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DISCUSSION ON b-METRIC SPACES AND RELATED RESULTS IN METRIC AND G-METRIC SPACES

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Abstract. In the present manuscript, we employ the concepts of Θ-map and Φ-map to define a strong $(\theta, \phi)s$-contraction of a map f in a b-metric space (M, d_b). Then we prove and derive many fixed point theorems as well as we provide an example to support our main result. Moreover, we utilize our results to obtain many results in the settings of metric and G-metric spaces. Our results improve and modify many results in the literature.

1. Introduction

Let f be a self-map on a nonempty set M. $\iota \in M$ is said to be a fixed point of f if $f\iota = \iota$. If (M, d) is a metric space, f is called a contraction if there is a real number $\varpi \in [0, 1)$ such that for all $\iota, \iota^* \in M$ we have

$$d(f\iota, f\iota^*) \leq \varpi d(\iota, \iota^*)$$

and f is called a Kannan contraction if there is $r \in [0, \frac{1}{2})$ such that for all $\iota, \iota^* \in M$ we have

$$d(f\iota, f\iota^*) \leq r[d(\iota, f\iota) + d(\iota^*, f\iota^*)].$$

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The main result in the fixed point theory in distance spaces is the Banach contraction principle [7] which asserts the existence and uniqueness of fixed point for every contraction in a complete metric. The Banach contraction theorem has been modified and generalized in many directions for example see [1, 2, 4, 8, 9, 10, 11, 12, 13, 25, 26, 27, 28, 29, 30, 31].

Kannan [17] proved that every Kannan-type contraction has a unique fixed point in a complete metric space. It is worth to mention that Kannan's theorem is a significant result in analysis because it characterizes metric completeness. In the past decade Mustafa and Sims [23] introduced the concept of G-metric spaces and studied some results in fixed point field. Then after, many researchers proved several results concerning fixed point through G-metric spaces, for example see [3, 5, 15, 18, 19, 20, 21, 22].

2. Preliminary

Definition 2.1. ([6]) A function $d_b : D\times D \to [0,+\infty)$ is said to be a b-metric if there is $s \in [1,+\infty)$ such that d_b satisfying:

(d_1) $d_b(\iota_1, \iota_2) = 0 \ \ iff \ \ \iota_1 = \iota_2$,
(d_2) $d_b(\iota_1, \iota_2) = d_b(\iota_2, \iota_1), \ \forall \iota_1, \iota_2 \in D$,
(d_3) $d_b(\iota_1, \iota_2) \leq s[d_b(\iota_1, \iota_3) + d_b(\iota_3, \iota_2)], \ \forall \ \iota_1, \iota_2, \iota_3 \in D$.

The pair (M, d_b) is called a b-metric space.

Note that whenever $s = 1$, then (D, d_b) is a metric space. Hence forth, (D, d_b) stands for a b-metric spaces with base s and (D, d) stands for a metric space. If f is a self-mapping on D, and $\iota_0 \in D$, then the sequence $\{\iota_n\}$, where $\iota_n = f\iota_{n-1}$, for $n \in N$ is called the Picard sequence generated by f at ι_0. Also, we refer by F_f the set of all fixed pints of f in D.

Definition 2.2. ([33]) Let Θ denotes the set of all continuous functions $\theta : (0,+\infty) \to (1,+\infty)$ that meets the following conditions:

(Θ_1) θ is nondecreasing,
(Θ_2) for each sequence $\{\iota_n\}$ in $(0,+\infty)$, $\lim_{p\to+\infty} \theta(\iota_p) = 1$ iff $\lim_{p\to+\infty} \iota_p = 0$.

Definition 2.3. ([33]) Let Φ denotes the set of all continuous functions $\phi : [1,+\infty) \to [1,+\infty)$ that meets the following conditions:

(Φ_1) ϕ is nondecreasing,
(Φ_2) for each $\iota > 1$, $\lim_{p\to+\infty} \phi^p(\iota) = 1$.

Remark 2.4. ([33]) If $\phi \in \Phi$, then $\phi(1) = 1$, and also $\phi(\iota) < \iota$ for each $\iota > 1$.

To facilitate our subsequent argument we call a function θ as a Θ-map if $\theta \in \Theta$ and a function ϕ as a Φ-map if $\phi \in \Phi$.

3. Fixed point in b-metric setting

Definition 3.1. Suppose f is a self-mapping on (M, d_b). Then, f is said to be a strong $(\theta, \phi)s$-contraction if there are a Θ-map θ and a Φ-map ϕ such that for all $\iota_1, \iota_2 \in M$

$$d_b(f\iota_1, f\iota_2) \neq 0 \Rightarrow \theta s d_b(f\iota_1, f\iota_2) \leq \phi\theta\Lambda(\iota_1, \iota_2), \tag{3.1}$$

where

$$\begin{aligned}\Lambda(\iota_1, \iota_2) = \max\{&d_b(\iota_1, \iota_2), d_b(\iota_1, f\iota_1), d_b(\iota_2, f\iota_2),\\ &\frac{1}{2s}d_b(\iota_1, f\iota_2), \frac{1}{2s}d_b(\iota_2, f\iota_1),\\ &\frac{1}{3s}[d_b(\iota_1, f\iota_2) + d_b(\iota_2, f\iota_1) + d_b(\iota_1, f\iota_1)],\\ &\frac{1}{3s}[d_b(\iota_1, f\iota_2) + d_b(\iota_2, f\iota_1) + d_b(\iota_2, f\iota_2)]\}.\end{aligned} \tag{3.2}$$

