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CONVERGENCE AND DECAY ESTIMATES FOR A NON-AUTONOMOUS DISPERSIVE-DISSIPATIVE EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. This paper deals with the long − time behavior of global bounded solutions for a non-autonomous dispersive-dissipative equation with time-dependent nonlinear damping terms under the null Dirichlet boundary condition. By a new Lyapunov functional and Łojasiewicz-Simon inequality, we show that any global bounded solution converges to a steady state and get the rate of convergence as well, which depends on the decay of the non-autonomous term $g(x,t)$, when damping coefficients are integral positive and positive-negative, respectively.

1. Introduction

We consider a non-autonomous semilinear dispersive-dissipative equation with time-dependent nonlinear damping terms

(1.1)
$$u_{tt} - \Delta u_{tt} - \Delta u + k_1(t)h(u_t) - k_2(t)\Delta u_t + f(x,u) = g(x,t),\ (x,t) \in \Omega \times [0,\infty),$$

subject to the null Dirichlet boundary and initial conditions

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, the function $u_0, u_1 : \Omega \to R$ are given initial data, the nonlinear function f, h, and non-autonomous term g will be specified later.

Our model (1.1) is closely related to equation

$$u_{tt} - \Delta u_{tt} - \Delta u = 0,$$

which is very interesting not only from the point of general theory of PDE, but also from the applications in dynamics. For example, the propagation of transverse homogeneous waves in the oscillation of viscoelastic solids ([11]) and

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the longitudinal vibration of a bar ([15, p.428]; [16]; [17]) for one-dimensional case.

In the past decades, there have been many results on well-posedness and qualitative properties to the dispersive-dissipative equations with nonlinearity. We refer readers to [2,4,14,18,19] for the equations with constant coefficients. Especially, the existence of local solution and global solution ([4,14,18]), the estimate of exponentially decay rate for global solutions with positive definite energy ([2,18]) and blow-up property of solutions with arbitrarily positive initial energy ([4,19]). In this paper, we would like to investigate the asymptotic behavior of global solutions to the initial boundary problem of nonlinear dispersive-dissipative wave equation with time-dependent damping, particularly, the convergence to steady states of all global bounded solutions and the estimate of convergence rate.

For autonomous case ($g = 0$), after 30 years development, there have been many results on convergence to steady states of solutions, see Jendoubi [12], Haraux and Jendoubi [5,6] (linear damping with constant coefficient); Hassen and Haraux [8] (nonlinear damping with constant coefficient), et al. Recently, Jiao [13] investigated the following wave equation with time-dependent damping and analytic nonlinearity

$$u_{tt} - \Delta u + k(t)u_t = f(u), \ \ (x,t) \in \Omega \times [0, \infty).$$

Under Dirichlet boundary condition, he established the result that global solutions converged to a steady state as time went to infinite when $k(t)$ is integrally positive, by virtue of generalized Łojasiewicz-Simon inequality. Furthermore, he presented the Dirichlet initial boundary value problem of a class of wave equations with nonlinear interior damping and analytic nonlinear source term

$$u_{tt} - \Delta u + k(t)h(u_t) = f(u), \ \ (x,t) \in \Omega \times [0, \infty),$$

where h satisfied **(H1)** $h \in C^1(R)$ is a monotone increasing function such that $0 < m_1 \le h'(s) \le m_2 < \infty$, $\forall s \in R$. **(H2)** $h(0) = 0$. He also gave the similar result of an abstract damping wave equation with analytic nonlinear source term.

For the study of non-autonomous case ($g \neq 0$), Chill and Jendoubi [1] considered the wave equation with linear damping

$$u_{tt} - \Delta u + u_t + f(u) = g(x,t), \ \ (x,t) \in \Omega \times [0, \infty).$$

Under the assumption that f is analytic and g satisfies

$$\sup_{t \in R^+} (1+t)^{1+\gamma} \int_t^\infty \|g(s)\|_2^2 ds < \infty,$$

they proved the global bounded solution converged to a steady state. Later, Hassen [9] improved the result of [1] and established the exponential decay rate. Hassen and Chergui [10] investigated the following non-autonomous wave equation with nonlinear damping

$$u_{tt} - \Delta u + |u_t|^\alpha u_t + f(u) = g(x,t), \quad (x,t) \in \Omega \times [0,\infty),$$

under the null Dirichlet boundary condition, where α is a small positive constant. By assuming f is analytic and $g(\cdot,t)$ tends to 0 sufficiently fast in $L^2(\Omega)$ as t tends to ∞, i.e.

$$\|g(\cdot,t)\|^2_{L^2(\Omega)} \leq \frac{C}{(1+t)^{1+\delta+\alpha}}, \ \forall t \in R^+,$$

they proved the solution converged to a steady state without convergence rate. Furthermore, we refer to [20] for the non-autonomous semilinear viscoelastic equation. The key point is that all these papers used an inequality, so called Łojasiewicz-Simon inequality, to obtain their results. However, it requests the nonlinearity $f(x,s)$ is analytic with respect to s.

