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COMMUTATIVE ELLIPTIC OCTONIONS

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Abstract. In this article, the matrix representation of commutative elliptic octonions and their properties are described. Firstly, definitions and theorems are given for the commutative elliptic octonion matrices using the elliptic quaternion matrices. Then the adjoint matrix, eigenvalue and eigenvector of the commutative elliptic octonions are investigated. Finally, $\alpha = -1$ for the Gershgorin Theorem is proved using eigenvalue and eigenvector of the commutative elliptic octonion matrix.

1. Introduction

A four-dimensional hyper-complex number, the quaternion algebra was described by W.R. Hamilton in 1853, [5]. Later in 1936, the real quaternion matrices were introduced by L.A. Wolf, [12]. After the quaternion algebra was used in many important fields such as modern mathematics, quantum physics, painting and signal processing, and matrix analysis, [8, 9]. But it is well known that the main obstacle in the study of the quaternion, the quaternion algebra does not provide commutative property of multiplication. So F. Catoni, R. Cannata, V. Catoni and P. Zampetti defined commutative quaternions in 2006, [2]. Later, H.H. Kösal and M. Tosun studied the properties of the commutative quaternion matrices and the complex adjoint matrix for a commutative quaternion matrix by using determinant calculus with adjoint matrix, [7].

If $i^2 = \alpha \in R$ $(\alpha < 0)$ is taken instead of $i^2 = -1$ in the commutative quaternions, the set of the elliptic quaternion is defined by the following equation

$$H_p = \{a = a_0 + a_1 i + a_2 j + a_3 k \mid \; a_i \in, \;\; 0 \leq i \leq 3\}$$

where $ij = ji = k, \quad ik = ki = \alpha j, \quad jk = kj = i, \quad i^2 = k^2 = \alpha, \quad j^2 = 1$, [6]. After the properties of the matrix structure in the elliptic quaternion were studied and the complex adjoint matrix of the elliptic quaternion was described, [6, 3]. With these studies, the elliptic quaternion algebra has had an important field of study in the literature.

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The octonion algebra O, a superset of the quaternion algebra, was obtained by Cayley-Dickson construction, [1]. After matrix representations of octonions was studied by Y. Tian, [11]. Then A. Cihan and M.A. Güngör given by the commutative octonion matrices and some properties of these matrices, [4]. Finally, Y. Song defined the set $SL_n(R)$ that is the most general of commutative octonion set, [10].

In this article, we examine commutative elliptic octonion matrices. First, commutative elliptic octonions CO_p and some of their properties are given. Then, some theorems and properties are shown for commutative elliptic octonion matrices by using the properties of elliptic quaternion matrices. Later, the elliptic quaternion adjoint matrix for the commutative elliptic octonion matrix is defined and the eigenvalues and eigenvectors of commutative elliptic octonion matrices are calculated with the help of the elliptic quaternion adjoint matrix. With the help of these calculations, $\alpha = -1$ for Gershgorin Theorem is proved. Finally, an example that related to the Gershgorin Theorem is given.

2. Commutative Elliptic Octonions

In this article we will explore how to construct the commutative elliptic octonion algebra. Since a commutative elliptic octonion can be written as a hyperbolic number that is elliptic quaternion their coefficients, a commutative elliptic octonion a is expressed by the following equation

$$a = a' + a''e$$

where $e^2 = 1$ and $a',\ a'' \in H_p$. Let

$$\begin{aligned} a' &= a_0 + a_1 i + a_2 j + a_3 k \in H_p \\ a'' &= a_4 + a_5 i + a_6 j + a_7 k \in H_p \end{aligned}$$

then a commutative elliptic octonion is written

$$a = (a_0 + a_1 i + a_2 j + a_3 k) + (a_4 + a_5 i + a_6 j + a_7 k)\, e.$$

As a result of these, a set of the commutative elliptic octonion is denoted by

$$\begin{aligned} CO_p = \{ a = {} & a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 \\ & + a_7 e_7 \mid a_i \in R,\ 0 \le i \le 7 \}, \end{aligned}$$

and the bases of the commutative elliptic octonions are defined by the following equations

$$\begin{aligned} & e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k,\ e_4 = e,\ e_5 = ie = ei,\ e_6 = je = ej,\ e_7 = ke = ek, \\ & e_0^2 = 1,\ e_1^2 = \alpha,\ e_2^2 = 1,\ e_3^2 = \alpha,\ e_4^2 = 1,\ e_5^2 = \alpha,\ e_6^2 = 1,\ e_7^2 = \alpha. \end{aligned}$$

If these equations are taken into account, the multiplication rule of the commutative elliptic octonion bases are given by the following table:

