

A NOTE ON *-CONFORMAL AND GRADIENT *-CONFORMAL η -RICCI SOLITONS IN α -COSYMPLECTIC MANIFOLDS

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Abstract. In the present paper we study the properties of α -cosymplectic manifolds endowed with *-conformal η -Ricci solitons and gradient *-conformal η -Ricci solitons.

1. Introduction

In 1959, the notion of *-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana ([24]) and further studied by Hamada ([9]) on real hypersurfaces of non-flat complex space forms. A Riemannian metric g on a smooth manifold M is called a *-Ricci soliton if there exists a smooth vector field φ (called soliton vector field) and a real number λ such that

$$(\mathcal{L}_\varphi g)(\varphi_1, \varphi_2) + 2S^*(\varphi_1, \varphi_2) = -2\lambda g(\varphi_1, \varphi_2),$$

where

$$S^*(\varphi_1, \varphi_2) = g(Q^*\varphi_1, \varphi_2) = \text{Trace} \{ \phi \circ R(\varphi_1, \phi\varphi_2) \},$$

where Q^* is the *-Ricci operator ([13]).

In 2004, the concept of conformal Ricci flow was developed by Fischer ([6]) as a variation of the classical Ricci flow equation. The conformal Ricci flow on a smooth closed connected oriented n -dimensional Riemannian manifold M is defined by the equations

$$\frac{\partial g}{\partial t} = -2\left(S + \frac{g}{n}\right) - pg$$

and $r = -1$, where p denotes for time dependent scalar field (which is non-dynamical), S is the Ricci tensor and r is the scalar curvature on M .

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Recently, the authors Haseeb and Prasad ([12]) studied $*$ -conformal η -Ricci soliton (in short, $(g^*, \varphi, \lambda, \mu)$) in ϵ -Kenmotsu manifolds. A Riemannian metric g on M is called $*$ -conformal η -Ricci soliton if

$$(1) \quad \frac{1}{2} \mathcal{L}_\varphi g + S^* = -(\lambda - \frac{1}{2n}(pn + 2))g - \mu\eta \otimes \eta,$$

where \mathcal{L}_φ is the Lie derivative along φ , S^* is the $*$ -Ricci tensor and λ, μ are real constants.

If an α -cosymplectic manifold (briefly, M^α) satisfies (1), then the manifold is said to admit $(g^*, \varphi, \lambda, \mu)$. For more details about the related studies in the context of contact Riemannian geometry, we recommend the papers ([3], [7], [11], [15]–[17], [21], [22], [26]–[28], [31], [32]).

We organize our work as follows: In Section 2, we collect the basic results and some basic definitions of M^α . We discussed $(g^*, \varphi, \lambda, \mu)$ and gradient $(g^*, \varphi, \lambda, \mu)$ on M^α in Section 3 and Section 4, respectively.

2. Preliminaries

Let M be an $n(= 2m + 1)$ -dimensional differentiable manifold equipped with a triplet (ϕ, ξ, η) , where ϕ is a tensor field of type $(1, 1)$, ξ is the structure vector field (or Reeb vector field) and η is a 1-form such that

$$(2) \quad \eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

which implies

$$\phi\xi = 0, \quad \eta(\phi\varphi_1) = 0, \quad \text{rank}(\phi) = n - 1$$

for all $\varphi_1 \in \chi(M)$, where $\chi(M)$ denotes the collection of all smooth vector fields of M ([4]). Then the manifold is said to have an almost contact metric structure (ϕ, ξ, η, g) when it admits a Riemannian metric g such that

$$(3) \quad g(\phi\varphi_1, \phi\varphi_2) = g(\varphi_1, \varphi_2) - \eta(\varphi_1)\eta(\varphi_2), \quad g(\varphi_1, \xi) = \eta(\varphi_1)$$

for all $\varphi_1, \varphi_2 \in \chi(M)$. The 2-form Φ of M defined by

$$\Phi(\varphi_1, \varphi_2) = g(\varphi_1, \phi\varphi_2)$$

is called the fundamental 2-form of the almost contact metric manifold. A manifold with the structure (ϕ, ξ, η, g) is said to be almost cosymplectic if $d\eta = d\Phi = 0$, where d is the exterior differential operator ([8]). The manifold (M, ϕ, ξ, η, g) is said to be normal if the Nijenhuis tensor

$$\begin{aligned} N_\phi(\varphi_1, \varphi_2) &= [\phi\varphi_1, \phi\varphi_2] - \phi[\phi\varphi_1, \varphi_2] - \phi[\varphi_1, \phi\varphi_2] \\ &\quad + \phi^2[\varphi_1, \varphi_2] + 2d\eta(\varphi_1, \varphi_2)\xi \end{aligned}$$

vanishes for any vector fields φ_1 and φ_2 . A normal almost cosymplectic manifold is called a cosymplectic manifold.

