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EXISTENCE AND MULTIPLICITY OF SOLUTIONS OF $p(x)$-TRIHARMONIC PROBLEM

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Abstract. In this paper, we study the following nonlinear problem:

$$\begin{cases} -\Delta_p^3(x)u = \lambda V_1(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0 & \text{on } \partial\Omega, \end{cases}$$

under adequate conditions on the exponent functions p, q and the weight function V_1. We prove the existence and nonexistence of eigenvalues for $p(x)$-triharmonic problem with Navier boundary value conditions on a bounded domain in $\mathbb{R}^N$. Our technique is based on variational approaches and the theory of variable exponent Lebesgue spaces.

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1. Introduction

We study the properties of the eigenvalue of the $p(x)$-triharmonic problem:

$$\begin{cases} -\Delta_p^3(x)u = \lambda V_1(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $(N > 3)$, $p, q \in C(\overline{\Omega})$, $1 < p(x) < \frac{N}{3}, 1 < q(x) < \frac{N}{3}$ for all $x \in \overline{\Omega}$, λ is a nonnegative real parameter, V_1 is an indefinite weight function that can change the sign in Ω, $\Delta^3_{p(x)}u := \operatorname{div}\big(\Delta(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u)\big)$ is $p(x)$-triharmonic operator. Note that $p(x)$-triharmonic operator which is not consistent and is related to the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. It is also worth mentioning that the problems with the growth conditions $p(x)$-triharmonic have more complicated nonlinearities than the constant cases. Indeed, firstly the problem is not homogeneous, and secondly, the Lagrange multiplier theorem is not be useful in such a case because $p(x)$ is variable. We find this kind of problem in the modeling of electrorheological fluids [12, 13] and of elastic mechanics. For more details, we invite the reader to an overview of references [3, 4, 9, 15].

In the literature, several authors treat the eigenvalues of biharmonic problems for example Ge et al. [8] considered the eigenvalues of the $p(x)$-biharmonic problem with an indefinite weight:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where p, q are continuous functions and V is an indefinite weight function. Under appropriate conditions on p and q, they showed the existence of a continuous family of eigenvalues of the problem.

In [1] Ayoujil studied a class of $p(\cdot)$-biharmonic of the form

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and he established the existence and non-existence of eigenvalues for a $p(x)$-biharmonic equation function of weight on a bounded domain in $\mathbb{R}^{\mathbb{N}}$.

In this paper, if not otherwise stated, we will always suppose that exponent $p(x)$ is continuous on $\overline{\Omega}$ with

$$p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < \frac{N}{3},$$

and $p^*(x)$ denotes the critical variable exponent related to $p(x)$, defined for all $x \in \overline{\Omega}$ by the pointwise relation $p_3^*(x) = \frac{Np(x)}{N-3p(x)}$.

Let us introduce some conditions for Problem (1.1) as follows:

$(\mathbf{H}_1)$ $p^+ < q^- \leq q^+ < p^*(x)$, $r_1(x) > \frac{p_3^*(x)}{p_3^*(x)-p(x)}$;

$(\mathbf{H}_2)$ $V_1 \in L^{r_1(x)}(\Omega)$.

Based on the use of Mountain Pass lemma here, Problem (1.1) is stated in the framework of the generalized Sobolev space:

$$X := W_0^{1,p(\cdot)}(\Omega) \cap W^{3,p(\cdot)}(\Omega)$$

equipped with the norm:

$$\|u\| = \inf \left\{ \mu > 0 : \int_\Omega \left(\left| \frac{\nabla \Delta u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

X endowed with the above norm is a separable and reflexive Banach space.

The paper is structured as follows. In Section 2, we present a mathematical background of variable exponent Lebesgue spaces and Sobolev spaces. In Section 3, we give our main results and the proofs.

