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SIGNATURAL APPLICATIONS OF THE FRICKE GROUP $\Gamma_F(N)$

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Abstract. In this paper, we establish the Fricke Group $\Gamma_F(N)$ which is a new special group of Non-Euclidean Crystallograhic (NEC) group. We obtain this group whose congruence subgroup $\Gamma_0(N)$ is expanded with Fricke reflection $F(z) = \frac{1}{N\bar{z}}$. Then, we research and calculate the structure of signature and fundamental domain of this group. And then, we calculate the number of boundary components in the signature for this group. Finally, we find the $2, 3, \infty$ valued link periods of these boundary components with the H. Jaffee technique.

1. Introduction

A Fuchsian group is a discrete subgroup of the hyperbolic group and it was researched by Poincaré [7]. He showed that how to obtain generators and relations. Fricke and Klein [5] obtained a canonical form. Then, Wilkie [11] defined an other discrete subgroup of the hyperbolic group, i.e. Non-Euclidean Crystallograhic group (NEC). You can find lots of works in the literature, i.e. [1]-[7] and [9]-[11]. Consequently, in this paper, firstly we obtain the Fricke Group $\Gamma_F(N)$ in terms of congruence subgroup $\Gamma_0(N)$ expanding with Fricke reflection $F(z) = \frac{1}{N\bar{z}}$. And it is a new special group of Non-Euclidean Crystallograhic group (NEC). Secondly, we research the structure of signature and fundamental domain of this group. Thirdly, we calculate the number of boundary components of signature for this group. And finally, we investigate the $2, 3, \infty$ valued link periods of these components with the H. Jaffee technique.

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2. Preliminaries

Definition 2.1. [10] *Let* $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \ c \equiv 0 \ mod\, N \right\}$, *where* $N \in \mathbb{Z}^+$ *and* $\Gamma = PSL(2,\mathbb{Z})$ *is a modular group. Then the following equality is called Fricke group*

$$\Gamma_F(N) = \left\langle \ \Gamma_0(N), F(z) \ \right\rangle = \left\langle \ \Gamma_0(N), z \longrightarrow \frac{1}{N\bar{z}} \ \right\rangle = \Gamma_0(N) \cup F.\Gamma_0(N).$$

We use the following notations throughout this paper.

1) Orbits $\Gamma_0(N)$ of rational number $\frac{k}{s} \in \mathbb{Q}$ is shown $\left[\frac{k}{s}\right]$ for $(k,s) = 1$.

2) $\left[\frac{k}{s}\right] := \left\{ \frac{u}{v} \in \mathbb{Q} \ : \ \exists\, T \in \Gamma_0(N) \text{ such that } T\left(\frac{k}{s}\right) = \frac{u}{v} \right\}$. Here $\frac{u}{v}$ is reduced in fraction form and $(u,v) = 1$ due to $detT = 1$ and $(k,s) = 1$.

3) The set of cusps for $\Gamma_0(N)$ is denoted by $\mathcal{B}$. For $\mathbb{H} := \{z \in \mathbb{C} : Imz > 0\}$, $\mathbb{H}^* := \mathbb{H} \cup \mathcal{B}$. And $X_0(N) := \mathbb{H}^* \diagup \Gamma_0(N)$, $X_F(N) := \mathbb{H}^* \diagup \Gamma_F(N)$.

Definition 2.2. [9] *Let* Λ *is a NEC group. If the following conditions are satisfy,*

1) $\bigcup_{T \in \Lambda} T(D) = \mathbb{H}$,

2) $D^\circ \cap T(D^\circ) = \emptyset$ *for all* $T \in \Lambda \setminus \{I\}$,

then the closed subset $D \subset \mathbb{H}$ *is called a fundamental domain for* Λ.

Definition 2.3. [9] *Let* Λ *is a NEC group. Then, the following representation is called a NEC signature of* Λ,

$$\sigma(\Lambda) = (g; \pm; [m_1, \cdots, m_r]; \{(n_{11}, \cdots, n_{1s_1}), \cdots, (n_{k1}, \cdots, n_{ks_k})\}).$$

Here, number $g \in \mathbb{N}$ *is called genius of orbital spaces, and* $\pm$ *is written in the signature according to the orientation of the orbital space. The* n*-th orders of the generating elliptical or parabolic elements of the* Λ *group are called special periods of the* Λ*. In addition, the* n *numbers giving the order of the elements formed by the resultant of the two reflections of the* Λ *are called the link period, and the* $C_i := (n_{i1}, n_{i2}, \cdots, n_{is_i})$ *expression is called the* i*-th boundary component of the signature.*

Definition 2.4. [10]

1) *Let* $\left[\frac{k}{s}\right]$ *is a cusp parabolic fixed point in* $X_0(N)$*. If there is a reflection* $W \in \Gamma_F(N)$*, such that* $W\left(\left[\frac{k}{s}\right]\right) = \left[\frac{k}{s}\right]$*, then* $\left[\frac{k}{s}\right]$ *is called a real cusp in* $X_F(N)$*.*

2) *If there is an elliptic element* $T \in \Gamma$*, such that* $T(z_0) = z_0$ *for* $z_0 \in \mathbb{H}$*, then the point* z_0 *is called elliptic fixed point or elpi point.*

Remark 2.5. *If there is a reflection $W \in \Gamma_F(N)$, such that z_0 is an elpi point and $W(z_0) = z_0$, then z_0 is called a imaginary elpi in $X_F(N)$. Imaginary elpi name is given by Aziz Büyükkaragöz*[3].

