



BI-UNIVALENT FUNCTIONS CONNECTED WITH THE MITTAG-LEFFLER-TYPE BOREL DISTRIBUTION BASED UPON THE LEGENDRE POLYNOMIALS

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Abstract. In this paper, we introduce new subclasses of analytic and bi-univalent functions associated with the Mittag-Leffler-type Borel distribution by using the Legendre polynomials. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses and obtain Fekete-Szegő problem for these subclasses. We also state certain new subclasses of Σ and initial coefficient estimates and Fekete-Szegő inequalities.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let \mathcal{A} denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}, \quad (1.1)$$

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and \mathcal{S} be the subclass of \mathcal{A} which are univalent functions in \mathbb{E} .

If \mathfrak{G} and \mathfrak{F} are analytic functions in \mathbb{E} , we say that \mathfrak{G} is subordinate to \mathfrak{F} , written $\mathfrak{G} \prec \mathfrak{F}$, if there exists a Schwarz function ω , which is analytic in \mathbb{E} , with $\omega(0) = 0$, and, $|\omega(z)| < 1$ for all $z \in \mathbb{E}$, such that $\mathfrak{G}(z) = \mathfrak{F}(\omega(z))$, $z \in \mathbb{E}$. Furthermore, if the function \mathfrak{F} is univalent in \mathbb{E} , then we have the following equivalence (see [6] and [24]):

$$\mathfrak{G}(z) \prec \mathfrak{F}(z) \Leftrightarrow \mathfrak{G}(0) = \mathfrak{F}(0) \text{ and } \mathfrak{G}(\mathbb{E}) \subset \mathfrak{F}(\mathbb{E}).$$

1.1. Quantum calculus. Jackson in 1909-1910 [18, 19] developed quantum calculus, popularly known as q -calculus. Since then it has found applications in physics, quantum mechanics, analytic number theory, Sobolev spaces, representation theory of groups, theta functions, gamma functions, operator theory, and more recently in geometric function theory. For the definitions and properties of q -calculus one may refer to [17, 20]. In fact, q -calculus methodology is centered on the idea of deriving q -analogues results without the use of limits. Let us first recall certain notations and definitions of the q -calculus.

Definition 1.1. Let $q \in (0, 1)$. The q -derivative (or q -difference operator) of a function f , defined on a subset Ω with $0 \in \Omega$ of \mathbb{C} , is given by

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

We note that $\lim_{q \rightarrow 1} (D_q f)(z) = f'(z)$ if f is differentiable at z .

For the function $f(z) = z^k$, we observe that

$$D_q z^k = \frac{1 - q^k}{1 - q} z^{k-1} = [k]_q z^{k-1}.$$

For a function f analytic in the open unit disc $\mathbb{E} := \{z : |z| < 1\}$, we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where

$$[k]_q := \frac{1 - q^k}{1 - q} = 1 + \sum_{j=1}^{k-1} q^j, \quad [0, q] := 0. \quad (1.2)$$

Clearly, for $q \rightarrow 1^-$, $[k]_q \rightarrow k$. For the definitions and properties of q -derivative and q -calculus, we may refer to [17, 18, 19, 20].

1.2. Mittag-Leffler function and Borel distribution: The study of operators plays an important role in geometric function theory in complex analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to better understand the geometric properties of such operators.

Let $\mathbf{E}_\alpha(z)$ and $\mathbf{E}_{\alpha,\beta}(z)$ be the function defined by

$$\mathbf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in \mathbb{C}, \Re(\alpha) > 0) \tag{1.3}$$

and

$$\mathbf{E}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0).$$

It can be written in other form:

$$\mathbf{E}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{k=2}^{\infty} \frac{z^{k-1}}{\Gamma(\alpha(k-1) + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0).$$

The function $\mathbf{E}_\alpha(z)$ was introduced by Mittag-Leffler [25] and is, therefore, known as the Mittag-Leffler function. A more general function $\mathbf{E}_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [30] and defined by