2. Preliminaries

As preliminaries, we need some results on the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$ and some properties. Let Ω be a bounded domain of $\mathbb{R}^N$ and denote

$$C_+(\overline{\Omega}) = \Big\{ h(x) : \quad h(x) \in C(\overline{\Omega}), \quad h(x) > 1, \quad \forall x \in \overline{\Omega} \Big\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max \Big\{ h(x) : \ x \in \overline{\Omega} \Big\}, \quad h^- = \min \Big\{ h(x) : \ x \in \overline{\Omega} \Big\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf \Big\{ \mu > 0 : \ \int_\Omega |\frac{u(x)}{\mu}|^{p(\cdot)} dx \leq 1 \Big\}.$$

Then $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space.

Proposition 2.1. ([14]) *Let $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ be separable, uniformly convex, reflexive and its conjugate space be $L^{q(\cdot)}(\Omega)$ where $q(\cdot)$ is the conjugate function of $p(\cdot)$, i.e.,*

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

Then for $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\Big|\int_\Omega uv dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

A fundamental tool in the manipulation of generalized Lebesgue spaces which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$, called the modular of the $L^{p(x)}(\Omega)$ space, defined by:

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

We remember the following, (see ([7, 11])) .

Proposition 2.2. *For all $u \in L^{p(x)}(\Omega)$, we have*

(1) $|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$ *if* $|u|_{p(x)} > 1$;

(2) $|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$ *if* $|u|_{p(x)} \leq 1$.

The Sobolev space with variable exponent $W^{k,p(\cdot)}(\Omega)$ is defined as

$$W^{k,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq k\Big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}} u$, with $\alpha = (\alpha_1, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(\cdot)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(\cdot)},$$

also becomes a separable and reflexive Banach space. For more details, see to ([14]). Denote

$$p_k^*(\cdot) = \begin{cases} \frac{Np(\cdot)}{N-kp(\cdot)} & \text{if } kp(\cdot) < N, \\ +\infty & \text{if } kp(\cdot) \geq N, \end{cases}$$

for any $k \geq 1$.

Proposition 2.3. ([2]) *For $p, q \in C_+(\overline{\Omega})$ such that $q(\cdot) \leq p_k^*(\cdot)$, there is a continuous embedding*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Similarly to Proposition 2.3, we have:

Proposition 2.4. ([6]) *Let $I_{p(x)}(u) = \int_\Omega |\nabla\Delta u(x)|^{p(x)}dx$. Then for $u \in X$, we have*

(1) *for $\|u\| \leq 1$, $\|u\|^{p^+} \leq I_{p(x)}(u) \leq \|u\|^{p^-}$;*
(2) *for $\|u\| \geq 1$, $\|u\|^{p^-} \leq I_{p(x)}(u) \leq \|u\|^{p^+}$.*

The following result (see ([2]), Theorem 3.2), which will be used later, is an embedding result between the spaces X and $L^{q(x)}(\Omega)$.

Theorem 2.5. *Let $p, q \in C_+(\overline{\Omega})$. Assume that*

$$p(x) < \frac{N}{3} \quad \text{and} \quad q(x) < p_3^*(x).$$

Then, there is a continuous and compact embedding X into $L^{q(x)}(\Omega)$.

We remember as well the next proposition, which will be needed later.

Proposition 2.6. ([5]) *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then , we have*

(1) *for $|u|_{p(x)q(x)} \leq 1$, $|u|^{p^+}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}$,*
(2) *for $|u|_{p(x)q(x)} >, 1$ $|u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}$.*

Let the functionals $I, J : X \to \mathbb{R}$ defined as

$$I(u) = \int_\Omega \frac{|\nabla\Delta u|^{p(x)}}{p(x)}dx, \quad \forall u \in X \tag{2.1}$$

and

$$J(u) = \int_\Omega \frac{V_1(x)|u|^{q(x)}}{q(x)}dx, \quad \forall u \in X. \tag{2.2}$$

Applying a standard argument, we can show the next lemma.