Theorem 2.6. [1] *Let $\left[\frac{x}{n}\right]$ is a cusp in $X_0(N)$ for $n \mid N$. Then,*

$$F\left(\left[\frac{x}{n}\right]\right) = \left[\frac{x}{n}\right] \Longleftrightarrow N = n^2.$$

Theorem 2.7. [1] *Let $n \in \mathbb{Z}^+, N = n^2$ and φ be an Euler function. If the number of solution congruence $x^2 \equiv -1 \ mod \ n$ is A and k is the smallest positive integer satisfying the congruence $4^k \equiv 1 \ mod \ n$, then*

1) *If n is odd, then there are A-piece periodic-circuit which is k real cusp, and again its $2k$ real cusp $\frac{\varphi(n)}{2k} - \frac{A}{2}$ piece in the $X_F(N)$.*
2) *If n is even, then there are A-piece periodic-circuit which is 1 real cusp, and again its 2 real cusp $\frac{\varphi(n)}{2} - \frac{A}{2}$ piece in the $X_F(N)$.*

Corollary 2.8. [1] *We obtain the following results from the above theorem.*

1) *If n is an odd number, the orbital graphs for real cusps are as follows:*
1.1) $\left[\frac{-x^{-1}}{n}\right], \left[\frac{x}{n}\right], \left[\frac{-(4x)^{-1}}{n}\right], \left[\frac{4x}{n}\right], \ldots, \left[\frac{-(4^{k-1}x)^{-1}}{n}\right], \left[\frac{4^{k-1}}{n}\right]$ *(k real cusp),*
1.2) $\left[\frac{x}{n}\right], \left[\frac{4x}{n}\right], \left[\frac{4^2x}{n}\right] \ldots, \left[\frac{4^{k-2}}{n}\right], \left[\frac{4^{k-1}x}{n}\right]$ *($2k$ real cusp).*
2) *If n is an even number, the orbital graphs for real cusps are as follows:*
2.1) *For $x \in \mathbb{Z}$,* $\left[\frac{x}{n}\right], \left[\frac{-x^{-1}}{n}\right]$ *(2 real cusp),*
2.2) *For $x^2 \equiv -1 mod \ n$ or $x \equiv -x^{-1} mod \ n$,* $\left[\frac{x}{n}\right]$ *(1 real cusp)*

Theorem 2.9. [9] *The following set is a fundamental domain for modular group Γ: $D = \{z \in \mathbb{H} : \mid z \mid \geq 1 \ and \ \mid Re(z) \mid \leq \frac{1}{2}\}$.*

Theorem 2.10. [9] *The following set is a fundamental domain for extended modular group $\hat{\Gamma}$: $\hat{D} = \{z \in \mathbb{H} : \mid z \mid \geq 1 \ and \ 0 \leq Re(z) \leq \frac{1}{2}\}$.*

3. Main Results

3.1. The Fricke Group $\Gamma_F(N)$ in terms of congruence subgroup $\Gamma_0(N)$

Lemma 3.1. *If $T(z) = z$ for all parabolic element $T \in \Gamma$ and $x \in \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, then $z \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.*

Proof. From $T \in \Gamma$ is a parabolic element and $z \in \mathbb{R}_\infty$, we can write the following equality:

$$T(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{Z}, \quad ad-bc = 1, \quad \mid a+d \mid = 2.$$

Because of $T(z) = z$, we obtain $\frac{az+b}{cz+d} = z$. Thus, $cz^2 + (d-a)z + b = 0$. This equation roots are

$$z_{1,2} = \frac{a-d \mp \sqrt{(d-a)^2+4bc}}{2c}.$$

So, we have $(d-a)^2 + 4bc = 0$ for $ad-bc = 1$ and $\mid a+d \mid = 2$. Consequently, $z_{1,2} = \frac{a-d}{2c} \in \hat{\mathbb{Q}}$, namely, $z \in \hat{\mathbb{Q}}$. □

Remark 3.2. *The Fricke Group $\Gamma_F(N)$ is important in real cusp. Since it leaves fixed under which conditions a cusp. For that reason, Necessary and sufficient conditions must be determined for the reflection $F(z) = \frac{1}{N\bar{z}}$. Because any reflection W in $\Gamma_F(N)$ will be $W = FK$ for $K \in \Gamma_0(N)$.*

Remark 3.3. *We should carefully calculate the representatives of real cusp in $X_F(N)$ such that $\left[\dfrac{x}{n}\right]$ according to Theorem 2.6. Now we consider $(x_1, n) = (x_2, n) = 1$ for $x_1, x_2 \in \mathbb{Z}$. Then we obtain the following relation from* [2] *and* [3, Theorem 3.4.1]

$$\left[\frac{x_1}{n}\right] = \left[\frac{x_1}{n}\right] \Longleftrightarrow x_1 \equiv x_2 \ mod\ n.$$

So, x numbers are less than n and can be chosen as prime numbers between them for cusps $\left[\dfrac{x}{n}\right]$.