In view of the works mentioned above, much less effort has been devoted to initial boundary value problem for a non-autonomous dispersive-dissipative equation with time-dependent nonlinear damping terms to our knowledge. The main difficulty is to construct an appropriate new Lyapunov function available to Łojasiewicz-Simon inequality. Our aim is to find the effect of the time-dependent damping terms and the decay of non-autonomous terms on convergence and the estimates of convergence rate.

The outline of the paper is as follows. In Section 2, we introduce the basic tools used in the statements and proofs of the main results. Section 3 is devoted to present the main results and proofs.

Throughout the present paper, the following notations are used for all statements.

- *We denote* $(\cdot,\cdot)_{H^1_0(\Omega)}$, $(\cdot,\cdot)_*$ *and* $(\cdot,\cdot)_2$ *(respectively* $\|\cdot\|_{H^1_0(\Omega)}$, $\|\cdot\|_*$ *and* $\|\cdot\|_2$*) as the inner products (respectively the norms) on the space* $H^1_0(\Omega)$, $H^{-1}(\Omega)$ *and* $L^2(\Omega)$. *The norm on* $L^p(\Omega)$ *is denoted by* $\|\cdot\|_p$.
- *Denote by* C *(somewhere* C_i*,*$(i \in N_+)$*) a generic constant may be different and depend on parameters and the measure of* Ω, *but can be chosen independently of* $t \in R^+$.

2. Preliminary

In this section, we prepare some material needed in the proof of our results. Firstly, we impose some assumptions on nonlinear weak damping function h and nonlinearity f as follows.

(H) $h \in C(R)$, $h(0) = 0$, *and there exist* $\alpha_1 \geq \alpha_2 > 0$ *such that*

$$\alpha_1 \leq h'(v) \leq \alpha_2, \ \forall v \in R.$$

(F1) *The function* f *is analytic in* s *and uniformly with respect to* $x \in \Omega$.

(F2) $sf(s) \geq 0$, $\forall s \in R$.

(F3) *$f(x,s)$ and $\frac{\partial f}{\partial s}(x,s)$ are bounded in $\Omega\times(-c,c)$ for all $c>0$ if $N=1,2$; or $f(x,0)\in L^\infty(\Omega)$ and there exist $\rho_0\geq 0$ and $\mu>0$ satisfying $(N-2)\mu<4$ such that*

$$|\frac{\partial f}{\partial s}(x,s)|\leq \rho_0(1+|s|^\mu),\ a.e.\ s\in(-\infty,\infty),$$

if $N\geq 3$.

Remark 2.1. *It follows clearly from (F2) that $F(x,s)=\int_0^s f(x,\tau)d\tau\geq 0$, $\forall s\in R$.*

In addition, we assume that, for some $\gamma>0$, the function $g\in L^2(R^+;L^2(\Omega))$ satisfies the following polynomial condition:

(G1) $\sup_{t\in R^+}(1+t)^{1+\gamma}\int_t^\infty \|g(s)\|_2^2 ds<\infty$,

or the exponential condition:

(G2) $\sup_{t\in R^+}e^{\gamma t}\int_t^\infty \|g(s)\|_2^2 ds<\infty$.

Similar to [3], we can get the result of existence and uniqueness of global weak solution by a Faedo-Galerkin method.

Proposition 2.2. *Suppose that nonlinear function f, h satisfy (H) and (F1)-(F3), the damping coefficients $k_i (i=1,2)$ meet some conditions, which will be given in Section 3, then for given initial data $(u_0,u_1)\in H_0^1(\Omega)\times H_0^1(\Omega)$, problem (1.1)-(1.3) admits a unique global solution u such that*

$$u\in C(R^+;H_0^1(\Omega)),\ u_t\in C(R^+;H_0^1(\Omega)).$$

The proofs of our convergence results depend on an appropriate new Lyapunov function, compactness properties, and Łojasiewicz-Simon inequality for the energy functional $E: H_0^1(\Omega)\to R$ defined by

$$E(u)=\frac{1}{2}\|\nabla u\|_2^2+\int_\Omega F(x,u)dx. \tag{2.1}$$

Proposition 2.3. *([11]) Suppose the assumptions (F1)-(F3) on f hold, then the energy function $E\in C^2(H_0^1(\Omega))$ satisfies the Łojasiewicz-Simon inequality near every equilibrium point $\phi\in H_0^1(\Omega)$, that is, for every $\phi\in\Sigma$,*

$$\Sigma=\{\phi\in H^2(\Omega)\cap H_0^1(\Omega): -\Delta\phi+f(x,\phi)=0\},$$

there exist $\beta_\phi>0$, $\sigma_\phi>0$ and $0<\theta_\phi\leq\frac{1}{2}$ such that

$$|E(\phi)-E(\psi)|^{1-\theta_\phi}\leq\beta_\phi\|-\Delta\psi+f(x,\psi)\|_*,$$

for all $\psi\in H_0^1(\Omega)$ such that $\|\phi-\psi\|_{H_0^1(\Omega)}<\sigma_\phi$. The number θ_ϕ is called the Łojasiewicz exponent of E at ϕ.

