H_X := \{(x, h_X(x)) \mid x \in X\} \text{ and } G_X := \{(x, g_X(x)) \mid x \in X\}$$

be hesitant fuzzy ideals of X such that $h_X(0) = g_X(0)$ and $h_X(x) \subseteq g_X(x)$ for all $x \in X$. If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy p-ideal of X, then so is $G_X := \{(x, g_X(x)) \mid x \in X\}$.

Proof. Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy p-ideal of X. Using (7), (8) and (III), we have $0*(0*(x*(0*(0*x)))) = 0$ for all $x \in X$. It follows from hypothesis and (22) that

$$\begin{aligned} g_X(x * (0 * (0 * x))) &\supseteq h_X(x * (0 * (0 * x))) \\ &\supseteq h_X(0 * (0 * (x * (0 * (0 * x))))) \\ &= h_X(0) = g_X(0). \end{aligned}$$

Hence

$$\begin{aligned} g_X(x) &\supseteq g_X(x * (0 * (0 * x))) \cap g_X(0 * (0 * x)) \\ &\supseteq g_X(0) \cap g_X(0 * (0 * x)) \\ &= g_X(0 * (0 * x)), \end{aligned}$$

and thus $G_X := \{(x, g_X(x)) \mid x \in X\}$ is a hesitant fuzzy p-ideal of X by Corollary 3.12. □

4. Hesitant fuzzy quasi-associative ideals

Definition 4.1. *An A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a hesitant fuzzy quasi-associative ideal of X related to A (briefly, A-hesitant fuzzy quasi-associative ideal of X) if it satisfies* (19) *and*

$$(25) \qquad (\forall x, y, z \in A) \, (h_A(x * z) \supseteq h_A(x * (y * z)) \cap h_A(y)) \, .$$

An A-hesitant fuzzy quasi-associative ideal of X with $A = X$ is called a hesitant fuzzy quasi-associative ideal of X.

Example 4.2. *(1) The hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ in Example 3.2 is a hesitant fuzzy quasi-associative ideal of X.*

(2) Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [\tfrac{1}{3}, 1]), (a, [\tfrac{1}{3}, 1]), (b, \{\tfrac{1}{3}, 1\})\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy quasi-associative ideal of X.

Theorem 4.3. *Every A-hesitant fuzzy quasi-associative ideal is both an A-hesitant fuzzy subalgerba and an A-hesitant fuzzy ideal, that is, every A-hesitant fuzzy quasi-associative ideal is a closed A-hesitant fuzzy ideal.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy quasi-associative ideal of X. If we put $z := 0$ in (25) and use (2), then

$$\begin{aligned} h_A(x) &= h_A(x * 0) \\ &\supseteq h_A(x * (y * 0)) \cap h_A(y) \\ &= h_A(x * y) \cap h_A(y) \end{aligned} \tag{26}$$

for all $x, y \in A$. Hence $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X. Now, we put $z := y$ in (25). Then

$$\begin{aligned} h_A(x * y) &\supseteq h_A(x * (y * y)) \cap h_A(y) \\ &= h_A(x * 0) \cap h_A(y) \\ &= h_A(x) \cap h_A(y) \end{aligned} \tag{27}$$

for all $x, y \in A$ by (III) and (2). Thus $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgerba of X. □

Corollary 4.4. *Every hesitant fuzzy quasi-associative ideal is both a hesitant fuzzy subalgerba and a hesitant fuzzy ideal, that is, every hesitant fuzzy quasi-associative ideal is a closed hesitant fuzzy ideal.*

The following example shows that the converse of Corollary 4.4 is not true in general.

Example 4.5. *(1) Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table.*

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0, \tfrac{3}{4}]), (a, (\tfrac{1}{4}, \tfrac{3}{4})), (b, (\tfrac{1}{4}, \tfrac{3}{4})), (c, (\tfrac{1}{4}, \tfrac{3}{4}))\right\}.$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is both a hesitant fuzzy subalgerba and a hesitant fuzzy ideal of X. But it is not a hesitant fuzzy quasi-associative ideal of X since

$$\begin{aligned} h_X(c * a) &= h_X(b) = (\tfrac{1}{4}, \tfrac{3}{4}) \\ &\not\supseteq [0, \tfrac{3}{4}] = h_X(0) \\ &= h_X(c * (0 * a)) \cap h_X(0). \end{aligned} \tag{28}$$

(2) The hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ which is given in Example 3.6 is both a hesitant fuzzy subalgerba and a hesitant fuzzy ideal of X. But it is not a hesitant fuzzy quasi-associative ideal of X since $h_X(c * a) \not\supseteq h_X(c * (0 * a)) \cap h_X(0)$.

We provide conditions for a hesitant fuzzy ideal to be a hesitant fuzzy quasi-associative ideal. We need the following lemma.