Corollary 3.4. *Let $n \in \mathbb{Z}^+$ and $N = n^2$;*

1) *The reflection $F(z) = \frac{1}{N\bar{z}}$ is exactly $\varphi(n)$-pieces leaves fixed to the cusp $\left[\frac{x}{n}\right]$. Where φ is the Euler function. So $X_F(N)$ has exactly $\varphi(n)$-pieces real cusps.*

2) *Each real cusp in $X_F(N)$ corresponds to the ∞-valued link period in any boundary component of the signature $\sigma(\Gamma_F(N))$.*

Remark 3.5. *If $N \neq n^2$ for all $n \in \mathbb{Z}^+$, then there is not ∞-valued link period in the boundary component of the signature $\sigma(\Gamma_F(N))$. However, there are some special cases, namely, for $N = 2, 3, 4, 5, 9, 16, 25, 49, 64, 81, 100$.*

Lemma 3.6. [8] *Let $n \in \mathbb{Z}^+, x \leq n$ and $(x, n) = 1$. In this case, solution of the congruence $x^2 \equiv 1 \ mod\ n$ has got 2^{r+s} values, such that*

$$n = 2^{\alpha_1} p_2^{\alpha_2} \dots p_{r+1}^{\alpha_{r+1}} \qquad and \qquad s = \begin{cases} 0, & if\ \alpha_1 = 1, \\ 1, & if\ \alpha_1 = 2, \\ 2, & if\ \alpha_1 \geq 3. \end{cases}$$

Corollary 3.7. *If $(a,n)=1$ and the congruence $x^2 \equiv a \bmod n$ is solvability, then there are just $A = 2^{r+s}$ solutions, such that*

$$s = \begin{cases} 0, & if\ 4 \nmid n, \\ 1, & if\ 4 \parallel n, \\ 2, & if\ 8 \mid n. \end{cases}$$

Where r is a number of different odd prime divisors of n.

Theorem 3.8. *The signature of groups Γ and $\hat{\Gamma}$ are respectively,*

1) $\sigma(\Gamma_0(1)) = \sigma(\Gamma) = (0\ ;\ +\ ;\ [2,3,\infty])$,

2) $\sigma(\Gamma_F(1)) = \sigma(\hat{\Gamma}) = (0\ ;\ +\ ;\ [\quad]\ ;\ \{(2,3,\infty)\})$.

Proof. The proof of this theorem is clear from Theorem 2.9 and Theorem 2.10. □

3.2. Signature and Fundamental Domain of the Fricke Group $\Gamma_F(N)$ in the Special Cases

Theorem 3.9. *The signatures of the groups $\Gamma_0(2)$ and $\Gamma_F(2)$ are :*

1) *Signature of the group $\Gamma_0(2)$ is $\sigma(\Gamma_0(2)) = (0\ ;\ +\ ;\ [2,\infty,\infty])$,*

2) *Signature of the group $\Gamma_F(2)$ is $\sigma(\Gamma_F(2)) = (0\ ;\ +\ ;\ [\infty]\ ;\ \{(2)\})$.*

Proof. Now we prove the above claims.

1) According to definition of $\Gamma_0(2)$,

$$K(z) = \frac{z-1}{2z-1} = z \Rightarrow z - 1 = 2z^2 - z \Rightarrow 2z^2 - 2z + 1 = 0.$$

We calculate the above value of the quadratic equation in $\mathbb{H}$. In this case, we take $z_1 = \frac{-1}{2} + \frac{i}{2}$ and $z_2 = \frac{1}{2} + \frac{i}{2}$.

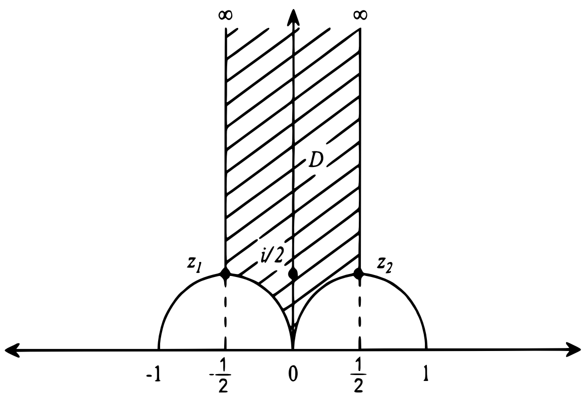

FIGURE 1. A fundamental domain D of $\Gamma_0(2)$

So, we have the followings,

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \Gamma_0(2) \text{ is a parabolic element, then } T(z) = \frac{z}{2z+1} \Longrightarrow T(z_1) = z_2,$$

$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \in \Gamma_0(2)$ is an elliptic element of order 2, then

$$K(z) = \frac{z-1}{2z-1} \Longrightarrow K(z_2) = z_2,$$

$P_1(z)$ is a parabolic element, $P_1(0) = 0$ and $P_2(z)$ is a parabolic element, $P_2(\infty) = \infty$. Consequently, signature of the group $\Gamma_0(2)$ is

$$\sigma(\Gamma_0(2)) = (0\ ;\ +\ ;\ [2, \infty, \infty]) = (0\ ;\ 2, \infty, \infty).$$