A manifold (M, ϕ, ξ, η, g) is said to be an almost α -Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, being $\alpha(\neq 0) \in \mathbb{R}$.

An almost α -Kenmotsu and almost cosymplectic manifolds were combined into a new class by Kim and Pak ([14]) and called as an almost M^α , where $\alpha \in \mathbb{R}$. By joining these two classes a new notion of an almost M^α was introduced, which is defined by the following formula

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi$$

for any $\alpha \in \mathbb{R}$. A normal almost M^α is called an M^α . An M^α is cosymplectic (resp. α -Kenmotsu) under the condition $\alpha = 0$ (resp. $\alpha \neq 0$) for $\alpha \in \mathbb{R}$. For detailed study of M^α , we refer to [1], [2], [5], [18], [19], and [23].

In M^α , we have

$$(4) \quad (\nabla_{\varphi_1} \phi) \varphi_2 = \alpha(g(\phi \varphi_1, \varphi_2) \xi - \eta(\varphi_2) \phi \varphi_1)$$

(refer to [20]). From (4), it is easy to see that

$$(5) \quad \nabla \xi = -\alpha \phi^2 = \alpha(I - \eta \otimes \xi),$$

$$(6) \quad \nabla \eta = \alpha(g - \eta \otimes \eta),$$

where ∇ denotes the Riemannian connection of the manifold and α is a real number.

In M^α , the following relations hold:

$$(7) \quad R(\xi, \varphi_1) \varphi_2 = \alpha^2(\eta(\varphi_2) \varphi_1 - g(\varphi_1, \varphi_2) \xi),$$

$$(8) \quad R(\varphi_1, \varphi_2) \xi = \alpha^2(\eta(\varphi_1) \varphi_2 - \eta(\varphi_2) \varphi_1),$$

$$(9) \quad S(\varphi_1, \xi) = -\alpha^2(n-1)\eta(\varphi_1),$$

$$(10) \quad Q\xi = -\alpha^2(n-1)\xi$$

for all $\varphi_1, \varphi_2 \in \chi(M)$, where R is the curvature tensor.

An M^α is said to be an η -Einstein manifold if its Ricci tensor S is of the form

$$(11) \quad S(\varphi_1, \varphi_2) = \rho_1 g(\varphi_1, \varphi_2) + \rho_2 \eta(\varphi_1) \eta(\varphi_2),$$

where ρ_1 and ρ_2 are smooth functions on M^α . If $\rho_2 = 0$, then M^α is said to be an Einstein. M^α satisfying the equation (11) is said to be proper η -Einstein if $\rho_2 \neq 0$.

Definition 2.1. A vector field φ_1 on an almost contact metric manifold M is said to be an infinitesimal contact transformation if there exist a smooth function \mathfrak{f} on M such that

$$(\mathcal{L}_{\varphi_1} \eta)(\varphi_2) = \mathfrak{f} \eta(\varphi_2)$$

for vector field $\varphi_2 \in \chi(M)$. In particular, if $\mathfrak{f} = 0$, then φ_1 is called a strict infinitesimal contact transformation on M (see [25, 29]).

Lemma 2.2. *In an n -dimensional M^α , we have*

$$(12) \quad (\nabla_{\varphi_1} Q)\xi = -\alpha Q\varphi_1 - (n-1)\alpha^3\varphi_1,$$

$$(13) \quad (\nabla_\xi Q)\varphi_1 = -2\alpha Q\varphi_1 - 2(n-1)\alpha^3\varphi_1,$$

where Q is the Ricci operator defined by $S(\varphi_1, \varphi_2) = g(Q\varphi_1, \varphi_2)$.

Proof. Differentiating (10) along φ_1 and using (5), we get (12). Next differentiating (8) then using (5) and (6), we find

$$(14) \quad (\nabla_{\varphi_4} R)(\varphi_1, \varphi_2)\xi = -\alpha R(\varphi_1, \varphi_2)\varphi_4 - \alpha^3(g(\varphi_4, \varphi_2)\varphi_1 - g(\varphi_4, \varphi_1)\varphi_2).$$

Let $\{e_i\}, i = 1, 2, 3, \dots, n$ be a local orthonormal basis on M . Putting $\varphi_1 = \varphi_4 = e_i$ in (14) and summing over i leads to

$$(15) \quad \sum_{i=1}^n g((\nabla_{e_i} R)(e_i, \varphi_2)\xi, \varphi_3) = \alpha S(\varphi_2, \varphi_3) + \alpha^3(n-1)g(\varphi_2, \varphi_3).$$