Table 1. The multiplication scheme of the commutative elliptic octonion units.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	αe_0	e_3	αe_2	e_5	αe_4	e_7	αe_6
e_2	e_2	e_3	e_0	e_1	e_6	e_7	e_4	e_5
e_3	e_3	αe_2	e_1	αe_0	e_7	αe_6	e_5	αe_4
e_4	e_4	e_5	e_6	e_7	e_0	e_1	e_2	e_3
e_5	e_5	αe_4	e_7	αe_6	e_1	αe_0	e_3	αe_2
e_6	e_6	e_7	e_4	e_5	e_2	e_3	e_0	e_1
e_7	e_7	αe_6	e_5	αe_4	e_3	αe_2	e_1	αe_0

Addition and multiplication of any commutative elliptic octonion $a = a' + a''e$, $b = b' + b''e \in CO_p$ are defined by

$$a + b = \left(a' + a''e\right) + \left(b' + b''e\right) = \left(a' + b'\right) + \left(a'' + b''\right) e \tag{1}$$

and

$$a \times b = \left(a' + a''e\right) \times \left(b' + b''e\right) = \left(a'b' + a''b''\right) + \left(a'b'' + b'a''\right) e \tag{2}$$

respectively. Let $a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \in CO_p$, then the commutative elliptic octonion a expressed as

$$a = \mathrm{Re}a + \mathrm{Im}a$$

where $\mathrm{Re}a = a_0e_0$ and $\mathrm{Im}a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$ are called the real and imaginary parts of the commutative elliptic octonion, respectively.

For $a = a' + a''e \in CO_p$, there exists seven kinds of its conjugate:

$$\begin{aligned}
a^{o_1} &= a'^{(1)} + a''^{(1)}e,\\
a^{o_2} &= a'^{(2)} + a''^{(2)}e,\\
a^{o_3} &= a'^{(3)} + a''^{(3)}e,\\
a^{o_4} &= a' - a''e,\\
a^{o_5} &= a'^{(1)} - a''^{(1)}e,\\
a^{o_6} &= a'^{(2)} - a''^{(2)}e,\\
a^{o_7} &= a'^{(3)} - a''^{(3)}e,
\end{aligned} \tag{3}$$

where $(1), (2), (3)$ denotes conjugates of the elliptic quaternion $q = t + xi + yj + xk \in H_p$

$$\begin{aligned}
q^{(1)} &= t - xi + yj - zk,\\
q^{(2)} &= t + xi - yj - zk,\\
q^{(3)} &= t - xi - yj + zk,
\end{aligned}$$

[6].

Property 2.1. *Let $a = (a_0 + a_1i + a_2j + a_3k) + (a_4 + a_5i + a_6j + a_7k)\, e \in CO_p$. Then the following identities hold:*

1) $a + a^{o_1} + a^{o_2} + a^{o_3} + a^{o_4} + a^{o_5} + a^{o_6} + a^{o_7} = 8a_0,$

2) $a_0 = \frac{a + a^{o_1} + a^{o_2} + a^{o_3} + a^{o_4} + a^{o_5} + a^{o_6} + a^{o_7}}{8} e_0,$

3) $a_1 = \frac{a - a^{o_1} + a^{o_2} - a^{o_3} + a^{o_4} - a^{o_5} + a^{o_6} - a^{o_7}}{8} e_1,$

4) $a_2 = \frac{a + a^{o_1} - a^{o_2} - a^{o_3} + a^{o_4} + a^{o_5} - a^{o_6} - a^{o_7}}{8} e_2,$

5) $a_3 = \frac{a - a^{o_1} - a^{o_2} + a^{o_3} + a^{o_4} - a^{o_5} - a^{o_6} + a^{o_7}}{8} e_3,$

6) $a_4 = \frac{a + a^{o_1} + a^{o_2} + a^{o_3} - a^{o_4} - a^{o_5} - a^{o_6} - a^{o_7}}{8} e_4,$

7) $a_5 = \frac{a - a^{o_1} + a^{o_2} - a^{o_3} - a^{o_4} + a^{o_5} - a^{o_6} + a^{o_7}}{8} e_5,$

8) $a_6 = \frac{a + a^{o_1} - a^{o_2} - a^{o_3} - a^{o_4} - a^{o_5} + a^{o_6} + a^{o_7}}{8} e_6,$

9) $a_7 = \frac{a - a^{o_1} - a^{o_2} + a^{o_3} - a^{o_4} + a^{o_5} + a^{o_6} - a^{o_7}}{8} e_7.$

Proof. Proof can be easily seen by using the equation (1) and the equation (3). □

Definition 2.2. *Let $a \in CO_p$. The norm of $a = (a_0 + a_1 i + a_2 j + a_3 k) + (a_4 + a_5 i + a_6 j + a_7 k)\, e$ is defined as*

$$
\begin{aligned}
\|a\|^8 &= a \times a^{o_1} \times a^{o_2} \times a^{o_3} \times a^{o_4} \times a^{o_5} \times a^{o_6} \times a^{o_7} \\
&= \left[(a_0 + a_2 - a_4 - a_6)^2 - \alpha(a_1 + a_3 - a_5 - a_7)^2\right] \\
&\times \left[(a_0 - a_2 + a_4 - a_6)^2 - \alpha(a_1 - a_3 + a_5 - a_7)^2\right] \\
&\times \left[(a_0 - a_2 - a_4 + a_6)^2 - \alpha(a_1 - a_3 - a_5 + a_7)^2\right] \\
&\times \left[(a_0 + a_2 + a_4 + a_6)^2 - \alpha(a_1 + a_3 + a_5 + a_7)^2\right] \geq 0.
\end{aligned} \tag{4}
$$