Lemma 2.7. *Assume that ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_2$) hold. Then, the functionals I and J are well defined, I is coercive, and J is weakly continuous. Moreover, $I, J \in C^1(X, \mathbb{R})$ with the derivatives are respectively given by*

$$\langle I'(u), \phi\rangle = \int_\Omega |\nabla\Delta u|^{p(x)-2}\nabla\Delta u\nabla\Delta\phi dx \tag{2.3}$$

and

$$\langle J^{'}(u), \phi\rangle = \int_\Omega V_1(x)|u|^{q(x)-2}u\phi dx,$$

for all $u, \phi \in X$, where $\langle ., .\rangle$ denotes the duality between X and its dual space X^*.

We give an auxiliary result which will help us further in the demonstration.

Proposition 2.8. (i) *I is weakly lower semi-continuous, namely $u_n \rightharpoonup u$ implies that $I(u) \leq \liminf I(u_n)$.*
(ii) *I is a weakly-strongly continuous functional, namely $u_n \rightharpoonup u$ implies that $I(u_n) \longrightarrow I(u)$.*

Proof. (i) By coercivity, we get

$$\begin{aligned}0 &\leq \langle I(u_n - u), u_n - u\rangle \\ &= \langle I(u_n), u_n\rangle - \langle I(u_n), u\rangle - \langle I(u), u_n\rangle + \langle I(u), u\rangle.\end{aligned}$$

Hence,

$$\langle I(u_n), u\rangle + \langle I(u), u_n\rangle - \langle I(u), u\rangle \leq \langle I(u_n), u_n\rangle.$$

Now, I is continuous, so by $u_n \rightharpoonup u$ it follows that $\langle I(u_n), u\rangle \to \langle I(u), u\rangle$. Then,

$$\langle I(u_n), u\rangle + \langle I(u), u_n\rangle - \langle I(u), u\rangle \to \langle I(u), u\rangle \quad \text{as} \quad n \to \infty.$$

As consequence, we have

$$\begin{aligned}\langle I(u), u\rangle &= \lim \inf_{n\longrightarrow\infty} (\langle I(u_n), u\rangle + \langle I(u_n), u_n\rangle - \langle I(u), u\rangle) \\ &\leq \lim \inf_{n\longrightarrow\infty} \langle I(u_n), u_n\rangle.\end{aligned}$$

(ii) Let's consider $\{u_n\}$ a sequence in X such that $u_n \rightharpoonup u$ in X. Denote by $r_1^{'}(x)$ the conjugate exponent of the function $r_1(x)$ (*i.e.* $r_1^{'}(x) = \frac{r_1(x)}{r_1(x)-1}$). Hence, as $q(x)r_1^{'}(x) < p_3^*(x)$, Theorem 2.5 involves $u_n \rightharpoonup u$ in $L^{q(x)r^{'}(x)}(\Omega)$. This, together with the continuity of Nemytski operator $\mathcal{N}_{V_1,q}$ defined by $\mathcal{N}_{V_1,q}(u)(x) = V_1(x)|u(x)|^{q(x)}$ if $u \neq 0$ and $\mathcal{N}_{V_1,q}(u)(x) = 0$ if not, give $I(u_n) \to I(u)$. □

3. Main results

Definition 3.1. We say that $u \in X$ is a weak solution of Problem (1.1) if u satisfies

$$-\int_\Omega |\nabla\Delta u|^{p(x)-2}\nabla\Delta u\nabla\Delta v dx - \lambda\int_\Omega V_1(x)|u|^{q(x)-2}uv dx = 0, \tag{3.1}$$

for all $v \in X$.

The energy functional corresponding to Problem (1.1) is defined by $L_\lambda : X \to \mathbb{R}$,

$$L_\lambda(u) = I(u) - \lambda J(u).$$

We consider

$$F(u) = \int_\Omega |\nabla \Delta u|^{p(x)} dx$$

and

$$G(u) = \int_\Omega V_1(x)|u|^{q(x)} dx,$$

for every $(u, v) \in X$. Define

$$\lambda^* = \inf \Big\{ \frac{I(u)}{J(u)}, u \in X \text{ and } J(u) > 0 \Big\}$$

and

$$\lambda_* = \inf \Big\{ \frac{F(u)}{G(u)}, u \in X \text{ and } G(u) > 0 \Big\}.$$

We begin with the next lemma, which plays a fundamental role in the proof of Theorem 3.3.