2) We can write $\Gamma_F(2) = \langle\ \Gamma_0(2), z \to \frac{1}{2\bar{z}} \rangle$ from the definition of $\Gamma_F(2)$. D is known a fundamental domain of $\Gamma_0(2)$. Let we firstly determine the axis of reflection to find a fundamental domain D_F of $\Gamma_F(2)$.

$$F(z) = \frac{1}{2\bar{z}} = z \Longrightarrow \mid z \mid^2 = \frac{1}{2} \Longrightarrow \mid z \mid = \frac{1}{\sqrt{2}}.$$

Here the axis of reflection is an Euclidean semicircle with a central origin and radius $\frac{1}{\sqrt{2}}$. Consequently, $\sigma(\Gamma_F(2)) = (0; +; [\infty]; \{(2)\})$. And

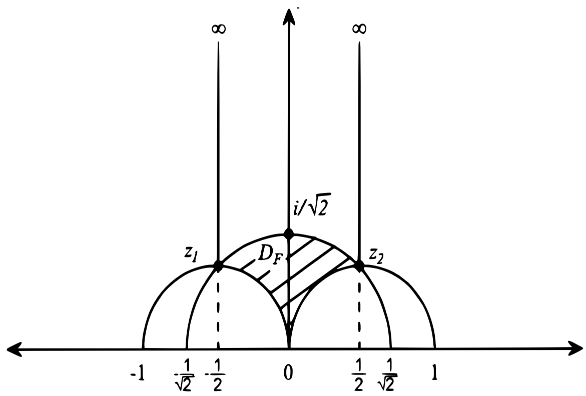

FIGURE 2. A fundamental domain D_F of $\Gamma_F(2)$

from $N = 2 \neq n^2$, there is not ∞-valued link period at the boundary component in $\Gamma_F(2)$.

□

Remark 3.10. *There is no boundary component in $\Gamma_0(2)$ because there is no reflection. Moreover, there are special periods for $\Gamma_0(2)$ due to $K^2 = P_1^\infty = P_2^\infty = I$.*

Remark 3.11. *The finite order elements of $\mathcal{G} = \hat{\Gamma}$ are either elliptical or reflection transformations. $P_1(0) = 0$ for $P_1(z)$, which is a parabolic element with infinite period. If $K \in \Gamma_0(2)$ is an elliptic and also $S \in \Gamma_F(2)$ is a reflection, then $(FW)^2 = I$ for $KS = W$. However, the point z_2 is fixed by the elliptic element $K(z) = \frac{z-1}{2z-1}$ with second order, hence z_2 becomes an imaginary elpi.*

Theorem 3.12. **1)** *The signature of $\Gamma_0(3)$ is $\sigma(\Gamma_0(3)) = (0\ ;\ +\ ;\ [3, \infty, \infty])$.*
2) *The signature of $\Gamma_F(3)$ is $\sigma(\Gamma_F(3)) = (0\ ;\ +\ ;\ [\infty]\ ;\ \{(3)\})$.*

Proof. **1)** According to the definition of $\Gamma_0(3)$, we can write as follows

$$K(z) = \frac{2z-1}{3z-1} = z \Longrightarrow 2z - 1 = 3z^2 - z \Longrightarrow 3z^2 - 3z + 1 = 0.$$

The roots of this quadratic equation are $\frac{1}{2} + \frac{\sqrt{3}}{6}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{6}i$. From these

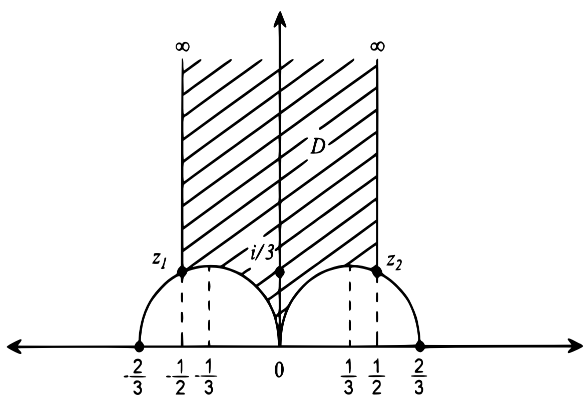

FIGURE 3. A fundamental domain D of $\Gamma_0(3)$

roots, evaluation is made only with the point in the upper half-plane of $\mathbb{H}$. In this case, let $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{6}i$ and $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}i$.

Parabolic element $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \in \Gamma_0(3)$, $T(z) = \frac{z}{3z+1}$ and $T(z_1) = z_2$.

Elliptical element of 3rd order $\begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix} \in \Gamma_0(3)$, $K(z) = \frac{2z-1}{3z-1}$ and $K(z_2) = z_2$. Then, $U_1(z) \in \Gamma_0(3)$ is a parabolic element and $U_1(0) = 0$. And then $U_2(z) \in \Gamma_0(3)$ is a parabolic element and $U_2(\infty) = \infty$. For that reason, $\sigma(\Gamma_0(3)) = (0\,;+;\,[3, \infty, \infty]) = (0\,; 3, \infty, \infty)$. Moreover, there is a relation $K^3 = U_1^\infty = U_2^\infty = I$.