Lemma 3.2. *Suppose f is a strong $(\theta, \phi)s$-contraction on (M, d_b). Then, for the Picard sequence $\{\iota_n\}$ with start point $\iota_0 \in M$, if $\iota_n \neq \iota_{n+1}$ for each $n \in \mathbb{N}$, then*

$$\lim_{n\to+\infty} d(\iota_n, \iota_{n+1}) = 0. \tag{3.3}$$

Proof. Let $\iota_0 \in M$ be arbitrary and consider the Picard sequence $\{\iota_n\}$ which starts at ι_0. If $\iota_n \neq \iota_{n+1}$ for all $n \geq 0$, then, $d_b(\iota_n, \iota_{n+1}) \neq 0$. So, by (3.1), we have

$$\begin{aligned}\theta s d_b(\iota_n, \iota_{n+1}) &= \theta s d_b(f\iota_{n-1}, f\iota_n)\\ &\leq \phi\theta\Lambda(\iota_{n-1}, \iota_n),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\Lambda(\iota_{n-1}, \iota_n) = \max\{&d_b(\iota_{n-1}, \iota_n), d_b(\iota_{n-1}, \iota_n), d_b(\iota_n, \iota_{n+1}),\\ &\frac{1}{2s}d_b(\iota_{n-1}, \iota_{n+1}), \frac{1}{2s}d_b(\iota_n, \iota_n),\\ &\frac{1}{3s}[d_b(\iota_{n-1}, \iota_{n+1}) + d_b(\iota_n, \iota_n) + d_b(\iota_{n-1}, \iota_n)],\\ &\frac{1}{3s}[d_b(\iota_{n-1}, \iota_{n+1}) + d_b(\iota_n, \iota_n) + d_b(\iota_n, \iota_{n+1})]\}\\ = \max\{&d_b(\iota_{n-1}, \iota_n), d_b(\iota_n, \iota_{n+1}), \frac{1}{2s}d_b(\iota_{n-1}, \iota_{n+1}),\\ &\frac{1}{3s}[d_b(\iota_{n-1}, \iota_{n+1}) + d_b(\iota_{n-1}, \iota_n)],\\ &\frac{1}{3s}[d(\iota_{n-1}, \iota_{n+1}) + d(\iota_n, \iota_{n+1})]\}.\end{aligned}$$

Now, we have to discuss the following cases:
Case (1): If $\Lambda(\iota_{n-1},\iota_n)=d_b(\iota_n,\iota_{n+1})$, then we have

$$\theta s d_b(\iota_n,\iota_{n+1})\le \phi\theta d_b(\iota_n,\iota_{n+1})<\theta d_b(\iota_n,\iota_{n+1}),$$

which is a contradiction.
Case (2): If $\Lambda(\iota_{n-1},\iota_n)=d_b(\iota_{n-1},\iota_n)$, then we have

$$\theta s d_b(\iota_n,\iota_{n+1})\le \phi\theta d_b(\iota_{n-1},\iota_n)<\theta d_b(\iota_{n-1},\iota_n).$$

So,

$$d_b(\iota_n,\iota_{n+1})<\frac{1}{s}d_b(\iota_{n-1},\iota_n). \tag{3.5}$$

Case (3): If $\Lambda(\iota_{n-1},\iota_n)=\frac{1}{2s}d_b(\iota_{n-1},\iota_{n+1})$, then we have

$$\theta s d_b(\iota_n,\iota_{n+1})\le \phi\theta\frac{1}{2s}d_b(\iota_{n-1},\iota_{n+1})<\theta\frac{1}{2s}d_b(\iota_{n-1},\iota_{n+1}).$$

So,

$$s d_b(\iota_n,\iota_{n+1})<\frac{1}{2s}d_b(\iota_{n-1},\iota_{n+1})\le\frac{1}{2}[d_b(\iota_{n-1},\iota_n)+d_b(\iota_n,\iota_{n+1})].$$

Thus,

$$d_b(\iota_n,\iota_{n+1})<\frac{1}{2s-1}d_b(\iota_{n-1},\iota_n). \tag{3.6}$$

Case (4): If $\Lambda(\iota_{n-1},\iota_n)=\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_{n-1},\iota_n)]$, then we have

$$\begin{aligned}\theta s d_b(\iota_n,\iota_{n+1})&\le \phi\theta\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_{n-1},\iota_n)]\\&<\theta\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_{n-1},\iota_n)].\end{aligned}$$

So, we obtain

$$\begin{aligned}s d_b(\iota_n,\iota_{n+1})&<\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_{n-1},\iota_n)]\\&\le\frac{1}{3s}[s[d_b(\iota_{n-1},\iota_n)+d_b(\iota_n,\iota_{n+1})]+d_b(\iota_{n-1},\iota_n)].\end{aligned}$$

Thus,

$$d_b(\iota_n,\iota_{n+1})<\frac{s+1}{s(3s-1)}d_b(\iota_{n-1},\iota_n). \tag{3.7}$$

Case (5): If $\Lambda(\iota_{n-1},\iota_n)=\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_n,\iota_{n+1})]$, then we have

$$\begin{aligned}\theta s d_b(\iota_n,\iota_{n+1})&\le \phi\theta\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_n,\iota_{n+1})]\\&<\theta\frac{1}{3s}[d_b(\iota_{n-1},\iota_{n+1})+d_b(\iota_n,\iota_{n+1})].\end{aligned}$$