Lemma 3.2. *Assume that* $(\boldsymbol{H}_1)$ *and* $(\boldsymbol{H}_2)$ *are verified and*

$$2q^+ - p^- < 2q^- \tag{3.2}$$

hold. Then

$$\lim_{\|u\| \longrightarrow 0} \frac{I(u)}{J(u)} = \infty \tag{3.3}$$

and

$$\lim_{\|u\| \longrightarrow \infty} \frac{I(u)}{J(u)} = \infty. \tag{3.4}$$

Proof. Since $J(u) = \int_\Omega \frac{V_1(x)|u|^{q(x)}}{q(x)} dx$,

$$\begin{aligned} |J(u)| &= \left| \int_\Omega \frac{V_1(x)|u|^{q(x)}}{q(x)} dx \right| \\ &\leq \int_\Omega \left| \frac{V_1(x)|u|^{q(x)}}{q(x)} \right| dx. \end{aligned}$$

By applying the Hölder's inequality, we get

$$|J(u)| \leq \frac{2}{q^-} |V_1|_{r_1(x)} \left| |u|^{q(x)} \right|_{r_1'(x)}.$$

Thanks to Proposition 2.6, it follows

$$|J(u)| \leq \frac{2}{q^-}|V_1|_{r_1(x)}|u|^{q^i}_{q(x)r_1'(x)}, \tag{3.5}$$

where $i = +$ if $|u|_{q(x)r_1'(x)} > 1$ and $i = -$ if $|u|_{q(x)r_1'(x)} < 1$.

On the one hand, using $(\mathbf{H}_1)$, we have $p(x) < q(x)r_1^{'}(x) < p^*(x)$. Hence, from Proposition 2.2, X is continuously embedded in $L^{q(x)r_1^{'}(x)}(\Omega)$. So, there exists $c_1 > 0$ such that

$$|J(u)| \quad \leq \quad \frac{2c_1}{q^-}|V_1|_{r_1(x)}|u|^{q^i}. \tag{3.6}$$

Then, we proceed as follows

$$\begin{aligned} I(u) &= \int_\Omega \frac{|\nabla\Delta u|^{p(x)}}{p(x)}dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla\Delta u|^{p(x)}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} \\ &\geq \frac{1}{p^+}\|u\|^{p^+}. \end{aligned}$$

For each $u \in X$ small enough with $\|u\| \leq 1$, by using (3.5) and (3.6), we infer

$$\frac{I(u)}{J(u)} \geq \frac{\frac{1}{p^+}\|u\|^{p^+}}{\frac{2c_1}{q^-}|V_1|_{r_1(x)}\|u\|^{q^i}}. \tag{3.7}$$

Since $p^+ < q^- \leq q^+$, passing to the limit as $\|u\| \longrightarrow 0$ in the above inequality, we conclude that assertion (3.3) stay true.

Next, we prove that assertion (3.4) remains true. From (3.2), there exists a positive constant δ such that $2q^+ - p^- < \delta < 2q^-$. Hence we get

$$p^- > 2(q^+ - \delta) > 2(q^- - \delta). \tag{3.8}$$

Let $s_1(x)$ be a measurable function such that

$$\frac{p^*(x)}{p^*(x) + \delta - q(x)} \leq s_1(x) \leq \frac{p^*(x)r_1(x)}{p^*(x) + \delta r_1(x)}, \tag{3.9}$$

for almost all $x \in \Omega$ and

$$\delta(\frac{s_1^+}{s_1^-} + 1) \leq q^-. \tag{3.10}$$

It's clear that $s_1 \in L^\infty(\Omega)$, $1 < s_1(x) < r_1(x)$. In addition, we have

$$\delta t_1(x) \leq p^*(x) \quad \text{and} \quad (q(x) - \delta)s_1^{'}(x) \leq p^*(x), \quad \forall\, x \in \overline{\Omega}, \tag{3.11}$$

where $t_1(x) := \frac{r_1(x)s_1(x)}{r_1(x)-s_1(x)}$ and $s_1^{'}(x) = \frac{s_1(x)}{s_1(x)-1}$.