From the second Bianchi's identity, we easily obtain

$$(16) \quad \sum_{i=1}^n g((\nabla_{e_i} R)(\varphi_3, \xi)\varphi_2, e_i) = g(\varphi_2, (\nabla_{\varphi_3} Q)\xi) - g(\varphi_2, (\nabla_\xi Q)\varphi_3).$$

By considering (15) in (16), it follows that

$$g((\nabla_\xi Q)\varphi_3, \varphi_2) = -2\alpha S(\varphi_2, \varphi_3) - 2\alpha^3(n-1)g(\varphi_2, \varphi_3),$$

from which (13) follows. \square

In M^α , we can easily prove the following lemmas:

Lemma 2.3. [10] *In M^α , we have*

$$\begin{aligned} \bar{R}(\varphi_1, \varphi_2, \phi\varphi_3, \phi\varphi_4) &= \bar{R}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) + \alpha^2[\Phi(\varphi_1, \varphi_3)\Phi(\varphi_2, \varphi_4) \\ &\quad - \Phi(\varphi_2, \varphi_3)\Phi(\varphi_1, \varphi_4) + g(\varphi_2, \varphi_3)g(\varphi_1, \varphi_4) - g(\varphi_1, \varphi_3)g(\varphi_2, \varphi_4)] \end{aligned}$$

for any $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \chi(M)$, where $\bar{R}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = g(R(\varphi_1, \varphi_2)\varphi_3, \varphi_4)$ and Φ is the fundamental 2-form of M .

Lemma 2.4. [10] *In an n -dimensional M^α , the $*$ -Ricci tensor is given by*

$$(17) \quad S^*(\varphi_2, \varphi_3) = S(\varphi_2, \varphi_3) + \alpha^2(n-2)g(\varphi_2, \varphi_3) + \alpha^2\eta(\varphi_2)\eta(\varphi_3)$$

for any $\varphi_2, \varphi_3 \in \chi(M)$, where S and S^* are the Ricci tensor and the $*$ -Ricci tensor of type $(0, 2)$ on M , respectively.

3. α -cosymplectic manifolds admitting $*$ -conformal η -Ricci solitons

In this section first we prove the following:

Theorem 3.1. *Let M^α be an n -dimensional manifold admitting $(g^*, \varphi, \lambda, \mu)$. Then either M^α is cosymplectic or $\lambda + \mu = \frac{1}{2n}(pn + 2)$.*

Proof. By using (17) in (1), we have

$$(18) \quad (\mathcal{L}_\varphi g)(\varphi_1, \varphi_2) = -2S(\varphi_1, \varphi_2) - 2[\lambda + (n-2)\alpha^2] \\ - \frac{1}{2n}(pn+2)g(\varphi_1, \varphi_2) - 2(\alpha^2 + \mu)\eta(\varphi_1)\eta(\varphi_2).$$

Taking covariant differentiation of (18) with respect to φ_3 , we get

$$(19) \quad (\nabla_{\varphi_3} \mathcal{L}_\varphi g)(\varphi_1, \varphi_2) = -2(\nabla_{\varphi_3} S)(\varphi_1, \varphi_2) - 2\alpha(\mu + \alpha^2)(g(\varphi_1, \varphi_3)\eta(\varphi_2) \\ + g(\varphi_2, \varphi_3)\eta(\varphi_1) - 2\eta(\varphi_1)\eta(\varphi_2)\eta(\varphi_3)).$$

Following Yano [30], the following formula

$$(\mathcal{L}_\varphi \nabla_{\varphi_1} g - \nabla_{\varphi_1} \mathcal{L}_\varphi g - \nabla_{[\varphi, \varphi_1]} g)(\varphi_2, \varphi_3) = -g((\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_3), \varphi_2) \\ - g((\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_2), \varphi_3)$$

is well-known for any $\varphi_1, \varphi_2, \varphi_3 \in \chi(M)$. As g is parallel with respect to ∇ , the above relation becomes

$$(20) \quad (\nabla_{\varphi_1} \mathcal{L}_\varphi g)(\varphi_2, \varphi_3) = g((\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_2), \varphi_3) + g((\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_3), \varphi_2)$$

for any $\varphi_1, \varphi_2, \varphi_3 \in \chi(M)$. Since $\mathcal{L}_\varphi \nabla$ is a symmetric tensor of type (1, 2), then from (20) it follows that

$$(21) \quad g((\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_2), \varphi_3) = \frac{1}{2}(\nabla_{\varphi_2} \mathcal{L}_\varphi g)(\varphi_1, \varphi_3) + \frac{1}{2}(\nabla_{\varphi_1} \mathcal{L}_\varphi g)(\varphi_2, \varphi_3) \\ - \frac{1}{2}(\nabla_{\varphi_3} \mathcal{L}_\varphi g)(\varphi_1, \varphi_2).$$