So,

$$sd_b(\iota_n, \iota_{n+1}) < \frac{1}{3s}[d_b(\iota_{n-1}, \iota_{n+1}) + d_b(\iota_n, \iota_{n+1})]$$
$$\leq \frac{1}{3s}[s[d_b(\iota_{n-1}, \iota_n) + d_b(\iota_n, \iota_{n+1})] + d_b(\iota_n, \iota_{n+1})].$$

Thus,

$$d_b(\iota_n, \iota_{n+1}) < \frac{s}{3s^2 - s - 1} d_b(\iota_{n-1}, \iota_n). \tag{3.8}$$

Hence, in all cases, we deduce that

$$d_b(\iota_n, \iota_{n+1}) < d_b(\iota_{n-1}, \iota_n),$$

and so $(d_b(\iota_n, \iota_{n+1}) : n \in N)$ is a nonincreasing sequence in $[0, +\infty)$. Thus, there is $r \geq 0$ such that $\lim_{n\to+\infty} d_b(\iota_n, \iota_{n+1}) = r$.

We claim that $r = 0$. If $r \neq 0$, then by taking the limit as $n \to +\infty$ in (3.4), we get

$$\theta sr \leq \phi\theta \max\left\{r, r, r, \frac{2s+1}{3s} r, \frac{2s+1}{3s} r\right\} = \phi\theta r < \theta r,$$

which is a contradiction. So, we have

$$\lim_{n\to+\infty} d_b(\iota_n, \iota_{n+1}) = 0. \tag{3.9}$$

□

Now, we are in a position to introduce our main result.

Theorem 3.3. *Suppose (M, d_b) is complete and f is a self-mapping on M. Suppose that θ is a Θ-map and ϕ is a Φ-map such that f is a strong $(\theta, \phi)s$-contraction. Then, F_f has exactly an element.*

Proof. Let $\iota_0 \in M$ be arbitrary and let $\{\iota_n\}$ be the Picard sequence generated by f at ι_0. If there is some $n_0 \in \mathbb{N}$ such that $\iota_{n_0} = \iota_{n_0+1}$, then ι_{n_0} is the fixed point of f. So, assume that for all $n \in N$ $\iota_n \neq \iota_{n+1}$. Therefore, by Lemma 3.2, we have $\lim_{n\to+\infty} d(\iota_n, \iota_{n+1}) = 0$.

Now, we show that $\{\iota_n\}$ is a Cauchy sequence. Suppose $\{\iota_n\}$ is not a Cauchy sequence, there is $\epsilon > 0$ and two subsequences $\{\iota_{n_k}\}$ and $\{\iota_{m_k}\}$ of $\{\iota_n\}$ such that n_k is chosen as the smallest index corresponding to m_k for which

$$d_b(\iota_{n_k}, \iota_{m_k}) \geq \epsilon, \; k < m_k < n_k. \tag{3.10}$$

This implies that

$$d_b(\iota_{n_k-1}, \iota_{m_k}) < \epsilon. \tag{3.11}$$

Now, by the triangle inequality and (3.11), we get

$$d_b(\iota_{n_k-1}, \iota_{m_k-1}) \leq s[d_b(\iota_{n_k-1}, \iota_{m_k}) + d_b(\iota_{m_k}, \iota_{m_k-1})] < s[\epsilon + d_b(\iota_{m_k}, \iota_{m_k-1})].$$

By taking the limit superior as $k \to +\infty$ and using Lemma 3.2, we get

$$\limsup_{k\to+\infty} d(\iota_{n_k-1}, \iota_{m_k-1}) \leq s\epsilon.$$

Also,

$$\begin{aligned}
d_b(\iota_{m_k-1}, \iota_{n_k}) &\leq s[d_b(\iota_{m_k-1}, \iota_{m_k}) + d_b(\iota_{m_k}, \iota_{n_k})] \\
&\leq s[d_b(\iota_{m_k-1}, \iota_{m_k}) + s[d_b(\iota_{m_k}, \iota_{n_k-1}) + d_b(\iota_{n_k-1}, \iota_{n_k})]] \\
&< s[d_b(\iota_{m_k-1}, \iota_{m_k}) + s[\epsilon + d_b(\iota_{n_k-1}, \iota_{n_k})]].
\end{aligned}$$

By taking the limit superior as $k \to +\infty$ and using Lemma 3.2, we get

$$\limsup_{k\to+\infty} d_b(\iota_{m_k-1}, \iota_{n_k}) \leq s^2\epsilon.$$

Now, standing on the above argument by substituting $\iota_1 = \iota_{n_k-1}$ and $\iota_2 = \iota_{m_k-1}$ in (3.1) we get