Let $u \in X$ with $\|u\| > 1$. From Hölder's inequality, we have

$$|J(u)| \quad \leq \quad \frac{2}{q^-}\left|V_1|u|^{\delta}\right|_{s_1(x)}\left||u|^{q(x)-\delta}\right|_{s_1^{'}(x)}. \tag{3.12}$$

Without loss of generality, we assume that $\left|V_1|u|^{\delta}\right|_{s_1(x)} > 1$. So, from Proposition 2.2 and from Hölder's inequality, we obtain

$$\begin{aligned} |J(u)| \quad &\leq \quad \frac{2}{q^-}\left(\left(\rho_{s_1(x)}|V_1|u|^{\delta}\right)\right)^{\frac{1}{s_1^-}}\left||u|^{q(x)-\delta}\right|_{s_1^{'}(x)} \\ &= \quad \frac{2}{q^-}\left(\int_{\Omega}\left||V_1|^{s_1(x)}|u|^{\delta s_1(x)}\right|\right)^{\frac{1}{s_1^-}}\left||u|^{q(x)-\delta}\right|_{s_1^{'}(x)} \\ &\leq \quad \frac{4}{q^-}\left||V_1|^{s_1(x)}\right|^{\frac{1}{s_1^-}}_{\frac{r_1(x)}{s_1(x)}}\left||u|^{\delta s_1(x)}\right|_{\frac{r_1(x)}{r_1(x)-s_1(x)}}\left||u|^{q(x)-\delta}\right|_{s_1^{'}(x)}. \end{aligned} \tag{3.13}$$

Taking into consideration Proposition 2.6, we write

$$\left||u|^{\delta s_1(x)}\right|^{\frac{1}{s_1^-}}_{\frac{r_1(x)}{r_1(x)-s_1(x)}} \leq |u|^{\frac{\delta s_1^+}{s_1^-}}_{\delta t_1(x)} + |u|^{\delta}_{\delta t_1(x)},$$

$$\left||u|^{q(x)-\delta}\right|_{s_1^{'}} \leq |u|^{q^+-\delta}_{(q(x)-\delta)s_1^{'}(x)} + |u|^{q^--\delta}_{(q(x)-\delta)s_1^{'}(x)}$$

and

$$\left||V_1|^{s_1(x)}\right|^{\frac{1}{s_1^-}}_{\frac{r_1(x)}{s_1(x)}} \leq |V_1|^{\nu_1}_{r_1(x)}$$

with

$$\nu_1 = \begin{cases} \frac{s_1^+}{s_1^-} & \text{if } |V_1|_{r_1(x)} > 1, \\ 1 & \text{if } |V_1|_{r_1(x)} \leq 1. \end{cases}$$

Therefore, we replace the above inequalities into (3.12) and then by Young's inequality, it follows

$$\begin{aligned} |J(u)| &\leq \frac{4}{q^-}|V_1|^{\nu_1}_{r_1(x)}\left(|u|^{\delta\frac{s_1^+}{s_1^-}}_{\delta t_1(x)} + |u|^{\delta}_{\delta t_1(x)}\right)\left(|u|^{q^+-\delta}_{(q(x)-\delta)s_1^{'}(x)} + |u|^{q^--\delta}_{(q(x)-\delta)s_1^{'}(x)}\right) \\ &\leq \frac{4}{q^-}|V_1|^{j}_{r_1(x)}\left(|u|^{2\delta\frac{s_1^+}{s_1^-}}_{\delta t_1(x)} + |u|^{2\delta}_{\delta t_1(x)} + |u|^{2(q^+-\delta)}_{(q(x)-\delta)s_1^{'}(x)} + |u|^{2(q^--\delta)}_{(q(x)-\delta)s_1^{'}(x)}\right). \end{aligned} \tag{3.14}$$