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0). \tag{1.4}$$

Observe that the function $\mathbf{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\begin{aligned} \mathbf{E}_{1,1}(z) &= e^z, & \mathbf{E}_{1,2}(z) &= \frac{e^z - 1}{z}, & \mathbf{E}_{2,1}(z^2) &= \cosh z, \\ \mathbf{E}_{2,1}(-z^2) &= \cos z, & \mathbf{E}_{2,2}(z^2) &= \frac{\sinh z}{z}, & \mathbf{E}_{2,2}(-z^2) &= \frac{\sin z}{z}, \\ \mathbf{E}_4(z) &= \frac{1}{2} \left(\cos z^{1/4} + \cosh z^{1/4} \right), \\ \mathbf{E}_3(z) &= \frac{1}{2} \left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos \left(\frac{\sqrt{3}}{2} z^{1/3} \right) \right]. \end{aligned}$$

We recall the error function \mathfrak{Erf} given by Abramowitz and Stegun[1, p. 297],

$$\mathfrak{Erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

the complement of the error function \mathfrak{Erf}_c defined by

$$\mathfrak{Erf}_c(z) := 1 - \mathfrak{Erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

and the normalized form of the error function \mathfrak{Erf} denoted by \mathfrak{Erf}^F (normalized with the condition $\mathfrak{Erf}'(0) = 1$) is given by

$$\mathfrak{Erf}^F(z) := \frac{\sqrt{\pi}z}{2} \mathfrak{Erf}(\sqrt{z}) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(2n-1)} z^n.$$

It is of interest to note that by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \cdot \mathfrak{Erf}_c(-z),$$

that is

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \left(1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1} \right).$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2, 3, 14, 15, 16, 21]. Observe that Mittag-Leffler function $\mathbf{E}_{\alpha,\beta}(z)$ does not belong to the family \mathcal{A} . Thus, it is natural to consider the following normalization of Mittag-Leffler functions as below:

$$E_{\alpha,\beta}(z) = z\Gamma(\beta)\mathbf{E}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1) + \beta)} z^k, \quad (1.5)$$

it holds for complex parameters α, β and $z \in \mathbb{C}$.

In this paper, we shall restrict our attention to the case of real-valued α, β and $z \in \mathbb{E}$.

A discrete random variable x is said to have a Borel distribution if it takes the values $1, 2, 3, \dots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \dots$, respectively, where λ is called the parameter.

Very recently, Wanas and Khuttar [29] introduced the Borel distribution (BD) whose probability mass function is

$$P(x = \rho) = \frac{(\rho\lambda)^{\rho-1} e^{-\lambda\rho}}{\rho!}, \quad \rho = 1, 2, 3, \dots$$

Wanas and Khuttar introduced a series $\mathcal{M}(\lambda; z)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{M}(\lambda; z) = z + \sum_{k=2}^{\infty} \frac{[\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)!} z^k, \quad (0 < \lambda \leq 1). \tag{1.6}$$

In [26], the authors defined the Mittag-Leffler-type Borel distribution as follows:

$$\mathcal{P}(\lambda, \alpha, \beta; \rho) = \frac{(\lambda\rho)^{\rho-1}}{E_{\alpha,\beta}(\lambda\rho) \Gamma(\alpha\rho + \beta)}, \quad \rho = 0, 1, 2, \dots,$$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$, ($\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$). Thus by using (1.5) and (1.6) and by convolution operator, we define the Mittag-Leffler-type Borel distribution series as below:

$$\mathcal{B}(\lambda, \alpha, \beta)(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda(k-1))! [\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)! E_{\alpha,\beta}(\lambda(k-1)) \Gamma(\alpha(k-1) + \beta)} z^k, \quad (0 < \lambda \leq 1).$$