2) According to definition of Fricke group $\Gamma_F(3) = \left\langle \Gamma_0(3), z \longrightarrow \frac{1}{3\bar{z}} \right\rangle$. We define a fundamental area D of $\Gamma_0(3)$. In this case, a fundamental area

D_F of $\Gamma_F(3)$ will be obtained by the reflection $F(z) = \frac{1}{3\bar{z}}$, namely,

$$F(z) = \frac{1}{3\bar{z}} = z \Longrightarrow z.\bar{z} = \frac{1}{3} \Longrightarrow \mid z \mid = \frac{1}{\sqrt{3}}.$$

So the reflection axis is the part of the Euclidean circle (i.e. Hyperbolic line) in $\mathbb{H}$ with a radius of $M(0,0)$ and a radius of $\frac{1}{\sqrt{3}}$. Consequently,

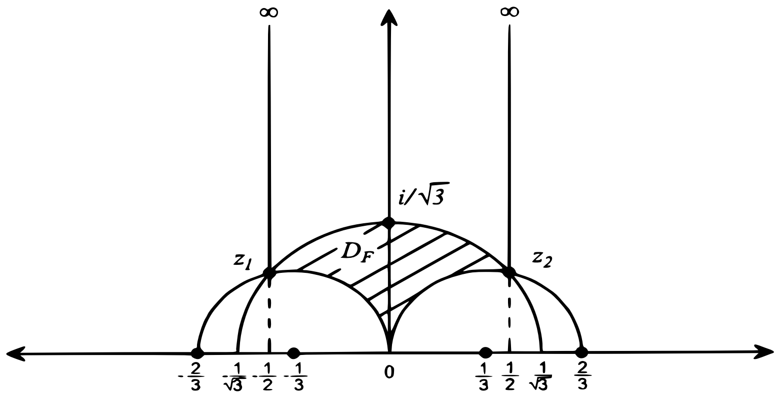

FIGURE 4. A fundamental domain D_F of $\Gamma_F(3)$

$\sigma(\Gamma_F(3)) = (0\ ;\ +\ ;\ [\infty]\ ;\ \{(3)\})$. Furthermore, there are not ∞-valued link period at the boundary component due to $N = 3 \neq n^2$ in the $\Gamma_F(3)$. It is clear that $U_1(0) = 0$ for parabolic element $U_1(z) \in \Gamma_0(3)$ with ∞ order.

□

Remark 3.13. *The point z_2 is fixed by the elliptic element $K(z) = \frac{2z-1}{3z-1}$ with third order. Namely, z_2 is an elpi point and it is fixed by $F(z) = \frac{1}{3\bar{z}}$. Therefore, z_2 is an imaginary elpi.*

Theorem 3.14. **1)** *Signature of* $\Gamma_0(4)$ *is* $\sigma(\Gamma_0(4)) = (0\ ;\ +\ ;\ [\infty,\infty,\infty])$.
2) *Signature of* $\Gamma_F(4)$ *is* $\sigma(\Gamma_F(4)) = (0\ ;\ +\ ;\ [\infty]\ ;\ \{(\infty)\})$.

Proof. **1)** $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(4)$ is a parabolic element, $T(z) = z+1$ and $T(\infty) = \infty$; $S = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \in \Gamma_0(4)$ is a parabolic element, $S(z) = \frac{z}{4z+1}$ and $S(0) = 0$; $K = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} \in \Gamma_0(4)$ is a parabolic element, $K(z) = \frac{-z-1}{4z+3}$, $K(-\frac{1}{2}) = -\frac{1}{2}$. Moreover, $T(-\frac{1}{2}) = -\frac{1}{2} + 1 = \frac{1}{2}$. As regards to this, there are three generator parabolic element of the group $\Gamma_0(4)$. Indeed, $\varepsilon_i = 0$, $\varepsilon_\rho = 0$ and $\sigma_\infty = 3$ owing to Theorem

3.1.10 [3]. Then, from Theorem 3.1.11 [3], signature of this group is $\sigma(\Gamma_0(4)) = (0\ ;\ +\ ;\ [\infty,\infty,\infty])$.

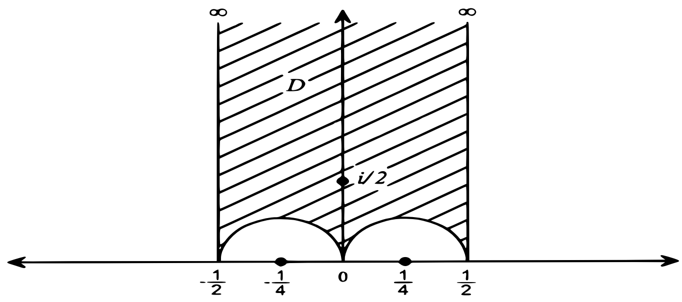

FIGURE 5. A fundamental domain D of $\Gamma_0(4)$

2) We define a domain D of $\Gamma_0(4)$ for the Fricke group such that

$$\Gamma_F(4) = \left\langle\ \Gamma_0(4), z \longrightarrow \frac{1}{4\bar{z}} \right\rangle.$$