Note that, we claim to prove convergence to equilibrium of any solution having relatively compact range in the energy space. The following assumption guarantees the boundedness of any global solution for problem (1.1)-(1.3):

(F4) *There exist $\lambda < \mu\lambda_1$ and $C > 0$ such that*

$$F(x,u) \le \frac{\lambda u^2}{2} + C, \ for\ all\ u \in R,$$

where $\lambda_1 > 0$ is the optimal constant of the following Poincaré inequality

$$\lambda_1 \|u\|_2^2 \le \|\nabla u\|_2^2, \ u \in H_0^1(\Omega).$$

Proposition 2.4. *Assume that (F4) holds and let u be a global solution of (1.1)-(1.3), then (u, u_t) is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$.*

Next, we present a lemma that plays a key role in the estimation of the rate of convergence, whose proof can be found in [9].

Lemma 2.5. *Let $\xi \in W_{loc}^{1,1}(R^+, R^+)$. We suppose that there exists constants $K_1 > 0$, $K_2 \ge 0$, $p > 1$ and $q > 0$ such that for almost $t \ge 0$ we have*

$$\frac{d}{dt}\xi(t) + K_1 \xi^p(t) \le K_2 (1+t)^{-q}.$$

Then there exists a positive constant $K > 0$ such that

$$\xi(t) \le K(1+t)^{-m}, \ m = \inf\{\frac{1}{p-1}, \frac{q}{p}\}.$$

3. Main results

In this section, we present the convergence result of every global solution for problem (1.1)-(1.3), when damping coefficients $k_1(t)$, $k_2(t)$ satisfy appropriate conditions.

3.1. The integrally positive case

Definition 3.1. *A function $k : [0, +\infty) \to [0, +\infty)$ is said to be integrally positive, if for every $\varepsilon > 0$, there exist $\eta > 0$ such that*

$$\int_t^{t+\varepsilon} k(s)ds \ge \eta, \ \forall t \ge 0.$$

Remark 3.2. *From the definition above, we note that the function h may vanish somewhere but not on any interval. Furthermore, it is clear that there exist a constant $\kappa > 0$ such that $k(t) > \kappa$ a.e. $t \in R$.*

Our first main result, which establishes the convergence result to the problem when damping coefficients $k_1(t)$, $k_2(t)$ are integrally positive, can be given as follows.

Theorem 3.3. *Suppose that $k_1(t)$, $k_2(t)$ are integrally positive, nonlinear functions h and f satisfy (H) and (F1)-(F3), respectively. Furthermore, assume that g satisfies (G1) or (G2). Let u be a global solution of problem (1.1)-(1.3) and assume also that*

(T1) (u,u_t) *is bounded in* $H_0^1(\Omega)\times H_0^1(\Omega)$;
(T2) $\{u(t):t\geq 0\}$ *is relatively compact in* $H_0^1(\Omega)$.
Then there exists $\phi\in\Sigma$ *such that*

$$\|u_t(t)\|_{H_0^1(\Omega)}+\|u(t)-\phi\|_{H_0^1(\Omega)}\to 0,$$

as $t\to\infty$. *Moreover, let* $\theta=\theta_\phi\in(0,\frac{1}{2}]$ *be the Łojasiewicz exponent of* E *at* ϕ. *Then the following assertions holds.*
(i) *If* $0<\theta<\frac{1}{2}$ *and* g *satisfies the polynomial growth (G1), then we have*

$$\|u(t)-\phi\|_{H_0^1(\Omega)}=o((1+t)^{-\chi}),\ t\to\infty,$$

where

$$\chi=\begin{cases}\inf\{\frac{\theta}{1-2\theta},\frac{\gamma}{2}\},\ if\ g\neq 0,\\ \frac{\theta}{1-2\theta},\ if\ g=0.\end{cases}$$

(ii) *If* $\theta=\frac{1}{2}$ *and* g *satisfies the exponential growth (G2), then we have*

$$\|u(t)-\phi\|_{H_0^1(\Omega)}=o(e^{-\zeta t}),\ t\to\infty,$$

where $\zeta>0$.

Let us recall the $\omega-$limit set of a continuous function $u:R^+\to H_0^1(\Omega)$, which is defined as

$$\omega(u)=\{\phi\in H_0^1(\Omega):\exists t_n\to+\infty\ s.t.\ \lim_{n\to\infty}\|u(t_n)-\phi\|_{H_0^1(\Omega)}=0\}.$$

From well-known results on dynamical systems [7], if u is a continuous function with relatively compact range, the corresponding $\omega-$limit set is a non-empty, compact and connected subset of $H_0^1(\Omega)$. Therefore, we present some auxiliary results as follows.