Using (19) in (21), we have

$$g((\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_2), \varphi_3) = (\nabla_{\varphi_3} S)(\varphi_1, \varphi_2) - (\nabla_{\varphi_2} S)(\varphi_3, \varphi_1) - (\nabla_{\varphi_1} S)(\varphi_2, \varphi_3) \\ - 2\alpha(\mu + \alpha^2)(g(\varphi_1, \varphi_2) - \eta(\varphi_1)\eta(\varphi_2))\eta(\varphi_3)$$

which by putting $\varphi_2 = \xi$ reduces to

$$(22) \quad g((\mathcal{L}_\varphi \nabla)(\varphi_1, \xi), \varphi_3) = (\nabla_{\varphi_3} S)(\varphi_1, \xi) - (\nabla_{\varphi_1} S)(\xi, \varphi_3) - (\nabla_\xi S)(\varphi_3, \varphi_1).$$

By considering (12) and (13) in (22), we obtain

$$(23) \quad (\mathcal{L}_\varphi \nabla)(\varphi_1, \xi) = 2\alpha Q\varphi_1 + 2\alpha^3(n-1)\varphi_1.$$

Taking the covariant derivative of (23) with respect to φ_2 , we have

$$(\nabla_{\varphi_2} \mathcal{L}_\varphi \nabla)(\varphi_1, \xi) = -\alpha(\mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_2) + 2\alpha\eta(\varphi_2)(\alpha Q\varphi_1 \\ + \alpha^3(n-1)\varphi_1) + 2\alpha(\nabla_{\varphi_2} Q)\varphi_1.$$

Again from [30], we have

$$(\mathcal{L}_\varphi R)(\varphi_1, \varphi_2)\varphi_3 + (\nabla_{\varphi_2} \mathcal{L}_\varphi \nabla)(\varphi_1, \varphi_3) - (\nabla_{\varphi_1} \mathcal{L}_\varphi \nabla)(\varphi_2, \varphi_3) = 0.$$

Thus the last two equations give

$$(24) \quad (\mathcal{L}_\varphi R)(\varphi_1, \varphi_2)\xi = 2\alpha\eta(\varphi_1)(\alpha Q\varphi_2 + \alpha^3(n-1)\varphi_2) \\ - 2\alpha\eta(\varphi_2)(\alpha Q\varphi_1 + \alpha^3(n-1)\varphi_1) \\ - 2\alpha((\nabla_{\varphi_2} Q)\varphi_1 - (\nabla_{\varphi_1} Q)\varphi_2).$$

Setting $\varphi_2 = \xi$ in (24), it follows that

$$(25) \quad (\mathcal{L}_\varphi R)(\varphi_1, \xi)\xi = 0.$$

Taking the Lie derivative of $R(\varphi_1, \xi)\xi = \alpha^2(\eta(\varphi_1)\xi - \varphi_1)$ along the vector field φ , we have

$$(\mathcal{L}_\varphi R)(\varphi_1, \xi)\xi - 2\alpha^2\eta(\mathcal{L}_\varphi\xi)\varphi_1 + \alpha^2g(\mathcal{L}_\varphi\xi, \varphi_1)\xi = \alpha^2(\mathcal{L}_\varphi\eta)(\varphi_1)\xi$$

which by using (25) leads to either $\alpha^2 = 0$, i.e., $\alpha = 0$, therefore M reduces to a cosymplectic manifold, or

$$(26) \quad (\mathcal{L}_\varphi\eta)(\varphi_1)\xi - g(\mathcal{L}_\varphi\xi, \varphi_1)\xi + 2\eta(\mathcal{L}_\varphi\xi)\varphi_1 = 0.$$

Now taking the Lie derivative of $g(\varphi_1, \xi) = \eta(\varphi_1)$, we find

$$(27) \quad (\mathcal{L}_\varphi\eta)\varphi_1 = g(\varphi_1, \mathcal{L}_\varphi\xi) + (\mathcal{L}_\varphi g)(\varphi_1, \xi).$$

By putting $\varphi_2 = \xi$ in (18) and using (2), (3) and (9), we find

$$(28) \quad (\mathcal{L}_\varphi g)(\varphi_1, \xi) = -2(\lambda + \mu - \frac{1}{2n}(pn + 2))\eta(\varphi_1).$$