From (3.11), we infer by Theorem 2.5 that X is continuously embedded in both $L^{\delta\left(\frac{r_1(x)}{s_1(x)}\right)'}(\Omega)$ and $L^{(q(x)-\delta)s_1'(x)}(\Omega)$. Then, there exists positive constant c_1 such that

$$|J(u)| \leq \frac{4c_1}{q^-}|V_1|^{\nu}_{r_1(x)}\left(\|u\|^{2\delta\frac{s_1^+}{s_1^-}} + \|u\|^{2\delta}+\|u\|^{2(q^+-\delta)} + \|u\|^{2(q^--\delta)}\right) \tag{3.15}$$

Therefore, we get

$$\frac{I(u)}{J(u)} \geq \frac{q^-\|u\|^{p^-}}{4c_1p^+|V_1|^{\nu}_{r_1(x)}\left(\|u\|^{2\delta\frac{s_1^+}{s_1^-}} + \|u\|^{2\delta}+\|u\|^{2(q^+-\delta)} + \|u\|^{2(q^--\delta)}\right)}.$$

Combining (3.8) and (3.10), we conclude $p^- > 2(q^+ - \delta) > 2(q^- - \delta) > 2\delta\frac{s_1^+}{s_1^-} > 2\delta$. Hence, passing to the limit as $\|u\| \longrightarrow \infty$ in the above inequality, we conclude that relation (3.4) remains valid. □

The main results of this work are presented as follows.

Theorem 3.3. *Suppose $V_1 > 0$ on Ω. Assume that $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_2)$ are verified and satisfy* (3.2)*. Then, we have*

(i) $0 < \lambda_* \leq \lambda^*$,

(ii) λ^* *is an eigenvalue of Problem* (1.1),

(iii) *For each $\lambda > \lambda^*$ is an eigenvalue of Problem* (1.1) *while any $\lambda < \lambda^*$ is not an eigenvalue.*

Proof. (i) We want to show that $\lambda_* \geq 0$ and $\frac{q^-}{p^+}\lambda_* \leq \lambda^* \leq \frac{q^+}{p^-}\lambda_*$. Therefore, $\lambda_* \leq \lambda^*$ since $p^+ < q^-$. We use reasoning by absurdity and we suppose that $\lambda_* = 0$, so $\lambda^* = 0$. Let's consider $\{u_n\}$ a sequence in $X\backslash\{0\}$ such that

$$\lim_n \frac{I(u_n)}{J(u_n)} = 0.$$

As in (3.7), we obtain

$$\frac{I(u_n)}{J(u_n)} \geq C\|u_n\|^{p^+-q^-},$$

for some positive constant C. Since $p^+ < q^-$, we have $\|u_n\| \to \infty$. And we deduce from (3.3) that

$$\lim_n \frac{I(u_n)}{J(v_n)} = \infty,$$

which is a contradiction with the hypothesis.

(ii) Let $\{u_n\} \subset X\backslash\{0\}$ be a minimizing sequence for λ^*, that is,

$$\lim_n \frac{I(u_n)}{J(u_n)} = \lambda^*. \tag{3.16}$$

From (3.4), $\{u_n\}$ is bounded in X which is reflexive. Therefore, there exists $u \in X$ such that $u_n \rightharpoonup u$ in X. This together with Proposition 2.8 gives that

$$I(u_n) \to I(u) \tag{3.17}$$

and

$$\liminf I(u_n) \geq I(u). \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we get that if $u \neq 0$,

$$\frac{I(u)}{J(u)} = \lambda^*.$$

We try to show that u is non-trivial. Through using the reasoning by absurd and suppose that $u = 0$. Hence, $\lim I(u_n) = 0$ and so, by (3.16), we deduce

$$\lim I(u_n) = \lim \frac{I(u_n)}{J(u_n)} J(u_n) = 0.$$

From the above equation and Proposition 2.4 involves that $\|u_n\| \to 0$. According to (3.4), we get

$$\lim \frac{I(u_n)}{J(u_n)} = \infty,$$

which is a contradiction. As a consequence, $u \neq 0$.