Lemma 3.4. *Let* $u:R^+\to H_0^1(\Omega)$ *be a weak solution of problem (1.1)-(1.3) and suppose that the assumptions of Theorem 3.3 still hold. Then, we have*
(i) $u_t\in L^2(R^+;H_0^1(\Omega))$;
(ii) *the function* E *is a constant on* $\omega(u)$, *and* $\omega(u)\subseteq\Sigma$;
(iii) $\lim_{t\to\infty}\|\nabla u_t(t)\|_2=\lim_{t\to\infty}\|\nabla u(t)\|_2=0$.

Proof. Without less of generality, we assume $k_1(t),k_2(t)\geq\kappa>0$ for all $t\in R$, and let $\Theta:R^+\to R$ be the function defined by

$$\Theta(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+E(u(t))+C_0\int_t^\infty\|g(s)\|_2^2ds,\tag{3.1}$$

where $C_0=\frac{1}{4\varepsilon_0}$ and $0<\varepsilon_0<\alpha_1\kappa$. Multiplying (1.1) by u_t, integrating over Ω and then using integration by parts, we can obtain

$$\begin{aligned}\frac{d}{dt}\Theta(t)&\leq\varepsilon_0\|u_t\|_2^2-k_1(t)\int_\Omega h(u_t)u_tdx-k_2(t)\|\nabla u_t\|_2^2\\&\leq-(\alpha_1\kappa-\varepsilon_0)\|u_t\|_2^2-\kappa\|\nabla u_t\|_2^2.\end{aligned}\tag{3.2}$$

Therefore, the energy function Θ is decreasing and the limit

$$\lim_{t\to\infty}\Theta(t)=\inf_{t\geq 0}\Theta(t)=\Theta_\infty,$$

exist, since it is also bounded from below. From this and the inequality (3.2) we obtain (i).

Let $\phi\in\omega(u)$, then there exists an unbounded increasing sequence $\{t_n\}_{n\in N_+}$ in R^+ such that $u(t_n)\to\phi$ in $H_0^1(\Omega)$. Since $u_t\in L^2(R^+;H_0^1(\Omega))$, we have

$$u(t_n+s)=u(t_n)+\int_{t_n}^{t_n+s}u_\tau(\tau)d\tau\to\phi\ in\ H_0^1(\Omega),$$

for every $s\in[0,1]$. Hence $E(u(t_n+s))\to E(\phi)$ in R for every $s\in[0,1]$. Consequently, using the dominated convergence theorem,

$$E(\phi)=\lim_{n\to\infty}\int_0^1 E(u(t_n+s))ds.$$

Therefore, by integrating $\Theta(t_n+\cdot)$ over $(0,1)$, we derive that

$$E(\phi)=\lim_{n\to\infty}\int_0^1\Theta(u(t_n+s))ds=\Theta_\infty.$$

Here we have used (i) and the fact that u_t is bounded in $H_0^1(\Omega)$. Since ϕ was chosen arbitrarily in $\omega(u)$, E is a constant on $\omega(u)$ and $\omega(u)\subseteq\Sigma$. Moreover, since u has compact range in $H_0^1(\Omega)$, we obtain

$$\lim_{t\to\infty}E(u(t))=\Theta_\infty=E_\infty.$$

Then we obtain (iii) by (3.1) and the equality above. □

The proof of Theorem 3.3.

We divide the proof into 3 steps.

Step 1. Let us define the Lyapunov functional as

$$\Gamma_0(t)=\Theta(t)+\varepsilon(-\Delta u+f(x,u),u_t-\Delta u_t)_*,$$

where $\Theta(t)$ satisfies (3.1) and $\varepsilon>0$ is a constant which will be specified later.

Firstly, we estimate $\frac{d}{dt}\Gamma_0(t)$.

Using (3.2) and computing directly, we have

(3.3)

$$\begin{aligned}\frac{d}{dt}\Gamma_0(t)\leq&-(\alpha_1\kappa-\varepsilon_0)\|u_t\|_2^2-\kappa\|\nabla u_t\|_2^2+\varepsilon(-\Delta u_t+\frac{\partial f}{\partial u}(x,u)u_t,u_t-\Delta u_t)_*\\&-\varepsilon(-\Delta u+f(x,u),\Delta u-f(x,u)-k_1(t)h(u_t)+k_2(t)\Delta u_t+g(x,t))_*\\\leq&-(\alpha_1\kappa+\varepsilon_0)\|u_t\|_2^2-\kappa\|\nabla u_t\|_2^2-\varepsilon(-\Delta u_t+\frac{\partial f}{\partial u}(x,u)u_t,u_t-\Delta u_t)_*\\&-\frac{\varepsilon}{2}\|-\Delta u+f(x,u)\|_*^2+\frac{\varepsilon}{2}\|k_1(t)h(u_t)-k_2(t)\Delta u_t-g(x,t)\|_*^2.\end{aligned}$$