Again putting $\varphi_1 = \xi$ in (28) leads to

$$(29) \quad \eta(\mathcal{L}_\varphi\xi) = (\lambda + \mu - \frac{1}{2n}(pn + 2)).$$

By making use of (27) – (29), we get from (26) that

$$(\lambda + \mu - \frac{1}{2n}(pn + 2))\phi^2\varphi_1 = 0$$

from which it follows that

$$(30) \quad \lambda + \mu = \frac{1}{2n}(pn + 2),$$

where $\phi^2\varphi_1 \neq 0$. This completes the proof of Theorem 3.1. \square

Lemma 3.2. *If an n -dimensional non-cosymplectic M^α admits $(g^*, \varphi, \lambda, \mu)$, then we have*

$$(31) \quad (i) \ \eta(\mathcal{L}_\varphi\xi) = 0 \quad \text{and} \quad (ii) \ (\mathcal{L}_\varphi g)(\varphi_1, \xi) = 0.$$

Proof. As a consequence of equations (28), (29) and (30), the result follows. \square

Setting $\varphi_2 = \varphi_1 = e_i$ in (11), where $\{e_i, i = 1, 2, \dots, n\}$ represents a set of orthonormal vector fields of M , and taking the summation over $i(1 \leq i \leq n)$ we have

$$(32) \quad r = \rho_1 n + \rho_2.$$

On the other hand, putting $\varphi_1 = \varphi_2 = \xi$ in (11) and making use of (2), (3), we also have

$$(33) \quad -(n-1)\alpha^2 = \rho_1 + \rho_2.$$

Hence from (32) and (33) it follows that

$$\rho_1 = \alpha^2 + \frac{r}{n-1}, \quad \rho_2 = -\alpha^2 n - \frac{r}{n-1}.$$

So the Ricci tensor S of an M^α is given by

$$(34) \quad S(\varphi_1, \varphi_2) = (\alpha^2 + \frac{r}{n-1})g(\varphi_1, \varphi_2) - (\alpha^2 n + \frac{r}{n-1})\eta(\varphi_1)\eta(\varphi_2)$$

which is assumed to admit $(g^*, \varphi, \lambda, \mu)$. If we choose, $r = -n(n-1)\alpha^2$ then (34) becomes $S = -(n-1)\alpha^2 g$. That is, the manifold M^α under consideration is an Einstein manifold. If we consider a proper η -Einstein non-cosymplectic manifold, then $r \neq -n(n-1)\alpha^2$.

Again, taking the Lie derivative of (34) along φ , we find

$$(35) \quad (\mathcal{L}_\varphi S)(\varphi_1, \varphi_2) = (\alpha^2 + \frac{r}{n-1})(\mathcal{L}_\varphi g)(\varphi_1, \varphi_2) + \frac{\varphi(r)}{n-1}(g(\varphi_1, \varphi_2) - \eta(\varphi_1)\eta(\varphi_2)) \\ - (\alpha^2 n + \frac{r}{n-1})[\eta(\varphi_1)(\mathcal{L}_\varphi \eta)(\varphi_2) + \eta(\varphi_2)(\mathcal{L}_\varphi \eta)(\varphi_1)].$$

Setting $\varphi_2 = \xi$ in (35) and making use of the equations (2), (3) and (31) (ii), we have

$$(36) \quad (\mathcal{L}_\varphi S)(\varphi_1, \xi) = -(\alpha^2 n + \frac{r}{n-1})[\eta(\varphi_1)(\mathcal{L}_\varphi \eta)(\xi) + (\mathcal{L}_\varphi \eta)(\varphi_1)].$$

Since $g(\xi, \xi) = 1$, therefore $(\mathcal{L}_\varphi g)(\xi, \xi) = -2g(\mathcal{L}_\varphi \xi, \xi)$. This expression along with the equations (27) and (31) leads to $(\mathcal{L}_\varphi \eta)(\xi) = 0$. Substituting this value in (36), we obtain

$$(37) \quad (\mathcal{L}_\varphi S)(\varphi_1, \xi) = -(\alpha^2 n + \frac{r}{n-1})(\mathcal{L}_\varphi \eta)(\varphi_1).$$