Definition 2.3. *The identity of the commutative elliptic octonion*

$$a = (a_0 + a_1 i + a_2 j + a_3 k) + (a_4 + a_5 i + a_6 j + a_7 k)\, e$$

is nearby to real vector $\mathbf{a} \in R^{8\times 1}$,

$$a \cong \mathbf{a} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T.$$

On the other hand, there is bijection transformation φ,

$$
\begin{aligned}
\varphi : CO_p &\to M \\
\mathrm{a} &\to \varphi(a)
\end{aligned}
$$

$$\varphi(a)=\begin{bmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 & a_4 & \alpha a_5 & a_6 & \alpha a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 & a_6 & \alpha a_7 & a_4 & \alpha a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & \alpha a_5 & a_6 & \alpha a_7 & a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & \alpha a_7 & a_4 & \alpha a_5 & a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}. \tag{5}$$

The matrix $\varphi(a)$ is called the basic matrix of the commutative elliptic octonion a.

Set M, defined by the base matrix of the commutative elliptic octonion, is given by the following equation

$$M=\left\{\begin{bmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 & a_4 & \alpha a_5 & a_6 & \alpha a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 & a_6 & \alpha a_7 & a_4 & \alpha a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & \alpha a_5 & a_6 & \alpha a_7 & a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & \alpha a_7 & a_4 & \alpha a_5 & a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}\;;\quad a_i\in R,\quad 0\le i\le 7,\quad \alpha<0\right\}.$$

Let a and b be two commutative elliptic octonion, then the multiplacition of these commutative elliptic octonions denoted by

$$a\times b=b\times a\cong\varphi(a)\,\mathbf{b}=\begin{bmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 & a_4 & \alpha a_5 & a_6 & \alpha a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 & a_6 & \alpha a_7 & a_4 & \alpha a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & \alpha a_5 & a_6 & \alpha a_7 & a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & \alpha a_7 & a_4 & \alpha a_5 & a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix}.$$

As can be seen here, in the clause with octonion, the basic matrix is divided into two as of right and left, whereas in commutative elliptic octonions the basic matrix is single.

Theorem 2.4. *Let $a,b\in CO_p$ and α, $\beta\in R$. Then the following identities hold:*

1) $a=b\Leftrightarrow\varphi(a)=\varphi(b)$,
2) $\varphi(a+b)=\varphi(a)+\varphi(b)$,
   $\varphi(a\times b)=\varphi(a)\,\varphi(b)$,
3) $\varphi(\alpha a+\beta b)=\alpha\varphi(a)+\beta\varphi(b)$,
4) $\|a\|^8=|\det(\varphi(a))|$,
5) $Trace(\varphi(a))=8a_0$.

Proof. Proof of 1), 2) and 3) can be easily seen with the equation (5). 4) can be obtained by the equations of (3), (4) and (5). Proof of 5) can be easily seen with the equation (5). □

Theorem 2.5. *Every commutative elliptic octonion can be represented by an elliptic quaternion matrix of type* 2×2.

Proof. Let $a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \in CO_p$ and

$$a = a' + a''e, \qquad e^2 = 1$$

where $a' = a_0 + a_1i + a_2j + a_3k$, $a'' = a_4 + a_5i + a_6j + a_7k \in H_p$.
Then for any $b \in CO_p$,

$$\begin{array}{l} \psi_a : CO_p \to CO_p \\ \qquad \mathrm{b} \to \psi_a(b) = a \times b \end{array}$$

$$\begin{array}{l} \psi_a(1) = a' + a''e \\ \psi_a(e) = (a' + a''e)\,e = a'e + a''e^2 = a'' + a'e \end{array}$$

with this transformation set N can be defined the following

$$N = \left\{ \begin{pmatrix} a' & a'' \\ a'' & a' \end{pmatrix} : \ a', a'' \in H_p \right\}.$$

Then CO_p and N are essentially same. ψ that is bijective and protecting the operation, is denoted by

$$\begin{array}{l} \psi : CO_p \to N \\ \qquad a = a' + a''e \to \psi(a) = \begin{pmatrix} a' & a'' \\ a'' & a' \end{pmatrix}. \end{array}$$