$$\begin{aligned}
\theta s\epsilon &\leq \theta s d_b(\iota_{n_k}, \iota_{m_k}) \\
&= \theta s d_b(f\iota_{n_k-1}, f\iota_{m_k-1}) \\
&\leq \phi\theta\Lambda(\iota_{n_k-1}, \iota_{m_k-1}),
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\Lambda(\iota_{n_k-1}, \iota_{m_k-1}) = \max\{&d_b(\iota_{n_k-1}, \iota_{m_k-1}), d_b(\iota_{n_k-1}, \iota_{n_k}), d_b(\iota_{m_k-1}, \iota_{m_k}), \\
&\frac{1}{2s}d_b(\iota_{n_k-1}, \iota_{m_k}), \frac{1}{2s}d_b(\iota_{m_k-1}, \iota_{n_k}), \\
&\frac{1}{3s}[d_b(\iota_{n_k-1}, \iota_{m_k}) + d_b(\iota_{m_k-1}, \iota_{n_k}) + d_b(\iota_{n_k-1}, \iota_{n_k})], \\
&\frac{1}{3s}[d_b(\iota_{n_k-1}, \iota_{m_k}) + d_b(\iota_{m_k-1}, \iota_{n_k}) + d_b(\iota_{m_k-1}, \iota_{m_k})]\} \\
< \max\{&d_b(\iota_{n_k-1}, \iota_{m_k-1}), d_b(\iota_{n_k-1}, \iota_{n_k}), d_b(\iota_{m_k-1}, \iota_{m_k}), \\
&\frac{\epsilon}{2s}, \frac{1}{2s}d_b(\iota_{m_k-1}, \iota_{n_k}), \\
&\frac{1}{3s}[\epsilon + d_b(\iota_{m_k-1}, \iota_{n_k}) + d_b(\iota_{n_k-1}, \iota_{n_k})], \\
&\frac{1}{3s}[\epsilon + d_b(\iota_{m_k-1}, \iota_{n_k}) + d_b(\iota_{m_k-1}, \iota_{m_k})]\}.
\end{aligned} \tag{3.13}$$

Hence, we have

$$\limsup_{k\to+\infty} \Lambda(\iota_{n_k-1}, \iota_{m_k-1}) \leq \max\left\{s\epsilon, 0, 0, \frac{\epsilon}{2s}, \frac{s\epsilon}{2}, \frac{\epsilon(s^2+1)}{3s}, \frac{\epsilon(s^2+1)}{3s}\right\} = s\epsilon.$$

By taking the limit superior as $k \to +\infty$ in (3.12) , we get

$$\theta s\epsilon \leq \phi\theta s\epsilon < \theta s\epsilon,$$

which is a contradiction. Therefore $\{\iota_n\}$ is a Cauchy sequence so there is $\iota' \in M$ such that $\lim_{n\to+\infty} d_b(\iota_n, \iota') = 0$.

Now, we will show that $f\iota' = \iota'$. To see this, we assume that $f\iota' \neq \iota'$. Now, by (3.1) we have

$$\begin{aligned}\theta s d_b(\iota_{n+1}, f\iota') &= \theta s d(f\iota_n, f\iota')\\ &\leq \phi\theta\Lambda(\iota_n, \iota'),\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\Lambda(\iota_n, \iota') = \max\{&d_b(\iota_n, \iota'), d_b(\iota_n, \iota_{n+1}), d_b(\iota', f\iota'),\\ &\frac{1}{2s}d_b(\iota_n, f\iota'), \frac{1}{2s}d_b(\iota', \iota_{n+1}),\\ &\frac{1}{3s}[d_b(\iota_n, f\iota') + d_b(\iota', \iota_{n+1}) + d_b(\iota_n, \iota_{n+1})],\\ &\frac{1}{3s}[d_b(\iota_n, f\iota') + d_b(\iota', \iota_{n+1}) + d_b(\iota', f\iota')]\}\\ \leq \max\{&d(\iota_n, \iota'), d_b(\iota_n, \iota_{n+1}), d_b(\iota', f\iota'),\\ &\frac{1}{2}[d_b(\iota_n, \iota') + d_b(\iota', f\iota')], \frac{1}{2s}d_b(\iota', \iota_{n+1}),\\ &\frac{1}{3s}[s[d_b(\iota_n, \iota') + d_b(\iota', f\iota')] + d_b(\iota', \iota_{n+1}) + d_b(\iota_n, \iota_{n+1})],\\ &\frac{1}{3s}[s[d_b(\iota_n, \iota') + d_b(\iota', f\iota')] + d_b(\iota', \iota_{n+1}) + d_b(\iota', f\iota')]\}.\end{aligned} \tag{3.15}$$

By using Lemma 3.2, we get

$$\begin{aligned}&\lim_{n\to+\infty} \Lambda(\iota_n, \iota')\\ &\leq \max\left\{0, 0, d_b(\iota', f\iota'), \frac{1}{2}d_b(\iota', f\iota'), 0, \frac{1}{3}d_b(\iota', f\iota'), \frac{s+1}{3s}d_b(\iota', f\iota')\right\}\\ &= d_b(\iota', f\iota').\end{aligned}$$

Thus, by taking the limit as $n \to +\infty$ in (3.14), we get

$$\theta s d_b(\iota', f\iota') \leq \phi\theta d_b(\iota', f\iota') < \theta d_b(\iota', f\iota'),$$

which is a contradiction. Hence $\iota' = f\iota'$.