Further, by the convolution operator we define

$$\begin{aligned} \mathcal{M}_{\alpha,\beta}^{\lambda} f(z) &= \mathcal{B}(\lambda, \alpha, \beta)(z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{(\lambda(k-1))! [\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)! E_{\alpha,\beta}(\lambda(k-1)) \Gamma(\alpha(k-1) + \beta)} a_k z^k \\ &= z + \sum_{k=2}^{\infty} \Upsilon_k a_k z^k, \end{aligned} \tag{1.7}$$

where $\alpha, \beta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $0 < \lambda \leq 1$ and

$$\Upsilon_k = \frac{(\lambda(k-1))! [\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)! E_{\alpha,\beta}(\lambda(k-1)) \Gamma(\alpha(k-1) + \beta)}. \tag{1.8}$$

Thus we have

$$D_q(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z)) = 1 + \sum_{k=2}^{\infty} [k]_q \frac{(\lambda(k-1))! [\lambda(k-1)]^{k-2} e^{-\lambda(k-1)}}{(k-1)! E_{\alpha,\beta}(\lambda(k-1)) \Gamma(\alpha(k-1) + \beta)} a_k z^{k-1}. \tag{1.9}$$

1.3. Legendre polynomials: Legendre polynomials, which are exceptional cases of Legendre functions, are familiarized in 1784 by the French mathematician Legendre (1752-1833). Legendre functions are a vital and important in problems including spherical coordinates. As well, the Legendre polynomials, $P_k(x)$, ($|x| < 1$), are designated via the following generating function (see

[13, 10, 22])

$$\mathcal{G}(x, z) = \frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{k=0}^{\infty} P_k(x)z^k. \quad (1.10)$$

Legendre polynomials are the everywhere regular solutions of Legendrâe differential equation that we can write as follows:

$$(1 - x^2) \frac{d^2}{dx^2} P_k(x) - 2x \frac{d}{dx} P_k(x) + m P_k(x) = 0,$$

where $m = k(k + 1)$ and $k = 0, 1, 2, \dots$. Taking $x = 1$ in (1.10) and by using geometric series, we see that $P_k(1) = 1$, so that the Legendre polynomials are normalized. Thus Let $\mathcal{G}(x, z)$ denote the class of analytic functions on Δ which are normalized by the conditions $\mathcal{G}(x, 0) = 0$ and $\mathcal{G}'(x, 0) = 1$.

Definition 1.2. Let $P_k(x)$ is Legendre polynomials of the first kind of order $k = 0, 1, 2, \dots$, the recurrence formula is

$$P_{k+1}(x) = \frac{2k + 1}{k + 1} x P_k(x) - \frac{k}{k + 1} P_{k-1}(x) \quad (1.11)$$

with

$$P_0(x) = 1 \quad \text{and} \quad P_1(x) = x.$$

The Koebe one quarter theorem (see [9]) proves that the image of \mathbb{E} under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfied

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{E})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.12)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{E} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{E} . Let Σ denote the class of bi-univalent functions in \mathbb{E} given by (1.1). For instance, the functions z , $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2} \log \frac{1+z}{1-z}$ are members of Σ . However, the Koebe function is not a member of Σ . For a brief history and interesting examples in the class Σ (see [4]). Brannan and Taha [5] (see also [7, 8, 11, 12, 28]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β ($0 \leq \beta < 1$), respectively (see [4]). Thus, following Brannan and Taha [5] a function $f \in \mathcal{A}$ is said to be in the

class $\mathcal{S}_\Sigma^*(\beta)$ of strongly bi-starlike functions of order β ($0 < \beta \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\beta\pi}{2} \quad (0 < \beta \leq 1; z \in \mathbb{E}) \tag{1.13}$$

and

$$\left| \arg \left(\frac{zg'(w)}{g(w)} \right) \right| < \frac{\beta\pi}{2} \quad (0 < \beta \leq 1; w \in \mathbb{E}), \tag{1.14}$$

where g is the extension of f^{-1} to \mathbb{E} is given by (1.12). The classes $\mathcal{S}_\Sigma^*(\beta)$ and $\mathcal{K}_\Sigma(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β ($0 < \beta \leq 1$), corresponding to the function classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $\mathcal{S}_\Sigma^*(\beta)$ and $\mathcal{K}_\Sigma(\beta)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [5] and [28]).