Furthermore there is the following relation between the norm of the commutative elliptic octonion a and $\psi(a)$,

$$\|a\|^8 = |\det \psi(a)|.$$

□

3. Commutative Elliptic Octonions Matrices

Now let's investigate the properties of the commutative elliptic octonion matrices whose elements are commutative elliptic octonion. $U_{m\times n}(CO_p)$ is defined the set of the commutative elliptic octonions matrix of $m \times n$ type.
Let $A = (a_{ij}) \in U_{m\times n}(CO_p)$, $B = (b_{ij}) \in U_{m\times n}(CO_p)$, $C = (c_{ij}) \in U_{n\times p}(CO_p)$ and a_{ij}, b_{ij}, $c_{jk} \in CO_p$ be, then addition and multiplication in the set of the commutative elliptic octonion matrix is defined by

$$\begin{array}{l} A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \in U_{m\times n}(CO_p), \\ AC = \left(\sum\limits_{j=1}^{n} a_{ij}c_{jk} \right) \in U_{m\times p}(CO_p), \end{array} \tag{6}$$

respectively. Considering equation (6), it is given by the following equations

$$\begin{aligned} aA &= Aa = (aa_{ij}), \\ (aA)\,C &= a\,(AC), \\ (Aa)\,C &= A\,(aC), \\ (ab)\,A &= a\,(bA), \end{aligned}$$

where $a, b \in CO_p$.

For $A = (a_{ij}) \in U_{m\times n}(CO_p)$ there exists seven kinds of the matrix A conjugates:

$$\begin{aligned} A^{o_1} &= ({a_{ij}}^{o_1}) \\ A^{o_2} &= ({a_{ij}}^{o_2}) \\ A^{o_3} &= ({a_{ij}}^{o_3}) \\ A^{o_4} &= ({a_{ij}}^{o_4}) \\ A^{o_5} &= ({a_{ij}}^{o_5}) \\ A^{o_6} &= ({a_{ij}}^{o_6}) \\ A^{o_7} &= ({a_{ij}}^{o_7}). \end{aligned} \tag{7}$$

$A^T = (a_{ji}) \in U_{n\times m}(CO_p)$ is the transpose of the matrix $A \in U_{m\times n}(CO_p)$ and $A^{o_{\dagger_i}} = (A^{o_i})^T \in U_{n\times m}(CO_p)$ is the i^{th} conjugate transpose of the matrix $A \in U_{m\times n}(CO_p)$. If $A \in U_{n\times n}(CO_p)$ is Hermitian matrix by i^{th} the conjugate, the equation $AA^{o_{\dagger_i}} = A^{o_{\dagger_i}}A$ is provided.

Theorem 3.1. *Let $A \in U_{m\times n}(CO_p), B \in U_{n\times p}(CO_p)$ and then the following properties hold:*

i. $(A^{o_i})^T = (A^T)^{o_i}$,
ii. $(AB)^{o_{\dagger_i}} = B^{o_{\dagger_i}} A^{o_{\dagger_i}}$,
iii. $(AB)^T = B^T A^T$,
iv. $(AB)^{o_i} = A^{o_i} B^{o_i}$,
v. If A and B are invertible, $(AB)^{-1} = B^{-1}A^{-1}$,
vi. If A is invertible, $\left(A^{o_{\dagger_i}}\right)^{-1} = \left(A^{-1}\right)^{o_{\dagger_i}}$,
vii. $(A^{o_i})^{o_j} = \begin{cases} A^{o_k}, & i \neq j \neq k \\ A, & i = j \end{cases}$

where A^{o_i} is i^{th} conjugate of the commutative elliptic octonion matrix $(1 \leq i, j, k \leq 7)$.

Proof. i, ii, iii, v, vi, vii and $viii$ can be easily proven. We will prove iv.

Let $A = A_1 + A_2 e \in U_{m\times n}(CO_p)$ and $B = B_1 + B_2 e \in U_{n\times p}(CO_p)$ be where A_1, A_2, B_1 and B_2 are the elliptic quaternion matrices.
Then the following equations are written for $i = 1$

$$\begin{aligned} (AB)^{o_1} &= [(A_1 + A_2 e)(B_1 + B_2 e)]^{o_1} \\ &= [(A_1B_1) + (A_2B_2) + [(A_1B_2) + (A_2B_1)]\,e]^{o_1} \\ &= [(A_1B_1 + A_2B_2) + (A_2B_1 + A_1B_2)\,e]^{o_1} \\ &= (A_1B_1 + A_2B_2)^{(1)} + (A_2B_1 + A_1B_2)^{(1)}e \\ &= \left({A_1}^{(1)}{B_1}^{(1)} + {A_2}^{(1)}{B_2}^{(1)}\right) + \left({A_2}^{(1)}{B_1}^{(1)} + {A_1}^{(1)}{B_2}^{(1)}\right) e \end{aligned}$$

and

$$\begin{aligned}A^{o_1}B^{o_1} &= (A_1 + A_2e)^{o_1}(B_1 + B_2e)^{o_1}\\ &= \left(A_1{}^{(1)} + A_2{}^{(1)}e\right)\left(B_1{}^{(1)} + B_2{}^{(1)}e\right)\\ &= \left(A_1{}^{(1)}B_1{}^{(1)} + A_2{}^{(1)}B_2{}^{(1)}\right) + \left(A_2{}^{(1)}B_1{}^{(1)} + A_1{}^{(1)}B_2{}^{(1)}\right)e\end{aligned}$$

where o_1 is conjugate of the commutative elliptic octonion and (1) is represented conjugate of the elliptic quaternion. Since the right-hand sides of the written equations are the same, the proof for iv is completed. The same way, the proof can be easily seen for $2 \le i \le 7$. □