According to this, we will calculate it for a fundamental domain D_F of $\Gamma_F(4)$ with the reflection $F(z) = \frac{1}{4\bar{z}}$.

$$F(z) = \frac{1}{4\bar{z}} = z \Longrightarrow z.\bar{z} = \frac{1}{4} \Longrightarrow |z| = \frac{1}{2}.$$

In this case, the reflection (symmetry) axis is $M(0,0)$ and the radius is $\frac{1}{2}$ part of the Euclidean circle at $\mathbb{H}$. So this circle is also a hyperbolic line at $\mathbb{H}$. Accordingly, from Theorem 2.7 [1], the orbital space of $\Gamma_F(4)$ has one boundary component and one ∞-valued link period. Indeed, since $T(-\frac{1}{2}) = \frac{1}{2}$ and $S(0) = 0$, this group has one in $\hat{\mathbb{Q}}$. There is a cusp point. It should also be noted that $N = 4 = 2^2 = n^2$. Again, since the group does not have a generating elliptic element, there is no elpi point or even imaginary elpi.

□

Now let's give the following theorem according to special cases.

Theorem 3.15. *Let $n \in \mathbb{Z}^+$ and $2^2.3^2 \mid n^2$. Then boundary components in the $\Gamma_F(n^2)$ group's signature do not have a natural link period.*

Proof. Because of $n \in \mathbb{Z}^+, 2^2.3^2 \mid n^2$, there is a $k \in \mathbb{Z}$ such that $n^2 = 2^2.3^2k$. We have the followings for $\Gamma_0(N) = \Gamma_0(n^2)$ from $N = n^2 = 2^2.3^2k$ and Theorem 3.1.10 [3]. $\varepsilon_i = 0$ and $\varepsilon_\rho = 0$ from $4 \mid N$ and $9 \mid N$, respectively. Thus, there are no generators of orders two and three elliptic elements. And there is not finite order period from composite of reflections. Moreover, $\Gamma_F(n^2) = \Gamma_0(n^2) \cup F.\Gamma_0(n^2)$ for $\Gamma_F(n^2) = \left\langle\ \Gamma_0(n^2), z \longrightarrow \frac{1}{n^2\bar{z}}\ \right\rangle$ and $F(z) = \frac{1}{n^2\bar{z}}$. Finally, if there

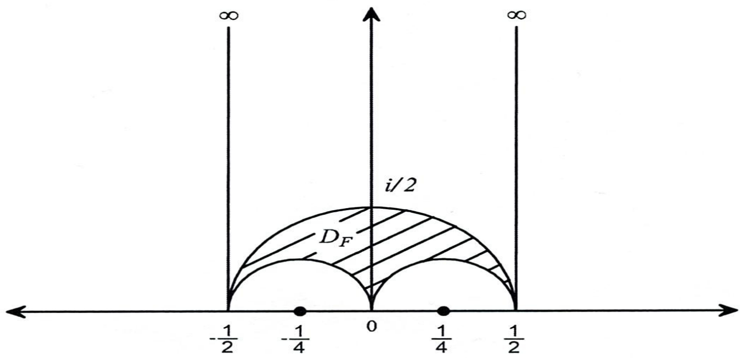

FIGURE 6. A fundamental domain D_F of $\Gamma_F(4)$

is no generators elliptic element in $\Gamma_0(n^2)$, then it is also there is no generators elliptic element $F\Gamma_0(n^2)$. Consequently, there is no natural link period at the boundary component of the signature of the group $\Gamma_F(n^2)$. $\square$

Corollary 3.16. *If set C is the set of boundary components in the group's signature, then*

1) $C = \{(\infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(9))$,
2) $C = \{(\infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(36))$,
3) $C = \{(\infty, \infty), (\infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(64))$,
4) $C = \{(\infty, \infty, \infty, \infty, \infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(81))$.

Corollary 3.17. *There is only one elliptic element of order two generator for* $\varepsilon_i = 1, \varepsilon_\rho = 0, \sigma_\infty = 2$ *in* $\Gamma_0(5)$. *And, for* $T = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \in \Gamma_0(5)$, *order*$T = 2$. *The point* $\frac{2}{5} + \frac{i}{5}$ *is fixed by the element* T *in the upper-half plane. So,* $\sigma(\Gamma_0(5)) = (0\,;\,+;\,[2, \infty, \infty])$, *and* $\sigma(\Gamma_F(5)) = (0\,;\,+\,;\,[\infty]\,;\,\{(2)\})$.

Corollary 3.18. *We obtain* $\varepsilon_i = 0$ *and* $\varepsilon_\rho = 0$ *for* $\Gamma_F(100), N = 10^2 = 100$ *and, we have* $\varepsilon_i = 0$ *and* $\varepsilon_\rho = 0$. *From Theorem 2.7,* $C = \{(\infty),\ (\infty),\ (\infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(100))$.

Now, let us include studies in $\Gamma_F(N)$ for the values $N = 25$ and $N = 49$ that do not satisfy the conditions of the Theorem 2.7. In other words, we research the generator elliptic and parabolic element in the groups $\Gamma_F(25)$ and $\Gamma_F(49)$, and find out ∞-valued link period as well as whether or 2 and 3 value link periods in the boundary components of groups' signatures. To do this, firstly we generalize the Fricke group's elements.