Noting that, for $N \geq 3$ and $0 < \mu < \frac{4}{N-2}$, using (F3) and the boundedness of u in $H_0^1(\Omega)$, we can derive

$$
\begin{aligned}
\|\frac{\partial f}{\partial u}(x,u)u_t\|_* \leq & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\leq 1} (\int_\Omega |u_t\varphi|dx + \int_\Omega |u|^\mu |u_t||\varphi|dx) \\
\leq & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\leq 1} (\|u_t\|_2\|\varphi\|_2 + \|u_t\|_{\frac{2N}{N-2}}\|\varphi\|_{\frac{2N}{N-2}}\|u^\mu\|_{\frac{N}{2}}) \\
\leq & C\|\nabla u_t\|_2.
\end{aligned} \tag{3.4}
$$

For $N = 1, 2$, we can also derive the estimate above by (F3) (setting $\mu = 1$). Moreover, computing directly, we have the following estimate

$$
\begin{aligned}
\|\Delta u_t\|_* \leq & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\leq 1} \int_\Omega |\Delta u_t \varphi| dx \\
\leq & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\leq 1} \int_\Omega |\nabla u_t \cdot \nabla\varphi| dx \\
\leq & C \sup_{\|\varphi\|_{H_0^1(\Omega)}\leq 1} \|\nabla u_t\|_2\|\nabla\varphi\|_2 \\
\leq & C\|\nabla u_t\|_2.
\end{aligned} \tag{3.5}
$$

Similarly, we have

$$
\|\Delta u\|_* \leq C\|\nabla u\|_2. \tag{3.6}
$$

Combining (3.3)-(3.5) and choosing $\varepsilon > 0$ small enough, then we have

$$
\frac{d}{dt}\Gamma_0(t) \leq -C_1\{\|\nabla u_t\|_2^2 + \| - \Delta u + f(x,u)\|_*^2\} + C_2\|g(t)\|_2^2. \tag{3.7}
$$

Now, let us define the functional

$$
\Gamma(t) = \Gamma_0(t) + C_2\int_t^\infty \|g(s)\|_2^2 ds.
$$

Then by (3.7), we see that

$$
\begin{aligned}
\frac{d}{dt}\Gamma(t) \leq & - C_1\{\|\nabla u_t\|_2^2 + \| - \Delta u + f(x,u)\|_*^2\} \\
\leq & - C\{\|\nabla u_t\|_2 + \| - \Delta u + f(x,u)\|_*\}^2,
\end{aligned} \tag{3.8}
$$

for all $t \geq T$. By virtue of (3.3)-(3.6), using the boundedness of $f(x,u)$ in (F3), we see that

$$
(-\Delta u + f(x,u), u_t - \Delta u_t)_* \leq C(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2).
$$

Therefore, $\Gamma(t) \geq 0$, for $\varepsilon > 0$ small enough. Then $\Gamma(t)$ is non-negative and non-increasing on $[T,\infty)$, and so that $\Gamma(t)$ has a limit at infinity.

Step 2. Now, we consider the following two possibilities:

Case 1. If the function g satisfies the polynomial growth (G1), then, for $\theta = \theta_\phi$ as in Proposition 2.3. Let $\theta_0 \in (0,\theta]$ be such that $(1+\gamma)(1-\theta_0) > 1$, i.e.

$\theta_0 < \frac{\gamma}{1+\gamma}$. Note that Łojasiewicz-Simon inequality is satisfied with θ replaced by θ_0. Then, by applying the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned}[\Gamma(t) - E(\phi)]^{1-\theta_0} \leq C\{&\|u_t\|_2^{2(1-\theta_0)} + \|\nabla u_t\|_2^{2(1-\theta_0)} + |E(u) - E(\phi)|^{1-\theta_0} \\ &+ \| - \Delta u + f(x,u)\|_* + \|u_t - \Delta u_t\|_*^{\frac{1-\theta_0}{\theta_0}} \\ &+ (\int_t^\infty \|g(s)\|_2^2 ds)^{1-\theta_0}\}.\end{aligned} \tag{3.9}$$

Noting that $2(1-\theta_0) > 1$ and $\frac{1-\theta_0}{\theta_0} > 1$, which together with Lemma 3.4 (iii) imply that

$$\begin{aligned}[\Gamma(t) - E(\phi)]^{1-\theta_0} \leq & C\{\|\nabla u_t\|_2 + |E(u) - E(\phi)|^{1-\theta_0} + \| - \Delta u + f(x,u)\|_* \\ &+ (\int_t^\infty \|g(s)\|_2^2 ds)^{1-\theta_0}\} \\ \leq & C\{\|\nabla u_t\|_2 + |E(u) - E(\phi)|^{1-\theta_0} + \| - \Delta u + f(x,u)\|_* \\ &+ (1+t)^{-(1+\gamma)(1-\theta_0)}\},\end{aligned} \tag{3.10}$$

if we choosing $T > 0$ large enough.