Theorem 3.2. *Let $A, B \in U_{n\times n}(CO_p)$. If $AB = I$, then $BA = I$.*

Proof. Let $A = A_1 + A_2e \in U_{n\times n}(CO_p)$ and $B = B_1 + B_2e \in U_{n\times n}(CO_p)$ be where $A_1,\ A_2,\ B_1,\ B_2$ are $n \times n$ type of the elliptic quaternion matrices. Considering the equation (6), the following equation is written

$$AB = (A_1B_1 + A_2B_2) + (A_1B_2 + A_2B_1)e = I_n.$$

From the equation AB we have

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix} = \begin{pmatrix} I_n & 0 \end{pmatrix},$$

so

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

Since $\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$ and $\begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix}$ are the elliptic quaternion matrices, we obtain

$$\begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

On the other hand, we can write $B_1A_1 + B_2A_2 = I_n, \quad B_1A_2 + B_2A_1 = 0,$

$$(B_1A_1 + B_2A_2) + (B_1A_2 + B_2A_1)e = I_n.$$

Consequently we obtain $BA = I$. Thus the proof is completed.

□

Definition 3.3. *Let $A = A_1 + A_2e \in U_{n\times n}(CO_p)$ and $\eta(A)$ be a $2n \times 2n$ type of the elliptic quaternion matrix. $\eta(A)$ is denoted by*

$$\eta(A) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}.$$

Further $\eta(A)$ is refer to the elliptic quaternion adjoint matrix of the matrix A.

Conclusion 3.4. *Let $A \in U_{n\times n}(CO_p)$ and $\eta(A)$ be elliptic quaternion adjoint matrix of the matrix A. Then, the determinant of the matrix A is described by the following equation*

$$\det(A) = \det(\eta(A))$$

where $\det(\eta(A))$ is the determinant of the matrix $\eta(A)$.

Example 3.5. *Let $A = \begin{bmatrix} 1+i+j+k+e & ei \\ ej & ek \end{bmatrix}$ be a commutative elliptic octonion matrix. Then the elliptic quaternion adjoint matrix of the matrix A is written by the following equation*

$$\eta(A) = \begin{bmatrix} 1+i+j+k & 0 & 1 & i \\ 0 & 0 & j & k \\ 1 & i & 1+i+j+k & 0 \\ j & k & 0 & 0 \end{bmatrix}.$$

Hence $\det(A) = \det(\eta(A)) = -\alpha(2+4i+2j+4k+2\alpha+2\alpha j)$ is seen easily.

Theorem 3.6. *Let $A, B \in U_{n\times n}(CO_p)$. Then the following properties hold:*
$i.\, \eta(I_n) = I_{2n}$,
$ii.\, \eta(A+B) = \eta(A) + \eta(B)$,
$iii.\, \eta(AB) = \eta(A)\eta(B)$,
$iv.$ If $A^{-1} \neq 0, \quad \eta(A^{-1}) = (\eta(A))^{-1}$,
$v.\, \eta\left(A^{o_{\dagger_i}}\right) = (\eta(A))^{o_{\dagger_i}},\ 1 \leq i \leq 7$,
$vi.$ $\det(AB) = \det(A)\det(B)$
If $A^{-1} \neq 0, \quad \det(A^{-1}) = (\det(A))^{-1}$.

Proof. The proof of i, ii, iv, vi can be easily seen. iii and v will show to be proven.

iii. Let $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ and $B = B_1 + B_2 e \in U_{n\times n}(CO_p)$.

Then the elliptic quaternion adjoint matrix of $AB = (A_1B_1 + A_2B_2) + (A_1B_2 + A_2B_2)\,e$ are written

$$\eta(AB) = \begin{pmatrix} A_1B_1 + A_2B_2 & A_1B_2 + A_2B_2 \\ A_1B_2 + A_2B_2 & A_1B_1 + A_2B_2 \end{pmatrix}.$$

On the other hand the elliptic quaternion adjoint matrix of $A = A_1 + A_2 e$ and $B = B_1 + B_2 e$ are expressed as

$$\eta(A) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}, \quad \eta(B) = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix},$$

respectively.
If $\eta(A)\eta(B)$ are calculated, then the following equation is written

$$\eta(A)\eta(B) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix} = \begin{pmatrix} A_1B_1 + A_2B_2 & A_1B_2 + A_2B_2 \\ A_1B_2 + A_2B_2 & A_1B_1 + A_2B_2 \end{pmatrix}.$$

Consequently $\eta(AB) = \eta(A)\eta(B)$ is obtained and the proof is completed.