Next contracting (24) along φ_1 , we get

$$(\mathcal{L}_\varphi S)(\varphi_2, \xi) = 2\alpha^2 \eta(Q\varphi_2) + 2\alpha^4(n-1)\eta(\varphi_2) \\ - 2\alpha^2\{\alpha^2 n(n-1) + r\}\eta(\varphi_2) - \alpha\varphi_2(r),$$

since $\varphi_2(r) = 2\text{div}(Q)\varphi_2$. In view of equation (9), the above equation becomes

$$(38) \quad (\mathcal{L}_\varphi S)(\varphi_1, \xi) = -2\alpha^2\{\alpha^2 n(n-1) + r\}\eta(\varphi_1) - \alpha\varphi_1(r).$$

From (37) and (38), we have

$$(39) \quad (\alpha^2 n + \frac{r}{n-1})(\mathcal{L}_\varphi \eta)(\varphi_1) = 2\alpha^2\{\alpha^2 n(n-1) + r\}\eta(\varphi_1) + \alpha\varphi_1(r),$$

which by taking $\varphi_1 = \xi$ and using $(\mathcal{L}_\varphi \eta)(\xi) = 0$ gives

$$(40) \quad 2\alpha^2\{r + \alpha^2 n(n-1)\} + \alpha\xi(r) = 0.$$

Using (40) in (39), we get

$$(\alpha^2 n + \frac{r}{n-1})\mathcal{L}_\varphi \eta = \alpha\{Dr - \xi(r)\xi\},$$

where $r \neq -n(n-1)\alpha^2$, and D denotes the gradient operator. Thus, we can state:

Theorem 3.3. *Let M^α be a non-cosymplectic proper η -Einstein manifold admitting $(g^*, \varphi, \lambda, \mu)$. Then the soliton vector field φ of $(g^*, \varphi, \lambda, \mu)$ is strictly infinitesimal contact transformation if and only if the gradient of scalar curvature of M^α is pointwise colinear with the Reeb vector field ξ .*

Now, let a non-cosymplectic M^α admit $(g^*, \varphi, \lambda, \mu)$ such that $\varphi = h\xi$, where h is a function. Then (1) holds and thus we have

$$(41) \quad \frac{1}{2} \mathcal{L}_{h\xi} g + S^* = -(\lambda - \frac{1}{2n}(pn+2))g - \mu\eta \otimes \eta.$$

By applying Lie derivative and Levi-Civita connection properties in (41), we have

$$\begin{aligned} & \frac{1}{2} hg(\nabla_{\varphi_1} \xi, \varphi_2) + \frac{1}{2} (\varphi_1 h) \eta(\varphi_2) + \frac{1}{2} hg(\varphi_1, \nabla_{\varphi_2} \xi) + \frac{1}{2} (\varphi_2 h) \eta(\varphi_1) \\ & + S^*(\varphi_1, \varphi_2) = -(\lambda - \frac{1}{2n}(pn+2))g(\varphi_1, \varphi_2) - \mu\eta(\varphi_1)\eta(\varphi_2) \end{aligned}$$

which by using (17) takes the form

$$(42) \quad \begin{aligned} & hg(\nabla_{\varphi_1} \xi, \varphi_2) + (\varphi_1 h) \eta(\varphi_2) + hg(\varphi_1, \nabla_{\varphi_2} \xi) + (\varphi_2 h) \eta(\varphi_1) = -2S(\varphi_1, \varphi_2) \\ & -2[\lambda + (n-2)\alpha^2 - \frac{1}{2n}(pn+2)]g(\varphi_1, \varphi_2) - 2(\mu + \alpha^2)\eta(\varphi_1)\eta(\varphi_2). \end{aligned}$$

Putting $\varphi_2 = \xi$ and using (2)-(5) and (9), (42) reduces to

$$(43) \quad (\varphi_1 h) + [(\xi h) + 2\lambda + 2\mu - \frac{1}{n}(pn+2)]\eta(\varphi_1) = 0.$$

Again putting $\varphi_1 = \xi$ in (43) and using (2), we get

$$(44) \quad (\xi h) = -[\lambda + \mu - \frac{1}{2n}(pn+2)].$$

Combining the equations (43) and (44), we find

$$(45) \quad (\varphi_1 h) = -[\lambda + \mu - \frac{1}{2n}(pn+2)]\eta(\varphi_1).$$

As the manifold admits $(g^*, \varphi, \lambda, \mu)$, therefore by virtue of (30), from (45) we get $\varphi_1(h) = 0$, that is, h is constant. Thus from (5) and (42) we obtain

$$(46) \quad \begin{aligned} S(\varphi_1, \varphi_2) &= -[\alpha^2(n-2) + h\alpha + \lambda - \frac{1}{2n}(pn+2)]g(\varphi_1, \varphi_2) \\ &\quad -[\mu + \alpha^2 - h\alpha]\eta(\varphi_1)\eta(\varphi_2). \end{aligned}$$