Since $\phi \in \omega(u)$, there exist $\{t_n\}_{n\geq 1} : t_n \to \infty$, such that

$$u(t_n) \to \phi, n \to \infty, \ in \ H_0^1(\Omega). \tag{3.11}$$

And we also get

$$\lim_{n\to\infty} E(t_n) = E(\phi). \tag{3.12}$$

It has been proved in Step1 that $\Gamma(t)$ has a limit at infinity and by means of (3.11), we have for all $\delta > 0$, $\delta \ll \sigma_\phi$, there exists $N > 0$ such that $t_N > T$ and

$$\|u(t_N) - \phi\|_{H_0^1(\Omega)} < \frac{\delta}{3}, \tag{3.13}$$

and for $\forall t \geq t_N$, we have

$$\frac{C_3}{\theta_0}\{[\Gamma(t_N) - E(\phi)]^{\theta_0} - [\Gamma(t) - E(\phi)]^{\theta_0}\} < \frac{\delta}{3C_4}, \tag{3.14}$$

$$\frac{C_3}{\theta_0}\{|(1+t_N)^{1-(1+\gamma)(1-\theta_0)} - (1+t)^{1-(1+\gamma)(1-\theta_0)}|\} < \frac{\delta}{3C_4}, \tag{3.15}$$

and

$$\Gamma(t) \geq E(\phi). \tag{3.16}$$

Let

$$\bar{t} = \sup\{t \geq t_N : \|u(s) - \phi\|_{H_0^1(\Omega)} < \sigma_\phi, \forall s \in [t_N, t]\}.$$

Then by Proposition 2.3 and (3.10), for all $t \in [t_N, \bar{t})$, we see that

$$[\Gamma(t) - E(\phi)]^{1-\theta_0} \leq C\{\|\nabla u_t\|_2 + \|-\Delta u + f(x,u)\|_* + (1+t)^{-(1+\gamma)(1-\theta_0)}\}. \tag{3.17}$$

Moreover, by computing directly, we can derive

$$-\frac{d}{dt}[\Gamma(t)-E(\phi)]^{\theta_0} = -\theta_0[\Gamma(t)-E(\phi)]^{\theta_0-1}\frac{d}{dt}\Gamma(t). \tag{3.18}$$

Combining (3.8), (3.17) and (3.18), we see that

$$\begin{aligned} -\frac{d}{dt}[\Gamma(t)-E(\phi)]^{\theta_0} \geq & \frac{\theta_0 C\{\|\nabla u_t\|_2 + \|-\Delta u + f(x,u)\|_*\}^2}{\|\nabla u_t\|_2 + \|-\Delta u + f(x,u)\|_* + (1+t)^{-(1+\gamma)(1-\theta_0)}} \\ \geq & \theta_0 C\{\|\nabla u_t\|_2 + \|-\Delta u + f(x,u)\|_* \\ & - (1+t)^{-(1+\gamma)(1-\theta_0)}\}. \end{aligned} \tag{3.19}$$

Integrating (3.19) over $[t_N, \bar{t})$ to obtain

$$\begin{aligned} \int_{t_N}^{\bar{t}} \|\nabla u_t\|_2 dt \leq & \int_{t_N}^{\bar{t}} \{\|\nabla u_t\|_2 + \|-\Delta u + f(x,u)\|_*\} \\ \leq & \frac{C_3}{\theta_0}\{|[\Gamma(t_N)-E(\phi)]^{\theta_0} - [\Gamma(t)-E(\phi)]^{\theta_0}| \\ & + |(1+t_N)^{1-(1+\gamma)(1-\theta_0)} - (1+t)^{1-(1+\gamma)(1-\theta_0)}|\}. \end{aligned} \tag{3.20}$$

Assuming $\bar{t} < \infty$, then by (3.13)-(3.15) and (3.20), we get

$$\begin{aligned} \|u(\bar{t})-\phi\|_{H_0^1(\Omega)} \leq & \int_{t_N}^{\bar{t}} \|u_t\|_{H_0^1(\Omega)} dt + \|u(t_N)-\phi\|_{H_0^1(\Omega)} \\ \leq & C_4 \int_{t_N}^{\bar{t}} \|\nabla u_t\|_2 dt + \|u(t_N)-\phi\|_{H_0^1(\Omega)} \\ < & \delta, \end{aligned}$$

which contradicts the definition of $\bar{t}$. Therefore, $\bar{t} = \infty$. Then it follows from (3.20) that

$$\int_{t_N}^{\infty} \|\nabla u_t\|_2 dt < \infty,$$

which implies the integrability of u in $H_0^1(\Omega)$. Since the compactness of the range of u, we have

$$\lim_{t\to\infty} \|u(t)-\phi\|_{H_0^1(\Omega)} = 0.$$

Case 2. When the growth condition in g is exponential, we replace the term $(1+t)^{-(1+\gamma)(1-\theta_0)}$ by $e^{-\gamma(1-\theta_0)}$, which is integrable too, and then the same conclusion holds.