v. Let $A \in U_{n\times n}(CO_p)$. Then using by the transpose conjugate and the elliptic adjoint matrix definitions of the matrix A, can be written by the following equations

$$A^{\overset{o}{\dagger}_1} = (A_1 + A_2 e)^{\overset{o}{\dagger}_1} = A_1{}^{\dagger_1} + A_2^{\dagger_1} e$$

and

$$\eta\left(A^{\overset{o}{\dagger}_1}\right) = \begin{pmatrix} A_1^{\overset{o}{\dagger}_1} & A_2^{\overset{o}{\dagger}_1} \\ A_2^{\overset{o}{\dagger}_1} & A_2^{\overset{o}{\dagger}_1} \end{pmatrix}.$$

On the other hand the elliptic quaternion matrix and conjugate of the elliptic quaternion matrix of the matrix A are defined by

$$\eta(A) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \text{ and } (\eta(A))^{\overset{o}{\dagger}_1} = \begin{pmatrix} A_1^{\overset{o}{\dagger}_1} & A_2^{\overset{o}{\dagger}_1} \\ A_2^{\overset{o}{\dagger}_1} & A_2^{\overset{o}{\dagger}_1} \end{pmatrix},$$

repectively. So $\left(\eta(A)^{\overset{o}{\dagger}_1}\right) = (\eta(A))^{\overset{o}{\dagger}_1}$ is written. Similarly, the proofs of $2 \le i \le 7$ can be proven.

□

Definition 3.7. *Let $A \in U_{n\times n}(CO_p)$, $\lambda \in CO_p$ and $x \in U_{n\times 1}(CO_p)$. If $0 \ne x$ is provided that $Ax = \lambda x$, λ is called the eigenvalue of the matrix A. Further x is called the eigenvector that is corresponding to the eigenvalue λ. The set of eigenvalues of the matrix A denoted by*

$$\xi(A) = \{\lambda \in CO_p : \ Ax = \lambda x \ , \exists x \ne 0\}$$

and $\xi(A)$ is called a spectrum of the matrix A.

Theorem 3.8. *Let $A = (a_{ij}) \in U_{n\times n}(CO_p)$. The following are equivalent:*

i.A is invertible,
ii.$Ax = 0$ has a unique solution,
iii.$\det(\eta(A)) \ne 0$ i.e $\eta(A)$ is invertible,

Proof. i. $\Rightarrow$ *ii.*, it is obvious.
ii. $\Rightarrow$ *iii.*, let $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ and $x = x_1 + x_2 e \in U_{n\times 1}(CO_p)$. Then

$$\begin{aligned} Ax &= (A_1 + A_2 e)(x_1 + x_2 e) \\ &= (A_1 x_1 + A_2 x_2) + (A_1 x_2 + A_2 x_1) e. \end{aligned}$$

Because of $Ax = 0$, it can be written

$$(A_1 x_1 + A_2 x_2) = 0 \text{ and } (A_1 x_2 + A_2 x_1) = 0.$$

Thus

$$Ax = 0 \text{ if and only if } \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, there is equation in the following

$$\eta(A) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Since $Ax = 0$ has a unique solution, $\eta(A)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ has a unique solution. Thus, since $\eta(A)$ is elliptic quaternion matrix, $\eta(A)$ is invertible.

iii. $\Rightarrow$ *i.*, if $\eta(A)$ is invertible, then for $A = A_1 + A_2 e \in U_{n\times n}(CO_p)$ there exist a elliptic quaternion matrix $\begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix}$ that is provided the equation $\begin{bmatrix} B_1 & B_2 \\ B_2 & B_1 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Thus, it can written $B_1A_1 + B_2A_2 = I$ and $B_1A_2 + B_2A_1 = 0$. Using by this equation, it is obtained

$$(B_1A_1 + B_2A_2) + (B_1A_2 + B_2A_1)\, e = I.$$

That is $BA = I$ for $B = B_1 + B_2 e$. So the matrix A is invertible by Theorem 3.2 and the proof is completed.
Also, as a consequence of the Theorem 3.8, A has no zero eigenvalue. $\square$

If $\alpha = -1$ is taken in the set of the commutative elliptic octonion, the set of the commutative octonion is obtained. Based on this situation, let's give the eigenvalue and eigenvector definitions that are important for commutative octonions. CO and $U_{n\times n}(CO)$ are denoted by the set commutative octonion and the set of $n \times n$ type of commutative octonion matrix, respectively. H and H_n^n are denoted by the set commutative quaternion and the set of $n \times n$ type of commutative quaternion matrix, respectively.