Definition 3.19. *Let* $n \in \mathbb{Z}^+$ *and* $N = n^2$. *Then, the below groups'*

$$\Gamma_F(N) = \Gamma_F(n^2) = \left\langle\ \Gamma_0(n^2), z \longrightarrow \frac{1}{n^2\bar{z}}\ \right\rangle$$

elements as follows,

$$\begin{aligned}
First\ Step &: \begin{pmatrix} a & b \\ cn^2 & d \end{pmatrix} \in \Gamma_0(n^2) \ and\ ad - bcn^2 = 1. \\
Second\ Step &: \begin{pmatrix} 0 & 1/n \\ n & 0 \end{pmatrix} = F \ and\ \det F = -1. \\
Third\ \ Step &: \begin{pmatrix} 0 & 1/n \\ n & 0 \end{pmatrix} \begin{pmatrix} a & b \\ cn^2 & d \end{pmatrix} = \begin{pmatrix} cn & d/n \\ an & bn \end{pmatrix} = M \\
& \quad and\ \det M = -1.
\end{aligned}$$

We suppose that the element of $M \in \Gamma_F(N)$ is expressed the reflection elements of this group in the most general sense. In this case, $bn + cn = 0$ from $izM = 0$. So, $c = -b$, namely, the reflections are $\begin{pmatrix} -bn & d/n \\ an & bn \end{pmatrix}$ and $-b^2n^2 - ad = -1$, i.e. $b^2n^2 + ad = 1$.

If we have generator reflection elements of the group, then we can apply the Generalized Hoare Uzzel Theorem. Now let's express the parabolic maps in $\Gamma_0(n^2)$ determined by fixed point elements on $\mathbb{Q}$.

Theorem 3.20. *Let $T, K \in \Gamma_0(n^2)$ are parabolic elements. Then, the following maps for $n \in \mathbb{Z}^+$ and $1 \leq k < n^2$*

$$T = \begin{pmatrix} n^2k - 1 & -k^2 \\ n^4 & -(n^2k+1) \end{pmatrix} \ or\ K = \begin{pmatrix} n^2k + 1 & -k^2 \\ n^4 & -(n^2k-1) \end{pmatrix}$$

leave fixed to the rational numbers $\frac{1}{n^2}, \frac{2}{n^2}, \ldots, \frac{n^2-1}{n^2}$.

Proof. Let $k \in \mathbb{Z}^+$ and $1 \leq k < n^2$. Then we determine the maps that leave fixed to the fraction $\left(\frac{k}{n^2}\right) \in \mathbb{Q}$.

$$\begin{aligned}
(n^2z - k)^2 = 0 &\implies n^4z^2 - 2n^2kz + k^2 = 0 \\
&\implies n^4z^2 - [(n^2k+1) + (n^2k-1)]z + k^2 = 0.
\end{aligned}$$

I. Case: $n^4z^2 - (n^2k+1)z = (n^2k-1)z - k^2$,

$$\implies z = \frac{(n^2k-1)z - k^2}{n^4z - (n^2k+1)} \implies T = \begin{pmatrix} n^2k - 1 & -k^2 \\ n^4 & -(n^2k+1) \end{pmatrix}.$$

II. Case: $n^4z^2 - (n^2k-1)z = (n^2k+1)z - k^2$,

$$\implies z = \frac{(n^2k+1)z - k^2}{n^4z - (n^2k-1)} \implies T = \begin{pmatrix} n^2k + 1 & -k^2 \\ n^4 & -(n^2k-1) \end{pmatrix}.$$

Thus, T and K parabolic maps leave fixed to the rational fractions that $\frac{k}{n^2}$. For example, one of the parabolic elements in $\Gamma_0(25)$ subgroup that leaves the fraction $\frac{4}{25}$ is fixed by T and K as follow,

$$T = \begin{pmatrix} 99 & -16 \\ 625 & -101 \end{pmatrix} \text{ and } K = \begin{pmatrix} 101 & -16 \\ 625 & -99 \end{pmatrix} = T^{-1}.$$

□

Remark 3.21. *We assume that p is a prime number. While we solve the $a^2 \mp a + 1 \equiv 0 \ mod\ p^2$, firstly we must find the solution of the congruence $a^2 \mp a + 1 \equiv 0 \ mod\ p$.*

Now we calculate the $f^{'}(x)$ for $f(x) = x^2 \mp x + 1$. In this case, we obtain y from $\frac{f(x)}{p} + f^{'}(x)y \equiv 0 \ mod\ p$. If $(f^{'}(x), p) = 1$, then there is only one solution. If it is $y \equiv y_o \ mod\ p$ then the solution of congruence $a^2 \mp a + 1 \equiv 0 \ mod\ p^2$ is $a \equiv x + y_o p \ mod\ p^2$. Similarly, the congruence $\mp(a^2 + 1) \equiv 0 \ mod\ p^2$ can also be solved [4].

Calculations can be made on the generator elements of $\Gamma_F(25)$ and $\Gamma_F(49)$ from the Remark 3.21.