Therefore we have:

Theorem 3.4. *If a non-cosymplectic M^α admits $(g^*, \varphi, \lambda, \mu)$ and $\varphi = h\xi$, then φ is a constant multiple of ξ and M is an η -Einstein manifold.*

In particular if $\varphi = \xi$, then (46) turns to

$$\begin{aligned} S(\varphi_1, \varphi_2) &= -[\alpha^2(n-2) + \alpha + \lambda - \frac{1}{2n}(pn+2)]g(\varphi_1, \varphi_2) \\ &\quad - [\mu + \alpha^2 - \alpha]\eta(\varphi_1)\eta(\varphi_2). \end{aligned}$$

Thus we have

Corollary 3.5. *Let M^α admit $(g^*, \varphi, \lambda, \mu)$. If $\varphi = \xi$, then M^α is an η -Einstein manifold.*

4. Gradient \ast -conformal η -Ricci solitons on α -cosymplectic manifolds

This section is concerned with the study of gradient $(g^*, \varphi, \lambda, \mu)$ within the context of M^α .

Definition 4.1. *A Riemannian metric g on M is called a gradient $(g^*, \varphi, \lambda, \mu)$ if it satisfies*

$$(47) \quad Hess f + S^* = -(\lambda - \frac{1}{2n}(pn+2))g - \mu\eta \otimes \eta$$

for some smooth function f , where $Hess f$ (Hessian of f) is defined by $Hess f = \nabla \nabla f$. It is noticed that if we choose $\varphi = Df$ in equation (1), where D stands for the gradient operator of g , then we get the equation (47).

If $S^* = (\lambda - \frac{1}{2n}(pn+2))g + \mu\eta \otimes \eta$ for smooth functions λ, μ on M , then the manifold is called \ast -conformal η -Einstein manifold. Further if $\mu = 0$, that is, $S^* = (\lambda - \frac{1}{2n}(pn+2))g$, then the manifold reduced to \ast -conformal Einstein manifold.

Let M be an M^α with g as a gradient $(g^*, \varphi, \lambda, \mu)$. Then from (47) it follows that

$$(48) \quad \nabla_{\varphi_1} Df + Q^* \varphi_1 + (\lambda - \frac{1}{2n}(pn+2))\varphi_1 + \mu\eta(\varphi_1)\xi = 0$$

for all $\varphi_1 \in \chi(M)$. First we prove the following lemmas for further use:

Lemma 4.2. *In an n -dimensional M^α , we have*

$$(49) \quad (\nabla_{\varphi_1} Q^*)\xi - (\nabla_\xi Q^*)\varphi_1 = -\{\alpha^3(n-1) + \frac{\alpha r}{n-1} + \frac{\xi r}{n-1}\}(\varphi_1 - \eta(\varphi_1)\xi).$$

Proof. By using (34) in (17), we have

$$S^*(\varphi_2, \varphi_3) = \{\alpha^2(n-1) + \frac{r}{n-1}\}(g(\varphi_2, \varphi_3) - \eta(\varphi_2)\eta(\varphi_3)).$$

It yields

$$(50) \quad Q^* \varphi_2 = \{\alpha^2(n-1) + \frac{r}{n-1}\}(\varphi_2 - \eta(\varphi_2)\xi).$$

Differentiating (50) along φ_1 , we get

$$(51) \quad (\nabla_{\varphi_1} Q^*)\varphi_2 = \frac{\varphi_1(r)}{n-1}(\varphi_2 - \eta(\varphi_2)\xi) - \left\{ \alpha^2(n-1) + \frac{r}{n-1} \right\} [(\nabla_{\varphi_1} \eta)(\varphi_2)\xi + \eta(\varphi_2)\nabla_{\varphi_1} \xi]$$

which by replacing φ_2 by ξ and using (2), (5) and (6) reduces to

$$(52) \quad (\nabla_{\varphi_1} Q^*)\xi = -\left\{ \alpha^3(n-1) + \frac{\alpha r}{n-1} \right\} (\varphi_1 - \eta(\varphi_1)\xi).$$

Again replacing φ_1 by ξ in (51) and using (5) and (6), we find

$$(53) \quad (\nabla_{\xi} Q^*)\varphi_1 = \frac{\xi r}{n-1}(\varphi_1 - \eta(\varphi_1)\xi).$$

By subtracting (53) from (52), (49) follows. \square

Lemma 4.3. In M^α , we have

$$(54) \quad R(\varphi_1, \varphi_2)Df = (\nabla_{\varphi_2} Q^*)\varphi_1 - (\nabla_{\varphi_1} Q^*)\varphi_2 + \mu\alpha(\eta(\varphi_1)\varphi_2 - \eta(\varphi_2)\varphi_1).$$