Now, assume that there is $\iota'' \in M$ such that $f\iota'' = \iota''$. If $\iota' \neq \iota''$, then by (3.1) we get

$$\begin{aligned}\theta s d_b(\iota', \iota'') &= \theta s d_b(f\iota', f\iota'')\\ &\leq \phi\theta\Lambda(\iota', \iota''),\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\Lambda(\iota', \iota'') = \max\{\ & d_b(\iota', \iota''), d_b(\iota', \iota'), d_b(\iota'', \iota''), \\ & \frac{1}{2s} d_b(\iota', \iota''), \frac{1}{2s} d_b(\iota'', \iota'), \\ & \frac{1}{3s}[d_b(\iota', \iota'') + d_b(\iota'', \iota') + d_b(\iota', \iota')], \\ & \frac{1}{3s}[d_b(\iota', \iota'') + d_b(\iota'', \iota') + d_b(\iota'', \iota'')]\ \} \\ = d_b(\iota', \iota''). &\end{aligned} \tag{3.17}$$

Thus,

$$\theta s d_b(\iota', \iota'') \leq \phi\theta d_b(\iota', \iota'') < \theta d_b(\iota', \iota''),$$

which is a contradiction and so $\iota' = \iota''$. this completes the proof. □

4. Consequence results in metric and G-metric spaces

4.1. Fixed point in metric setting.

Definition 4.1. Suppose f is a self-mapping on (M, d). Then, f is said to be a strong (θ, ϕ)-contraction if there exist a Θ-map θ and a Φ-map ϕ such that for all $\iota_1, \iota_2 \in M$

$$d(f\iota_1, f\iota_2) \neq 0 \quad \Rightarrow \quad \theta d(f\iota_1, f\iota_2) \leq \phi\theta\Lambda(\iota_1, \iota_2), \tag{4.1}$$

where

$$\begin{aligned}\Lambda(\iota_1, \iota_2) = \max\{ & d(\iota_1, \iota_2), d(\iota_1, f\iota_1), d(\iota_2, f\iota_2), \frac{1}{2} d(\iota_1, f\iota_2), \frac{1}{2} d(\iota_2, f\iota_1), \\ & \frac{1}{3}[d(\iota_1, f\iota_2) + d(\iota_2, f\iota_1) + d(\iota_1, f\iota_1)], \\ & \frac{1}{3}[d(\iota_1, f\iota_2) + d(\iota_2, f\iota_1) + d(\iota_2, f\iota_2)]\}.\end{aligned} \tag{4.2}$$

A consequence result of Theorem 3.3 in the setting of metric spaces is the following theorem.

Theorem 4.2. *Suppose (M, d) is complete and f is a self mapping on (M, d). Suppose that θ is a Θ-map and ϕ is a Φ-map such that f is a strong (θ, ϕ)-contraction. Then, F_f has exactly one element.*

Now, we give an example to illustrate Theorem 4.2.

Example 4.3. Let $M = \{0, 1, 2, \cdots\}$ and $k \in (0, 1)$. Let $d : M \times M \to [0, +\infty)$ be defined by $d(\iota_1, \iota_2) = |\iota_1 - \iota_2|$, $\theta : (0, +\infty) \to (1, +\infty)$, $\phi : [1, +\infty) \to [1, +\infty)$ by $\theta(\iota) = e^{\iota}$, $\phi(\iota) = \iota^k$ respectively, and $f : M \to M$

by

$$f(n) = \begin{cases} 0, & n \in \{0,1,2\}, \\ 1, & n \geq 3. \end{cases}$$

Then

(1) (M,d) is a complete metric space,
(2) ϕ is a Φ-map and θ is a Θ-map,
(3) f is neither Banach contraction nor Kannan contraction,
(4) f is a strong (θ,ϕ)-contraction.

Proof. We show (3) and (4).

(3) Note that if $\iota_1 = 2$ and $\iota_2 = 3$, then $d(f\iota_1, f\iota_2) = 1 = d(\iota_1, \iota_2)$. Also, if $\iota_1 = 0$ and $\iota_2 = 3$, then

$$d(f\iota_1, f\iota_2) = 1 = \frac{1}{2}[d(\iota_1, f\iota_1) + d(\iota_2, f\iota_2)].$$

Hence, f is neither Banach contraction nor Kannan contraction.

(4) Since $d(f\iota_1, f\iota_2) \neq 0$, then $d(f\iota_1, f\iota_2) = 1$. So, if $\iota_1 \in \{0,1,2\}$, then it must be that $\iota_2 \geq 3$.

Now, $d(\iota_1,\iota_2) = \iota_2 - \iota_1$, $d(\iota_1, f\iota_1) = \iota_1$, $d(\iota_2, f\iota_2) = \iota_2 - 1$, $d(\iota_1, f\iota_2) = |\iota_1 - 1|$ and $d(\iota_2, f\iota_1) = \iota_1$. Therefore,

$$\begin{aligned} &\Lambda(\iota_1,\iota_2) \\ &= \max\left\{\iota_2 - \iota_1, \iota_2 - 1, \iota_1, \frac{|\iota_1 - 1|}{2}, \frac{\iota_2}{2}, \frac{\iota_2 + \iota_1 + |\iota_1 - 1|}{3}, \frac{2\iota_2 - 1 + |\iota_1 - 1|}{3}\right\}. \end{aligned}$$

If $\iota_1 = 0$, then $\Lambda(\iota_1,\iota_2) = \iota_2 \geq 3$.
If $\iota_1 = 1$, then $\Lambda(\iota_1,\iota_2) = \iota_2 - 1 \geq 2$.
If $\iota_1 = 2$, then $\Lambda(\iota_1,\iota_2) = \iota_2 - 1 \geq 2$.
Thus, in each case we have $\theta d(f\iota_1, f\iota_2) \leq \phi\theta\Lambda(\iota_1,\iota_2)$. Therefore, all hypothesis of Theorem 4.2 and the unique fixed point for f is 0. $\square$

Now, we introduce some results based on Theorem 4.2.