Step 3. We now prove the estimate of convergence rate.

Case 1. (`Polynomial decay`) Let $\theta \in (0, \frac{1}{2})$ and g satisfy the polynomial growth (G1). Noting that, inequality (3.8) and (3.17) still hold if we replace θ_0 by Łojasiewicz exponent θ. It follows from (3.17) and Young's inequality that

$$[\Gamma(t)-E(\phi)]^{2(1-\theta)} \leq C\{(\|\nabla u_t\|_2 + \|-\Delta u + f(x,u)\|_*)^2 + (1+t)^{-2(1+\gamma)(1-\theta)}\}, \tag{3.21}$$

for all $t > T$. Using (3.8) and (3.21) to obtain

$$\frac{d}{dt}[\Gamma(t) - E(\phi)] + C_4[\Gamma(t) - E(\phi)]^{2(1-\theta)} \leq C(1+t)^{-2(1+\gamma)(1-\theta)}, \tag{3.22}$$

for some constant $C_4 > 0$.

Then by Lemma 2.5 we see that for all $t > T$,

$$[\Gamma(t) - E(\phi)] \leq C(1+t)^{-m}, \tag{3.23}$$

where

$$m = \inf\{\frac{1}{1-2\theta}, 1+\gamma\}.$$

On the other hand, we can derive from (3.8) that

$$-\frac{d}{dt}[\Gamma(t) - E(\phi)] \geq C\|\nabla u_t\|_2^2.$$

Integrating the inequality above over $(t, 2t)(t > T)$, we can deduce

$$\int_t^{2t} \|\nabla u_\tau(\tau)\|_2^2 d\tau \leq C(1+t)^{-m}. \tag{3.24}$$

Note that for every $t \in R^+$, Hölder's inequality implies

$$\int_t^{2t} \|\nabla u_\tau(\tau)\|_2 d\tau \leq t^{\frac{1}{2}} \int_t^{2t} \|\nabla u_\tau(\tau)\|_2^2 d\tau. \tag{3.25}$$

Combining (3.24) and (3.25), we can get

$$\int_t^{2t} \|\nabla u_\tau(\tau)\|_2 d\tau \leq C(1+t)^{\frac{1-m}{2}},$$

for every $t > T$. Therefore we obtain for every $t > T$,

$$\int_t^{\infty} \|\nabla u_\tau(\tau)\|_2 d\tau \leq \sum_{i=0}^{\infty} \int_{2^i t}^{2^{i+1} t} \|\nabla u_\tau(\tau)\|_2 d\tau \leq C(2^i t)^{\frac{1-m}{2}} \leq C(1+t)^{\frac{1-m}{2}}.$$

Then,for all $t > T$,

$$\|u(\bar{t}) - \phi\|_{H_0^1(\Omega)} \leq C \int_t^{\infty} \|\nabla u_\tau(\tau)\|_2 d\tau \leq C(1+t)^{-\chi},$$

where

$$\chi = \inf\{\frac{\theta}{1-2\theta}, \frac{\gamma}{2}\}.$$

Case 2. (`Exponential decay`) Suppose that $\theta = \frac{1}{2}$ and g satisfies the exponential growth (G2). Then (3.22) becomes

$$\frac{d}{dt}[\Gamma(t) - E(\phi)] \leq -C_5[\Gamma(t) - E(\phi)] + C_6 e^{-\gamma t},$$

where $C_5 = \frac{1}{C_4}$ and C_4 can be chosen large enough to ensure that $C_5 < \gamma$.

Now, let

$$\Lambda(t) = \Gamma(t) - E(\phi) - C_6 e^{-C_5 t} \int_0^t e^{-(\gamma - C_5)\tau} d\tau.$$

Then

$$\begin{aligned}\frac{d}{dt}\Lambda(t) &= \frac{d}{dt}[\Gamma(t) - E(\phi)] - C_6 e^{-\gamma t} + C_5 C_6 e^{-C_5 t}\int_0^t e^{-(\gamma - C_5)\tau} d\tau \\ &\le \frac{d}{dt}[\Gamma(t) - E(\phi)] + C_5 C_6 e^{-C_5 t}\int_0^t e^{-(\gamma - C_5)\tau} d\tau \\ &= -C_5 \Lambda(t).\end{aligned}$$