Theorem 3.9. *$A \in U_{n\times n}(CO)$ has at the very most $2n$ commutative quaternion eigenvalues and $4n$ complex eigenvalues.*

Proof. Let $A = A_1 + A_2 e \in U_{n\times n}(CO)$ and $\lambda \in H$ be an eigenvalue of the matrix A. Hence there exist $x = x_1 + x_2 e \in U_{n\times 1}(CO)$ nonzero column vectors that is provided $Ax = \lambda x$. Therefore it can be written by the following equation

$$\begin{aligned} &(A_1 + A_2 e)(x_1 + x_2 e) = \lambda x_1 + \lambda x_2 e, \\ &A_1x_1 + A_2x_2 = \lambda x_1 \quad \text{and} \quad A_1x_2 + A_2x_1 = \lambda x_2 \end{aligned}$$

and this equation provided that

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus, it is seen that the commutative octonion matrix A has at the very most $2n$ commutative quaternion eigenvalues. Considering that the commutative quaternion matrix has at the very most $2n$ complex eigenvalues, it is seen that it has at the very most $4n$ complex eigenvalues for the commutative octonion matrix. $\square$

Corollary 3.10. *Let $A \in U_{n\times n}(CO)$ and*

$$\xi(\eta(A)) = \{\lambda \in H : \ \eta(A)\, y = \lambda y \ , \exists y \neq 0\}$$

be the set of eigenvalues of the adjoint matrix $\eta(A)$.

$$\xi(A) \cap H = \xi(\eta(A))$$

is provided where $\xi(A) = \{\lambda \in CO : \ Ay = \lambda y \ , \exists y \neq 0\}$ *is the set of eigenvalues of the adjoint matrix* A*.*

Theorem 3.11. *Let* $A = A_1 + A_2 e \in U_{n\times n}(CO)$ *and* $\lambda = \lambda_1 + \lambda_2 e$ *be an eigenvalue of* A*. Then for* λ *if and only if there exist* $x_1, x_2 \in H_1^n \quad (x_1 \neq 0, \ x_2 \neq 0)$ *such that*

$$\begin{bmatrix} A_1 - \lambda_1 I & A_2 - \lambda_2 I \\ A_2 - \lambda_2 I & A_1 - \lambda_1 I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Proof. Let $A = A_1 + A_2 e \in U_{n\times n}(CO)$ and $\lambda = \lambda_1 + \lambda_2 e$ be the eigenvalue of A if and only if there exists $x_1, x_2 \in H_1^n \quad (x_1 \neq 0, \ x_2 \neq 0)$ that provided the equation

$$(A_1 + A_2 e)(x_1 + x_2 e) = (\lambda_1 + \lambda_2 e)(x_1 + x_2 e).$$

Therefore it can be written

$$\begin{array}{l} A_1x_1 + A_1x_2e + A_2x_1e + A_2x_2 = \lambda_1x_1 + \lambda_1x_2e + \lambda_2x_1e + \lambda_2x_2 \\ (A_1x_1 + A_2x_2) + (A_1x_2 + A_2x_1)e = \lambda_1x_1 + \lambda_2x_2 + (\lambda_1x_2 + \lambda_2x_1)e \end{array}$$

and so the following equations are provided

$$\begin{array}{l} (A_1 - \lambda_1 I_n)x_1 + (A_2 - \lambda_2 I_n)x_2 = 0 \\ (A_1 - \lambda_1 I_n)x_2 + (A_2 - \lambda_2 I_n)x_1 = 0. \end{array}$$

Using these obtained equations, we may write

$$\begin{bmatrix} A_1 - \lambda_1 I & A_2 - \lambda_2 I \\ A_2 - \lambda_2 I & A_1 - \lambda_1 I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

□

Now let's give the Gershgorin Theorem for commutative octonion matrices, which is the special case of commutative elliptic octonion matrices.

Theorem 3.12 (Gershgorin Theorem for commutative octonions matrices)**.** *Let* $A = (a_{ij}) \in U_{n\times n}(CO)$*. Then*

$$\xi(A) \subseteq \bigcup_{i=1}^{n} \{a \in CO : \ \|a - a_{ii}\| \leq R_i\}$$

where $R_i = \sum\limits_{j=1,\, i\neq j}^{n} \|a_{ij}\|$ *and the set of eigenvalues of the adjoint matrix* A $\xi(A) = \{\lambda \in CO : \ Ay = \lambda y \ , \exists y \neq 0\}$.

Proof. Let $A \in M_{n\times n}(CO)$ and λ be eigenvalue of $A = (a_{ij})$. Besides, $x \neq 0$ is the corresponding eigenvector then $Ax = \lambda x$. Also x_i is component of x such that $\|x_i\| \geq \|x_j\|$ for all j then we have $\|x_i\| > 0$ and λx_i corresponds to the i^{th} component of vector Ax which means that

$$\lambda x_i = \sum_{j=1}^{n} a_{ij}x_j.$$

For this reason, we may write

$$\lambda x_i - a_{ii}x_i = \sum_{j=1, \mathrm{i}\neq \mathrm{j}}^{n} a_{ij}x_j \Rightarrow (\lambda - a_{ii})x_i = \sum_{j=1, \mathrm{i}\neq \mathrm{j}}^{n} a_{ij}x_j.$$