Corollary 4.4. *Suppose (M,d) is complete and f is a self-mapping on M such that for all $\iota_1, \iota_2 \in M$*

$$d(f\iota_1, f\iota_2) \neq 0 \;\;\Rightarrow\;\; d(f\iota_1, f\iota_2) \leq k\ \Lambda(\iota_1,\iota_2), \tag{4.3}$$

where $k \in (0,1)$. Then F_f has exactly one element.

Proof. Define $\theta : (0,+\infty) \to (1,+\infty)$ by $\theta(\iota) = e^{\iota}$, and $\phi : [1,+\infty) \to [1,+\infty)$ by $\phi(\iota) = \iota^k$. Then

$$d(f\iota_1, f\iota_2) \leq k\ \Lambda(\iota_1,\iota_2) \text{ if and only if } e^{d(f\iota_1,f\iota_2)} \leq e^{k\ \Lambda(\iota_1,\iota_2)} = (e^{\Lambda(\iota_1,\iota_2)})^k.$$

So, we have

$$\theta d(f\iota_1, f\iota_2) \leq \phi\theta k \ \Lambda(\iota_1, \iota_2).$$

Hence the result follows from Theorem 4.2 $\square$

Corollary 4.5. *Suppose (M, d) is complete and f is a self-mapping on M. Suppose that θ is Θ-map and ϕ is Φ-map such that for all $\iota_1, \iota_2 \in M$, we have*

$$d(f\iota_1, f\iota_2) \neq 0 \ \Rightarrow \ \theta d(f\iota_1, f\iota_2) \leq \theta\phi d(\iota_1, \iota_2). \tag{4.4}$$

Then F_f has exactly one element.

Corollary 4.6. *Suppose (M, d) is complete and f is a self-mapping on M such that for all $\iota_1, \iota_2 \in M$*

$$d(f\iota_1, f\iota_2) \neq 0 \ \Rightarrow \ d(f\iota_1, f\iota_2) \leq k \ d(\iota_1, \iota_2), \tag{4.5}$$

where $k \in (0, 1)$. Then F_f has exactly one element.

Corollary 4.7. *Suppose (M, d) is complete and f is a self-mapping on M. Assume that there exists $\lambda \in (0, 1)$ such that for all $\iota_1, \iota_2 \in M$*

$$d(f\iota_1, f\iota_2) \neq 0 \ \Rightarrow \ d(f\iota_1, f\iota_2) \leq \lambda \max\{d(\iota_1, f\iota_1), d(\iota_2, f\iota_2)\}. \tag{4.6}$$

Then F_f has exactly one element.

Corollary 4.8. *Suppose (M, d) is complete and f is a self-mapping on M. Assume that there exists $\alpha \in (0, \frac{1}{2})$ such that for all $\iota_1, \iota_2 \in M$*

$$d(f\iota_1, f\iota_2) \neq 0 \ \Rightarrow \ d(f\iota_1, f\iota_2) \leq \alpha[d(\iota_1, f\iota_1) + d(\iota_2, f\iota_2)]. \tag{4.7}$$

Then F_f has exactly one element.

4.2. **Fixed point in G-metric setting.**

Definition 4.9. ([23]) Let M be a nonempty set and let $d_G : M \times M \times M \to [0, +\infty)$ be a function satisfying:

- (G1) $d_G(\iota_1, \iota_2, \iota_3) = 0$ if $\iota_1 = \iota_2 = \iota_3$,
- (G2) $d_G(\iota_1, \iota_1, \zeta) > 0$ for all $\iota_1, \zeta \in M$ with $\iota_1 \neq \zeta$,
- (G3) $d_G(\iota_1, \iota_1, \zeta) \leq d_G(\iota_1, \zeta, \varsigma)$ for all $\iota_1, \zeta, \varsigma \in M$ with $\zeta \neq \varsigma$,
- (G4) $d_G(\iota_1, \zeta, \varsigma) = d_G(p\{\iota_1, \zeta, \varsigma\})$, where $p\{\iota_1, \zeta, \varsigma\}$ is the all possible permutations of $\iota_1, \zeta, \varsigma$ (symmetry),
- (G5) $d_G(\iota_1, \zeta, \varsigma) \leq d_G(\iota_1, a, a) + G(a, \zeta, \varsigma)$ for all $\iota_1, \zeta, \varsigma, a \in M$.

Then the function d_G is called a *generalized metric*, or a G-metric on M, and the pair (M, d_G) is called a G-metric space.

From now on, (M, d_G) stands for a G-metric space.

Definition 4.10. ([23]) Let $\{\iota_n\}$ be a sequence on (M, d_G). Then we say that $\{\iota_n\}$ is G-convergent to $\iota' \in M$ if $\lim_{n,m\to+\infty} d_G(\iota', \iota_n, \iota_m) = 0$, that is, for any $\epsilon > 0$, there exists $k \in N$ such that $d_G(\iota', \iota_n, \iota_m) < \epsilon$, for all $n, m \geq k$.