Proof. Differentiating (48) covariantly along φ_2 , we have

$$(55) \quad \nabla_{\varphi_2} \nabla_{\varphi_1} Df + \nabla_{\varphi_2} Q^* \varphi_1 + \left(\lambda - \frac{1}{2n}(pn+2) \right) \nabla_{\varphi_2} \varphi_1 + \mu \nabla_{\varphi_2} (\eta(\varphi_1)\xi) = 0.$$

By interchanging φ_1 and φ_2 in (55), we have

$$(56) \quad \nabla_{\varphi_1} \nabla_{\varphi_2} Df + \nabla_{\varphi_1} Q^* \varphi_2 + \left(\lambda - \frac{1}{2n}(pn+2) \right) \nabla_{\varphi_1} \varphi_2 + \mu \nabla_{\varphi_1} (\eta(\varphi_2)\xi) = 0.$$

Also from (48), we find

$$(57) \quad \nabla_{[\varphi_1, \varphi_2]} Df = -Q^*[\varphi_1, \varphi_2] - \left(\lambda - \frac{1}{2n}(pn+2) \right) [\varphi_1, \varphi_2] - \mu \eta([\varphi_1, \varphi_2])\xi.$$

By using (5) and (55)–(57), Lemma 4.3 follows. \square

Theorem 4.4. Let the metric of an M^α admit a gradient $(g^*, \varphi, \lambda, \mu)$. Then either M is cosymplectic or the gradient of the potential function is pointwise collinear with the Reeb vector field ξ .

Proof. Putting $\varphi_1 = \xi$ in (55), we have

$$R(\xi, \varphi_2)Df = (\nabla_{\varphi_2} Q^*)\xi - (\nabla_{\xi} Q^*)\varphi_2 + \mu(\varphi_2 - \eta(\varphi_2)\xi)$$

which by virtue of the Lemma 4.2 leads to

$$(58) \quad g(R(\xi, \varphi_2)Df, \xi) = 0.$$

By using (7), we have

$$(59) \quad g(R(\xi, \varphi_2)Df, \xi) = \alpha^2[\eta(\varphi_2)(\xi f) - (\varphi_2 f)].$$

From (58) and (59), we have

$$\alpha^2[\eta(\varphi_2)(\xi f) - (\varphi_2 f)] = 0.$$

Thus we have either $\alpha^2 = 0$ or $(\varphi_2 f) = \eta(\varphi_2)(\xi f)$. Hence for $\alpha^2 = 0$, i.e., $\alpha = 0$, the manifold M reduces to a cosymplectic. From the latter case, we find $Df = (\xi f)\xi$. This completes the proof. \square

We suppose that $\alpha \neq 0$, then from Theorem 4.4, we have $Df = (\xi f)\xi$. Taking covariant derivative of this expression along φ_1 , we have

$$(60) \quad \nabla_{\varphi_1} Df = (\varphi_1(\xi f))\xi + \alpha(\xi f)(\varphi_1 - \eta(\varphi_1)\xi),$$

which gives

$$g(\nabla_{\varphi_1} Df, \xi) = \varphi_1(\xi f),$$

where the equations (2) and (3) are used. Using the last equation in (60), we obtain

$$(61) \quad \nabla_{\varphi_1} Df = g(\nabla_{\varphi_1} Df, \xi)\xi + \alpha(\xi f)(\varphi_1 - \eta(\varphi_1)\xi).$$

From the equations (17) and (47), we conclude that

$$(62) \quad \nabla_{\varphi_1} Df = -Q\varphi_1 - [\alpha^2(n-2) + \lambda - \frac{1}{2n}(pn+2)]\varphi_1 - [\alpha^2 + \mu]\eta(\varphi_1)\xi,$$

which implies that

$$(63) \quad g(\nabla_{\varphi_1} Df, \xi) = [\frac{1}{2n}(pn+2) - (\lambda + \mu)]\eta(\varphi_1).$$

From the equations (30) and (61)–(63), we lead to

$$Q\varphi_1 = -[\alpha^2(n-2) + \lambda - \frac{1}{2n}(pn+2) + \alpha(\xi f)]\varphi_1 + [\alpha(\xi f) - (\alpha^2 + \mu)]\eta(\varphi_1)\xi,$$

which informs that the manifold M^α under the consideration is an η -Einstein. Hence, we can state:

Corollary 4.5. *Every M^α of dimension n endowed with a gradient \ast -conformal η -Ricci metric is η -Einstein, provided $\alpha \neq 0$.*

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