Corollary 3.22. *Let set C be the set of boundary components in the group's signature. Then,*

1.) $C = \{(\infty, \infty), (\infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(25))$,
2.) $C = \{(\infty, \infty, \infty, \infty, \infty, \infty)\}$ *is at the signature of* $\sigma(\Gamma_F(49))$.

Proof. **1.** We obtain $\varepsilon_i = 2$, $\varepsilon_\rho = 0$, $\sigma_\infty = 6$ in the group of $\Gamma_0(25)$. Thus, there are two second order generator elliptic elements on the $\Gamma_0(25)$. We have the followings due to Remark 3.21,

$T = \begin{pmatrix} 7 & -2 \\ 25 & -7 \end{pmatrix} \in \Gamma_0(25)$ and *orderT* $= 2$. So, T leaves the point $\frac{7}{25} + \frac{i}{25}$ fixed in the upper half-plane.

$K = \begin{pmatrix} -7 & -2 \\ 25 & 7 \end{pmatrix} \in \Gamma_0(25)$ and *orderK* $= 2$. So, K leaves the point $-\frac{7}{25} + \frac{i}{25}$ fixed in the upper half-plane. Consequently, we obtain the $C = \{(\cdots, \infty, \infty), (\cdots, \infty, \infty)\}$ at the signature of $\sigma(\Gamma_F(25))$. In this case, it is only one of the following situations will be true,

i) $C = \{(2, 2, \infty, \infty), (\infty, \infty)\}$,
ii) $C = \{(2, \infty, \infty), (2, \infty, \infty)\}$,
iii) $C = \{(\infty, \infty), (\infty, \infty)\}$.

We can write $S = \begin{pmatrix} -5b & d/5 \\ 5a & 5b \end{pmatrix}$ and $ad + 25b^2 = 1$ from definition 3.19 with reflection in the $\Gamma_F(25)$. Thus,

$$\begin{pmatrix} 0 & 1/5 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} -5b & d/5 \\ 5a & 5b \end{pmatrix} = \begin{pmatrix} a & b \\ -25b & d \end{pmatrix} = U \quad and \quad U \in \Gamma_0(25).$$

From this, the generator elliptic elements are second order from $a+d = 0$. And then, $U_1 = \begin{pmatrix} a & b \\ -25b & -a \end{pmatrix}$ *and* $U_2 = \begin{pmatrix} -a & b \\ -25b & a \end{pmatrix}$. In this case,

U_1 and U_2 maps cannot be found according to the specified conditions from the resultant of FS reflections. It is show that there is no 2 valued link period in the boundary components of the signature $\sigma(\Gamma_F(25))$.

2. We obtain $\varepsilon_i = 0$, $\varepsilon_\rho = 2$, $\sigma_\infty = 8$ at the group of $\Gamma_0(49)$. Thus, there are two generator elliptic elements of order three on the $\Gamma_0(25)$. We have the followings from Remark 3.21, $T = \begin{pmatrix} 18 & -7 \\ 49 & -19 \end{pmatrix} \in \Gamma_0(49)$, and $orderT = 3$. So, T leaves fixed to the point $\frac{37}{98} + \frac{\sqrt{3}}{98}i$ in the upper half-plane.

$K = \begin{pmatrix} -18 & -7 \\ 49 & 19 \end{pmatrix} \in \Gamma_0(49)$, and $orderK = 3$. So, K leaves fixed to the point $-\frac{37}{98} + \frac{\sqrt{3}}{98}i$ in the upper half-plane. Consequently, we obtain the $C = \{(.., \infty, \infty, \infty, \infty, \infty, \infty)\}$ at the signature of $\sigma(\Gamma_F(49))$. In this case, it is only one of the following situations will be true,

i) $C = \{(3, 3, \infty, \infty, \infty, \infty, \infty, \infty)\}$,

ii) $C = \{(3, \infty, \infty, \infty, \infty, \infty, \infty)\}$,

iii) $C = \{(\infty, \infty, \infty, \infty, \infty, \infty)\}$.

From the Definition 3.19 with reflection in the $\Gamma_F(49)$, we have

$$S = \begin{pmatrix} -7b & d/7 \\ 7a & 7b \end{pmatrix} \text{ and } ad + 49b^2 = 1.$$

Thus,

$$\begin{pmatrix} 0 & 1/7 \\ 7 & 0 \end{pmatrix} \begin{pmatrix} -7b & d/7 \\ 7a & 7b \end{pmatrix} = \begin{pmatrix} a & b \\ -49b & d \end{pmatrix} = V \quad \text{and} \quad V \in \Gamma_0(49).$$

From this, the generator elliptic elements is $a + d = \mp 1$ owing to third order. Therefore, we have $d = 1 - a$ or $d = -1 - a$. Consequently, we obtain

$$V_1 = \begin{pmatrix} a & b \\ -49b & 1-a \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} a & b \\ -49b & -1-a \end{pmatrix}.$$

In this case, V_1 and V_2 maps cannot be found according to the specified conditions from the resultant of FS reflections. It is show that there is no 3 valued link period in the boundary components of the signature $\sigma(\Gamma_F(49))$.

□

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