This yields

$$\Lambda(t) \le e^{-C_5 t},$$

and therefore

$$\Gamma(t) - E(\phi) \le e^{-C_5 t}\{1 + C_6 \int_0^t e^{-(\gamma - C_5)\tau} d\tau\} \le Ce^{-C_5 t}. \tag{3.26}$$

On the other hand, from the inequality (3.19) (when g satisfies the exponential growth and $\theta_0 = \theta = \frac{1}{2}$), we have for every $t > T$,

$$-\frac{d}{dt}[\Gamma(t) - E(\phi)]^{\frac{1}{2}} + Ce^{-\frac{\gamma t}{2}} \ge C\|\nabla u_t\|_2.$$

Integrating this inequality over the interval $[t, \infty)(t > T)$, we obtain

$$\|u(t) - \phi\|_{H_0^1(\Omega)} \le C\int_t^\infty \|\nabla u_\tau(\tau)\|_2 d\tau \le [\Gamma(t) - E(\phi)]^{\frac{1}{2}} + Ce^{-\frac{\gamma t}{2}}.$$

This inequality together with the inequality (3.26) imply the claim.

3.2. The case of positive-negative

We begin with the definition of positive-negative.

Definition 3.5. *Assume that $\{I_n\}_{n\in N}$ is a sequence of disjoint interval in $(0, \infty)$, $I_n = (a_n, b_n)$, where $a_1 = 0$, $b_n = a_{n+1}$ and $a_n \to \infty$ as $n \to \infty$. If $k : [0, +\infty) \to R$ satisfies: for all $t \in I_n$, there exists $0 < m_n \le M_n < \infty$ such that*

$$m_n \le k(t) \le M_n,$$

we call $k(t)$ is in the positive-negative case.

Remark 3.6. *Noting that this kind of intermitting damping may change sign at the discontinuous points. If all the discontinuous points $k(b_n) = 0$, we call this damping is in on-off case.*

Using the same method, we can get the convergence to equilibrium theorem when $k_1(t), k_2(t)$ are positive-negative.

Theorem 3.7. *Suppose that $k_1(t)$, $k_2(t)$ are positive-negative, nonlinear functions h and f satisfy (H) and (F1)-(F3), respectively. Furthermore, assume that g satisfies (G1) or (G2). Let u be a global solution of problem (1.1)-(1.3) and assume also that*

(T1) *(u, u_t) is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$;*

(T2) *$\{u(t) : t \geq 0\}$ is relatively compact in $H_0^1(\Omega)$.*

Then there exists $\phi \in \Sigma$ such that

$$\|u_t(t)\|_{H_0^1(\Omega)} + \|u(t) - \phi\|_{H_0^1(\Omega)} \to 0,$$

as $t \to \infty$. Moreover, let $\theta = \theta_\phi \in (0, \frac{1}{2}]$ be the Lojasiewicz exponent of E at ϕ. Then the following assertions holds.

(i) *If $0 < \theta < \frac{1}{2}$ and g satisfies the polynomial growth (G1), then we have*

$$\|u(t) - \phi\|_{H_0^1(\Omega)} = o((1+t)^{-\chi}),\ t \to \infty,$$

where

$$\chi = \begin{cases} \inf\{\frac{\theta}{1-2\theta}, \frac{\gamma}{2}\},\ if\ g \neq 0, \\ \frac{\theta}{1-2\theta},\ if\ g = 0. \end{cases}$$

(ii) *If $\theta = \frac{1}{2}$ and g satisfies the exponential growth (G2), then we have*

$$\|u(t) - \phi\|_{H_0^1(\Omega)} = o(e^{-\zeta t}),\ t \to \infty,$$

where $\zeta > 0$.

Proof. The proof is similar to Theorem 3.3, so we omit it. □

3.3. Boundedness of global solutions

In this subsection, we present the boundedness of global solutions to problem (1.1)-(1.3), under the assumption (F4). We give the proof of proposition 2.4 as follows.

Proof. Setting Θ to be the function provided by (3.1), which is nonincreasing by (3.2). Based on the condition (F3), we can see

$$|\int_\Omega F(x, u_0)dx| \leq C(1 + \|\nabla u_0\|_2^{\mu+2}),$$

where $C \geq 0$ is a constant depending on the constant in (F3), the measure of Ω and the constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{\mu+2}(\Omega)$. According to the inequality above and the definition of Θ, there exists a constant $C_7 \geq 0$ such that

$$\Theta(0) \leq C_7(1 + \|\nabla u_0\|_2^2 + \|\nabla u_1\|_2^2 + \|\nabla u_0\|_2^{\mu+2}). \tag{3.27}$$

On the other hand, it follows the definition of E and the condition (F4) that there exist positive constants C_8 and C_9 such that

$$\|\nabla u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 \leq C_8\Theta(t) + C_9. \tag{3.28}$$

Combining (3.27), (3.28) and using the nonincreasing property of E, we can derive the result. □

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