Taking norm of both sides in the above equation

$$\| (\lambda - a_{ii})\, x_i\| = \left\| \sum_{j=1, \mathrm{i}\neq \mathrm{j}}^{n} a_{ij} x_j \right\|$$

is obtained. Then make use of triangle inequality is accessible the following inequalities

$$\begin{aligned} \| (\lambda - a_{ii})\, x_i\| &\leq \sum_{j=1, \mathrm{i}\neq \mathrm{j}}^{n} \|a_{ij} x_j\|, \\ \| (\lambda - a_{ii})\| \, \|x_i\| &\leq \sum_{j=1, \mathrm{i}\neq \mathrm{j}}^{n} \|a_{ij}\| \, \|x_j\|\,, \\ \| (\lambda - a_{ii})\| &\leq \sum_{j=1, \mathrm{i}\neq \mathrm{j}}^{n} \|a_{ij}\| = R_i. \end{aligned}$$

So, we have

$$\xi\,(A) \subseteq \bigcup_{i=1}^{n} \{a \in CO : \ \ \|a - a_{ii}\| \leq R_i\}.$$

□

Example 3.13. *Let* $A = \begin{bmatrix} 1+i+j+k+e & ei \\ ej & ek \end{bmatrix}$; *A is a commutative octonion matrix. Then the adjoint matrix of A is*

$$\eta\,(A) = \begin{bmatrix} 1+i+j+k & 0 & 1 & i \\ 0 & 0 & j & k \\ 1 & i & 1+i+j+k & 0 \\ j & k & 0 & 0 \end{bmatrix}.$$

The set of eigenvalues of the matrix $\eta\,(A)$ *are*

$$\xi\,(\eta\,(A)) = \left\{ \begin{array}{l} \frac{1}{2}(2+i+j-\sqrt{4+4i+4j+6k}+2k),\ \frac{1}{2}(2+i+j+\sqrt{4+4i+4j+6k}+2k), \\ \frac{1}{2}(i+j-\sqrt{-4+4i-4j+6k}),\ \frac{1}{2}(i+j+\sqrt{-4+4i-4j+6k})\} \end{array} \right\}.$$

The Gershgorin disks are

$$D_1 = \{q \in H : \ \|q - (1+i+j+k)\| \leq 2\,\} \quad \text{and} \quad D_2 = \{q \in H : \ \|q\| \leq 2\,\}\,.$$

Thus, we obtain

$$\xi\,(A) \ \cap \ H = \xi\,(\eta\,(A)) \subseteq D_1 \cup D_2.$$

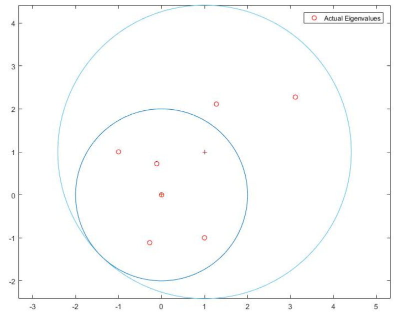
The image of the example above is A in the figure below.

References

[1] J. C. Baez, *The octonions*, Bull. Amer. Math. Soc., **39** (2002), no. 2, 145–205.

[2] F. Catoni, R. Cannata, and P. Zampetti *An introduction to commutative quaternions*, Adv. Appl. Clifford Algebras, **16** (2006), no. 1, 1–28.

[3] F. Catoni, R. Cannata, and P. Zampetti, *Commutative hypercomplex numbers and functions of hypercomplex variable: a matrix study*, Adv. Appl. Clifford Algebras **15** (2005), 183–212.



(A) Eigenvalues of commutative octonions

[4] A. Cihan and M. A. Güngör, *Commutative octonion matrices*, IECMSA, Skopje Macedonia, 2020.

[5] W. R. Hamilton, *Lectures on quaternions*, Hodges and Smith, Dublin, 1853.

[6] H. H. Kösal, *On the commutative quaternion matrices*, Ph.D. thesis, Sakarya University, 2016.

[7] H. H. Kösal and M. Tosun, *Commutative quaternions matrices*, Adv. Appl. Clifford Algebras, **24**, (2014), no. 3, 769–779.

[8] S. C. Pei, J. H. Chang, and J. J. Ding, *Commutative reduced biquaternions and their fourier transform for signal and image processing applications*, IEEE Trans. Signal Process. **52** (2004), no. 7, 2012–2031.

[9] D. A. Pinotsis, *Segre quaternions, spectral analysis and a four-dimensional Laplace equation*, Progress in Analysis and Its Applications (2010), 240–246.

[10] Y. Song, *Construction of commutative number systems*, Linear Multilinear A. (2020).

[11] Y. Tian, *Matrix representations of octonions and their applications*, Adv. Appl. Clifford Algebras **10** (2000), no. 61, 61–90.

[12] L. A. Wolf, *Similarity of matrices in which the elements are real quaternions*, Bull. Amer. Math. Soc. **42** (1936), 737–743.

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