Proposition 4.11. ([23]) *The followings are equivalent in* (M, d_G)*:*

(1) $\{\iota_n\}$ *is* G*-convergent to* $\iota' \in M$*.*
(2) $d_G(\iota_n, \iota_n, \iota') \to 0 \;\; as\, n \;\to +\infty$.
(3) $d_G(\iota_n, \iota', \iota') \to 0 \;\; as\, n \;\to +\infty$.

Definition 4.12. ([23]) A sequence $\{\iota_n\}$ in (M, d_G) is said to be G-Cauchy if a given $\epsilon > 0$, there is $k \in N$ such that $d_G(\iota_n, \iota_m, \iota_l) < \epsilon$ for all $n, m, l \geq k$.

Proposition 4.13. ([23]) *The following are equivalent in* (M, d_G)*:*

(1) *The sequence* $\{\iota_n\}$ *is* G*-Cauchy.*
(2) *For every* $\epsilon > 0$*, there is* $k \in \mathbb{N}$ *such that* $d_G(\iota_n, \iota_m, \iota_m) < \epsilon$, $\forall n, m \geq k$*.*

Definition 4.14. ([23]) A G-metric space (M, d_G) is said to be G-complete or complete G-metric if every G-Cauchy sequence in (M, d_G) is G-convergent in (M, d_G).

The following theorem is a relation between G-metric spaces and metric spaces.

Theorem 4.15. ([16]) *Suppose there is* (M, d_G) *and a function* $d : M \times M \to [0, +\infty)$ *is defined by* $d(\iota_1, \iota_2) = \max\{d_G(\iota_1, \iota_2, \iota_2), d_G(\iota_2, \iota_1, \iota_1)\}$*. Also, suppose* (ι_n) *is a sequence in* M*. Then*

(1) (M, d) *is a metric space;*
(2) $\{\iota_n\}$ *is* G*-convergent to* $\iota' \in M$ *if and only if* $\{\iota_n\}$ *is convergent to* ι' *in* (M, d)*;*
(3) $\{\iota_n\}$ *is* G*-Cauchy if and only if* $\{\iota_n\}$ *is Cauchy in* (M, d)*;*
(4) (M, d_G) *is* G*-complete if and only if* (M, d) *is complete.*

Jleli and Samet [16] remarks in their clever paper that some fixed point theorems in the setting of G-metric spaces can be deduced from proven results in metric spaces in a smart way. In this section, we utilize our results to get a consequence results in the concept of G-metric spaces using the method of Jleli and Samet.

Definition 4.16. ([32]) Suppose there is (M, d_G). A mapping $f : M \to M$ is said to be a generalized (θ, ϕ)-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $\iota_1, \iota_2, \iota_3 \in M$,

$$d_G(f\iota_1, f\iota_2, f\iota_3) \neq 0 \;\; \Rightarrow \;\; \theta d_G(f\iota_1, f\iota_2, f\iota_3) \leq \phi\theta N(\iota_1, \iota_2, \iota_3), \tag{4.8}$$

where

$$\begin{aligned}
&N(\iota_1, \iota_2, \iota_3)\\
&= \max\{d_G(\iota_1, \iota_2, \iota_3), d_G(\iota_1, f\iota_1, f\iota_1), d_G(\iota_2, f\iota_2, f\iota_2),\\
&\qquad \frac{1}{2}d_G(\iota_1, f\iota_2, f\iota_2), \frac{1}{2}d_G(\iota_2, f\iota_3, f\iota_3), \frac{1}{2}d_G(\iota_3, f\iota_1, f\iota_1),\\
&\qquad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_3, f\iota_3) + d_G(\iota_3, f\iota_1, f\iota_1)]\}.
\end{aligned} \tag{4.9}$$

We are in a position to give new proofs of the following two theorems in [20, 32] by using the technique of Jleli and Samet together with our results.

Theorem 4.17. *Let (M, d_G) be complete and let $f : M \to M$ be a generalized (θ, ϕ)-contraction. Then F_f has exactly one element ι' such that the sequence $(f^n\iota)$ converges to ι' for every $\iota \in M$.*

Proof. By letting $\iota_2 = \iota_3$ in (4.9), we get

$$\begin{aligned}
&N(\iota_1, \iota_2, \iota_2)\\
&= \max\{d_G(\iota_1, \iota_2, \iota_2), d_G(\iota_1, f\iota_1, f\iota_1), d_G(\iota_2, f\iota_2, f\iota_2),\\
&\qquad \frac{1}{2}d_G(\iota_1, f\iota_2, f\iota_2), \frac{1}{2}d_G(\iota_2, f\iota_1, f\iota_1), \frac{1}{2}d_G(\iota_2, f\iota_2, f\iota_2),\\
&\qquad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1)]\}.
\end{aligned} \tag{4.10}$$

Also, by exchanging ι_1 and ι_2, we get

$$\begin{aligned}
&N(\iota_2, \iota_1, \iota_1)\\
&= \max\{d_G(\iota_2, \iota_1, \iota_1), d_G(\iota_2, f\iota_2, f\iota_2), d_G(\iota_1, f\iota_1, f\iota_1),\\
&\qquad \frac{1}{2}d_G(\iota_1, f\iota_2, f\iota_2), \frac{1}{2}d_G(\iota_2, f\iota_1, f\iota_1), \frac{1}{2}d_G(\iota_1, f\iota_1, f\iota_1),\\
&\qquad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_1, f\iota_1, f\iota_1) + d_G(\iota_2, f\iota_1, f\iota_1)]\}.
\end{aligned} \tag{4.11}$$