(iii) Assume that $\lambda > \lambda^*$ is fixed and let $u \in X$ with $\|u\| > 1$. It follows from inequality (3.15) that

$$L_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda K_1 \left(\|u\|^{2\delta\frac{s^+}{s^-}} + \|u\|^{2\delta} + \|u\|^{2(q^+-\delta)} + \|u\|^{2(q^--\delta)} \right),$$

where $K_1 = \frac{4c_1}{q^-}|V|^{\nu}_{r_1(x)}$. As $p^- > 2(q^+ - \delta) > 2(q^- - \delta) > 2\delta\frac{s_1^+}{s_1^-}$, the inequality above involves that $L_\lambda(u) \to \infty$ as $\|u\| \to \infty$, that is, L_λ is coercive. Moreover, it results from Proposition 2.8 that the functional L_λ is weakly lower semi-continuous. As result we conclude from [[10], Proposition 1.2, Chapter 32], that there exists a global minimizer u_0 of L_λ in X. Since $\lambda > \lambda^*$, by definition of λ^* we verify that there is an element $v \in X\backslash\{0\}$ such that $\frac{I(u)}{J(u)} < \lambda$. Hence, $L_\lambda(v) < 0$ which ensures that

$$L_\lambda(u_0) = \inf_{u\in X\backslash\{0\}} L_\lambda(u) < 0.$$

Therefore, we deduce that $u_0 \neq 0$.

Now, suppose by contradiction that there exists $\lambda \in (0, \lambda^*)$ an eigenvalue of Problem (1.1). Therefore, there exists $u_\lambda \in X \backslash \{0\}$ such that

$$\langle I^{'}(u_\lambda), v\rangle = \lambda \langle J^{'}(u_\lambda), v\rangle, \ \ \forall v \in X.$$

In particular, for $v = u_\lambda$, we have

$$I(u_\lambda) = \lambda J(u_\lambda).$$

As $u_\lambda \neq 0$, we have $J(u_\lambda) > 0$. This, together with the fact $\lambda < \lambda_*$ gives

$$I(u_\lambda) > \lambda_* J(u_\lambda) > \lambda J(u_\lambda) = I(u_\lambda),$$

which is a contradiction. The proof has been completed. □

In the situation when V_1 is a sign-changing function, we define

$$X_1^+ = \big\{u \in X : \int_\Omega V_1(x)|u|^{q(x)}dx > 0\big\}$$

and

$$X_1^- = \big\{u \in X : \int_\Omega V_1(x)|u|^{q(x)} < 0\big\}.$$

And also, we define

$$\alpha^* = \inf_{u\in X^+} \frac{I(u)}{J(u)}, \quad \alpha_* = \inf_{u\in X^+} \frac{F(u)}{G(u)}, \tag{3.19}$$

$$\beta^* = \inf_{u\in X^-} \frac{I(u)}{J(u)}, \quad \beta_* = \inf_{u\in X^-} \frac{F(u)}{G(u)}. \tag{3.20}$$

Theorem 3.4. *Suppose that* ($\boldsymbol{H}_1$) *and* ($\boldsymbol{H}_2$) *are verified and*

$$\Big|\big\{x \in \Omega : \ V_1(x) > 0\big\}\Big| \neq 0 \tag{3.21}$$

are hold. Then, we get

(i) $\beta^* \leq \beta_* < 0 < \alpha_* \leq \alpha^*$,
(ii) α^* *(resp.* β^**) is a positive (resp. negative) eigenvalue of Problem* (1.1),
(iii) *any* $\lambda \in (-\infty, \beta^*) \cup (\alpha^*, \infty)$ *is an eigenvalue of Problem* (1.1) *while* $\lambda \in (\beta_*, \alpha^*)$ *is not an eigenvalue.*

Proof. Precise that if $\lambda > 0$ is an eigenvalue of Problem 1.1 with weight V_1, hence, $-\lambda$ is an eigenvalue of Problem 1.1 with weight V_1. Then, it is enough to show Theorem 3.3 only for $\lambda > 0$. Then, the Problem 1.1 has only to be considered in X^+ and in this situation, the same demonstration to that of Theorem 3.3 and thus it will be neglected here. □

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