Lemma 4.6 ([3])**.** *Every A-hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies:*

$$(\forall x, y \in A)\,(x \leq y \;\Rightarrow\; h_A(x) \supseteq h_A(y)). \tag{29}$$

Theorem 4.7. *If $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X, then the following assertions are equivalent:*

(1) *$H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy quasi-associative ideal of X.*
(2) *$(\forall x, y \in A)\,(h_A(x * (0 * y)) \subseteq h_A(x * y))$.*
(3) *$(\forall x, y, z \in A)\,(h_A(x * (y * z)) \subseteq h_A((x * y) * z))$.*

Proof. (1) $\Rightarrow$ (2). Assume that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy quasi-associative ideal of X. If we take $y := 0$ and $z := y$ in (25) and use (19), then

$$h_A(x * y) \supseteq h_A(x * (0 * y)) \cap h_A(0) = h_A(x * (0 * y))$$

for all $x, y \in A$.

(2) $\Rightarrow$ (3). Suppose that (2) is valid. Note that

$$\begin{aligned} &((x*y)*(0*z))*(x*(y*z)) \\ &= ((x*y)*(x*(y*z)))*(0*z) \\ &\leq ((y*z)*y)*(0*z) \\ &= (0*z)*(0*z) = 0 \end{aligned} \tag{30}$$

for all $x, y, z \in A$. It follows from (2), (18) and Lemma 4.6 that

(31)

$$\begin{aligned} h_A((x*y)*z) &\supseteq h_A((x*y)*(0*z)) \\ &\supseteq h_A(((x*y)*(0*z))*(x*(y*z))) \cap h_A(x*(y*z)) \\ &\supseteq h_A(0) \cap h_A(x*(y*z)) \\ &= h_A(x*(y*z)) \end{aligned}$$

(3) $\Rightarrow$ (1). Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X in which the condition (3) holds. Using (18), (4) and (3), we have

$$\begin{aligned} h_A(x*z) &\supseteq h_A((x*z)*y) \cap h_A(y) \\ &= h_A((x*y)*z) \cap h_A(y) \\ &\supseteq h_A(x*(y*z)) \cap h_A(y) \end{aligned} \tag{32}$$

for all $x, y, z \in X$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy quasi-associative ideal of X. □

Corollary 4.8. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, then the following assertions are equivalent:*

(1) *$H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy quasi-associative ideal of X.*
(2) $(\forall x, y \in X)\,(h_X(x*(0*y)) \subseteq h_X(x*y))$.
(3) $(\forall x, y, z \in X)\,(h_X(x*(y*z)) \subseteq h_X((x*y)*z))$.

Theorem 4.9. *If an A-hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies*

$$(\forall x, y \in A)\,(h_A(x) \subseteq h_A(x*y)), \tag{33}$$

then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy quasi-associative ideal of X.

Proof. For any $x, y, z \in A$, we have

$$\begin{aligned} h_A(x * z) &\supseteq h_A(x) \\ &\supseteq h_A(x * (y * z)) \cap h_A(y * z) \\ &\supseteq h_A(x * (y * z)) \cap h_A(y). \end{aligned} \tag{34}$$

Thus $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy quasi-associative ideal of X. □

Theorem 4.10. *If X is associative, then every A-hesitant fuzzy ideal is an A-hesitant fuzzy quasi-associative ideal.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of an associative BCI-algebra X and let $x, y, z \in A$. Then

$$\begin{aligned} h_A(x * (y * z)) \cap h_A(y) &= h_A((x * y) * z) \cap h_A(y) \\ &= h_A((x * z) * y) \cap h_A(y) \\ &\subseteq h_A(x * z). \end{aligned} \tag{35}$$

Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy quasi-associative ideal of X. □

Corollary 4.11. *If X is associative, then every hesitant fuzzy ideal is a hesitant fuzzy quasi-associative ideal.*

For any hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ on X, consider a set

$$H_a := \{x \in X \mid h_X(a) \subseteq h_X(x)\}.$$

Lemma 4.12 ([3]). *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, then the set H_a is an ideal of X for all $a \in X$.*

Theorem 4.13. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy quasi-associative ideal of X, then the set H_a is a quasi-associative ideal of X for all $a \in X$.*

Proof. Let $a, x, y \in X$ be such that $x * (0 * y) \in H_a$. Then

$$h_X(a) \subseteq h_X(x * (0 * y)) \subseteq h_X(x * y)$$

by Corollary 4.8, and so $x * y \in H_a$. Hence H_a is a quasi-associative ideal of X for all $a \in X$. □

Proposition 4.14. *Given an element $a \in X$, if $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy quasi-associative ideal of X, then the following conditions are valid:*

(1) $(\forall x, y \in X)\,(h_X(a) \subseteq h_X(x * y) \cap h_X(y) \;\Rightarrow\; h_X(a) \subseteq h_X(x))$,

(2)

$$(\forall x, y, z \in X)\,(h_X(a) \subseteq h_X(x * (y * z)) \cap h_X(y) \\ \Rightarrow\ h_X(a) \subseteq h_X(x * z))\,,$$

(3) $(\forall x, y \in X)\,(h_X(a) \subseteq h_X(x * (0 * y)) \ \Rightarrow\ h_X(a) \subseteq h_X(x * y))\,.$