Define $d : M \times M \to [0, +\infty)$ by $d(\iota_1, \iota_2) = \max\{d_G(\iota_1, \iota_2, \iota_2), d_G(\iota_2, \iota_1, \iota_1)\}$. Then we have

$$\begin{aligned}
\theta d(f\iota_1, f\iota_2) &= \theta \max\{d_G(f\iota_1, f\iota_2, f\iota_2), d_G(f\iota_2, f\iota_1, f\iota_1)\}\\
&= \max\{\theta d_G(f\iota_1, f\iota_2, f\iota_2), \theta d_G(f\iota_2, f\iota_1, f\iota_1)\}\\
&\leq \phi\theta \max\{N(\iota_1, \iota_2, \iota_2), N(\iota_2, \iota_1, \iota_1)\}\\
&\leq \phi\theta\Lambda(\iota_1, \iota_2).
\end{aligned} \tag{4.12}$$

Hence, f is a strong (θ, ϕ)-contraction and so the result follows from Theorem 4.2 and Theorem 4.15. $\square$

Theorem 4.18. *Let (M, d_G) be complete and let $f : M \to M$ be a self-mapping which satisfies the following condition for all $\iota_1, \iota_2 \in M$.*

$$\begin{aligned} &d_G(f\iota_1, f\iota_2, f\iota_2) \\ &\le \max\{ad_G(\iota_1, \iota_2, \iota_2), b[d_G(\iota_1, f\iota_1, f\iota_1) + 2d_G(\iota_2, f\iota_2, f\iota_2)], \\ &\qquad b[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1) + d_G(\iota_2, f\iota_2, f\iota_2)]\}, \end{aligned} \tag{4.13}$$

where $0 \le a < 1$ and $0 \le b < \frac{1}{3}$. Then F_f has exactly one element.

Proof. From condition (4.13), we have

$$\begin{aligned} &d_G(f\iota_1, f\iota_2, f\iota_2) \\ &\le \max\{ad_G(\iota_1, \iota_2, \iota_2), b[d_G(\iota_1, f\iota_1, f\iota_1) + 2d_G(\iota_2, f\iota_2, f\iota_2)], \\ &\qquad b[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1) + d_G(\iota_2, f\iota_2, f\iota_2)]\}. \end{aligned} \tag{4.14}$$

If $k = \max\{a, 3b\}$, then

$$\begin{aligned} &d_G(f\iota_1, f\iota_2, f\iota_2) \\ &\le k \max\{d_G(\iota_1, \iota_2, \iota_2), \frac{1}{3}[d_G(\iota_1, f\iota_1, f\iota_1) + 2d_G(\iota_2, f\iota_2, f\iota_2)], \\ &\quad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1) + d_G(\iota_2, f\iota_2, f\iota_2)]\}. \end{aligned} \tag{4.15}$$

By the same argument, we have

$$\begin{aligned} &d_G(f\iota_2, f\iota_1, f\iota_1) \\ &\le k \max\{d_G(\iota_2, \iota_1, \iota_1), \frac{1}{3}[d_G(\iota_2, f\iota_2, f\iota_2) + 2d_G(\iota_1, f\iota_1, f\iota_1)], \\ &\quad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1) + d_G(\iota_1, f\iota_1, f\iota_1)]\}. \end{aligned} \tag{4.16}$$

Define $d : M \times M \to [0, +\infty)$ by $d(\iota_1, \iota_2) = \max\{d_G(\iota_1, \iota_2, \iota_2), d_G(\iota_2, \iota_1, \iota_1)\}$. Then we have

$$\begin{aligned} &d(f\iota_1, f\iota_2) \\ &= \max\{d_G(f\iota_1, f\iota_2, f\iota_2), d_G(f\iota_2, f\iota_1, f\iota_1)\} \\ &\le k \max\{d_G(\iota_1, \iota_2, \iota_2), \frac{1}{3}[d_G(\iota_1, f\iota_1, f\iota_1) + 2d_G(\iota_2, f\iota_2, f\iota_2)], \\ &\quad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1) + d_G(\iota_2, f\iota_2, f\iota_2)], \\ &\quad d_G(\iota_2, \iota_1, \iota_1), \frac{1}{3}[d_G(\iota_2, f\iota_2, f\iota_2) + 2d_G(\iota_1, f\iota_1, f\iota_1)], \\ &\quad \frac{1}{3}[d_G(\iota_1, f\iota_2, f\iota_2) + d_G(\iota_2, f\iota_1, f\iota_1) + d_G(\iota_1, f\iota_1, f\iota_1)]\} \end{aligned} \tag{4.17}$$

$$\begin{aligned}
&\le k \max\{d(\iota_1,\iota_2), \frac{1}{3}[d(\iota_1, f\iota_1) + 2d(\iota_2, f\iota_2)], \frac{1}{3}[d(\iota_2, f\iota_2) + 2d(\iota_1, f\iota_1)],\\
&\quad \frac{1}{3}[d(\iota_1, f\iota_1) + d(\iota_2, f\iota_1) + d(\iota_2, f\iota_2)],\\
&\quad \frac{1}{3}[d(\iota_1, f\iota_1) + d(\iota_2, f\iota_1) + d(\iota_1, f\iota_1)]\}\\
&\le k\Lambda(\iota_1,\iota_2).
\end{aligned}$$

Hence, the result follows from Corollary 4.4 and Theorem 4.15. □

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