Proof. Since every hesitant fuzzy quasi-associative ideal is a hesitant fuzzy ideal, (1) is by [3, Theorem 3.28 and Theorem 3.29]. Let $x, y, z \in X$ be such that

$$h_X(a) \subseteq h_X(x * (y * z)) \cap h_X(y).$$

Then $x * (y * z) \in H_a$ and $y \in H_a$. Since H_a is a quasi-associative ideal of X by Theorem 4.13, it follows that $x * z \in H_a$ and that $h_X(a) \subseteq h_X(x * z)$. Hence (2) is valid. Finally let $x, y \in X$ be such that $h_X(a) \subseteq h_X(x * (0 * y))$. Then $x * (0 * y) \in H_a$, and so $x * y \in H_a$ since H_a is a quasi-associative ideal of X. Thus $h_X(a) \subseteq h_X(x * y)$. □

Theorem 4.15. *If a hesitant fuzzy set* $H_X := \{(x, h_X(x)) \mid x \in X\}$ *on* X *satisfies* (19), *(1) and (2) in Proposition 4.14, then the set* H_a *is a quasi-associative ideal of* X.

Proof. By [3, Theorem 3.29], we know that H_a is an ideal of X. Let $x, y, z \in X$ be such that $x * (y * z) \in H_a$ and $y \in H_a$. Then $h_X(a) \subseteq h_X(x * (y * z))$ and $h_X(a) \subseteq h_X(y)$, which imply that $h_X(a) \subseteq h_X(x * (y * z)) \cap h_X(y)$. Hence $h_X(a) \subseteq h_X(x * z)$ by (2), and thus $x * z \in H_a$. Therefore H_a is a quasi-associative ideal of X. □

Theorem 4.16. *If a hesitant fuzzy set* $H_X := \{(x, h_X(x)) \mid x \in X\}$ *on* X *satisfies* (19), *(1) and (3) in Proposition 4.14, then the set* H_a *is a quasi-associative ideal of* X.

Proof. By [3, Theorem 3.29], we know that H_a is an ideal of X. Let $x, y \in X$ be such that $x * (0 * y) \in H_a$. Then $h_X(a) \subseteq h_X(x * (0 * y))$, and so $h_X(a) \subseteq h_X(x * y)$ by (3), that is, $x * y \in H_a$. Therefore H_a is a quasi-associative ideal of X. □

Theorem 4.17. (Extension property for hesitant fuzzy quasi-associative ideals) *Let*

$$H_X := \{(x, h_X(x)) \mid x \in X\} \text{ and } G_X := \{(x, g_X(x)) \mid x \in X\}$$

be hesitant fuzzy ideals of X *such that* $h_X(0) = g_X(0)$ *and*

$$h_X(x) \subseteq g_X(x)$$

for all $x \in X$. *If*

$$H_X := \{(x, h_X(x)) \mid x \in X\}$$

is a hesitant fuzzy quasi-associative ideal of X, then so is

$$G_X := \{(x, g_X(x)) \mid x \in X\}.$$

Proof. Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy quasi-associative ideal of X. Using (18), (4), Corollary 4.8, (III) and given conditions, we have

$$\begin{aligned} g_X(x * y) &\supseteq g_X((x * y) * (x * (0 * y))) \cap g_X(x * (0 * y)) \\ &\supseteq h_X((x * y) * (x * (0 * y))) \cap g_X(x * (0 * y)) \\ &= h_X((x * (x * (0 * y))) * y) \cap g_X(x * (0 * y)) \\ &\supseteq h_X((x * (x * (0 * y))) * (0 * y)) \cap g_X(x * (0 * y)) \\ &= h_X((x * (0 * y)) * (x * (0 * y))) \cap g_X(x * (0 * y)) \\ &= h_X(0) \cap g_X(x * (0 * y)) \\ &= g_X(0) \cap g_X(x * (0 * y)) \\ &= g_X(x * (0 * y)) \end{aligned}$$

for all $x, y \in X$. It follows from Corollary 4.8 that $G_X := \{(x, g_X(x)) \mid x \in X\}$ is a hesitant fuzzy quasi-associative ideal of X. □

5. Conclusions

The hesitant fuzzy set is a useful generalization of the fuzzy set which is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. In this paper, we have introduced the notions of hesitant fuzzy p-ideals and hesitant fuzzy quasi-associative ideals in BCI-algebras, and have investigated their relations and properties. We have discussed relations between hesitant fuzzy ideals, hesitant fuzzy p-ideals and hesitant fuzzy quasi-associative ideals. We have provided conditions for a hesitant fuzzy ideal to be a hesitant fuzzy p-ideal (resp. a hesitant fuzzy quasi-associative ideal). We have characterized a p-semisimple BCI-algebra, and have considered characterizations of hesitant fuzzy quasi-associative ideals. We have established extension property for a hesitant fuzzy p-ideal and a hesitant fuzzy quasi-associative ideal. Future research will focus on applying the notions/contents to other types of ideals in BCK/BCI-algebras and related algebraic structures.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgement

The authors would like to thank the anonymous referees for their very valuable comments and recommendations

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