The object of the present paper is to introduce new classes of the function class Σ involving the q -analogue of convolution based upon the Legendre polynomials previous defined as in Definition 1.3, and find estimates on the coefficients $|a_2|$, and $|a_3|$ for functions in these new subclasses of the function class Σ and obtain Fekete-Szegő problem for these subclasses. Further by specializing the parameters η, γ we define new subclasses of Σ (and not studied so far in the literature) based on Mittag-Leffler functions associated with Borel distribution and state the coefficient estimate as corollaries in the last section.

Definition 1.3. Let $\eta \neq 0$ be a complex number and $f(z)$ given by (1.1), and $f(z) \in \mathcal{H}_\Sigma^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$ if the following conditions are satisfied:

$$1 + \frac{1}{\eta} \left(\frac{\gamma z D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right) \right) + \gamma D_q \left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right) + 1 - \gamma}{D_q \left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right)} - 1 \right) \prec \mathcal{G}(x, z) \tag{1.15}$$

and

$$1 + \frac{1}{\eta} \left(\frac{\gamma w D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right) \right) + \gamma D_q \left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right) + 1 - \gamma}{D_q \left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right)} - 1 \right) \prec \mathcal{G}(x, w) \tag{1.16}$$

with $\gamma > 0, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, 0 < \lambda \leq 1; 0 < q < 1; \eta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, where the function $g = f^{-1}$ is given by (1.12).

By specializing the parameters η, γ we define the following subclasses which are new not studied so far in the literature based on Mittag-Leffler functions associated with Borel distribution.

Remark 1.4. (i) As $q \rightarrow 1^-$, we obtain that $\lim_{q \rightarrow 1^-} \mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x) =: \mathcal{I}_{\Sigma}^{\lambda}(\eta, \gamma, \alpha, \beta, x)$, where $\mathcal{I}_{\Sigma}^{\lambda}(\eta, \gamma, \alpha, \beta, x)$ represents the functions $f \in \Sigma$ that satisfy the following conditions

$$1 + \frac{1}{\eta} \left(\frac{\gamma z \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)'' + \gamma \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)' + 1 - \gamma}{\left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)'} - 1 \right) \prec \mathcal{G}(x, z) \quad (1.17)$$

and

$$1 + \frac{1}{\eta} \left(\frac{\gamma w \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right)'' + \gamma \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right)' + 1 - \gamma}{\left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right)'} - 1 \right) \prec \mathcal{G}(x, w). \quad (1.18)$$

(ii) Fixing $\gamma = 1$, we obtain that $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta, 1, \alpha, \beta, x) =: \mathcal{K}_{\Sigma}^{q,\lambda}(\eta, \alpha, \beta, x)$, where $\mathcal{K}_{\Sigma}^{q,\lambda}(\eta, \alpha, \beta, x)$ represents the functions $f \in \Sigma$ that satisfy the following conditions

$$1 + \frac{1}{\eta} \left(\frac{z D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right) \right)}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)} \right) \prec \mathcal{G}(x, z) \quad (1.19)$$

and

$$1 + \frac{1}{\eta} \left(\frac{w D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right) \right)}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right)} \right) \prec \mathcal{G}(x, w). \quad (1.20)$$

(iii) Taking $\gamma = 1$ and $\eta = 1$, we obtain that $\mathcal{H}_{\Sigma}^{q,\lambda}(1, 1, \alpha, \beta, x) =: \mathcal{K}_{\Sigma}^{q,\lambda}(\alpha, \beta, x)$, where $\mathcal{K}_{\Sigma}^{q,\lambda}(\alpha, \beta, x)$ represents the functions $f \in \Sigma$ that satisfy the following conditions

$$1 + \left(\frac{z D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right) \right)}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)} \right) \prec \mathcal{G}(x, z) \quad (1.21)$$

and

$$1 + \left(\frac{w D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right) \right)}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right)} \right) \prec \mathcal{G}(x, w). \quad (1.22)$$

(iv) Assuming $q \rightarrow 1^-$ and $\gamma = \eta = 1$, we obtain that $\lim_{q \rightarrow 1^-} \mathcal{H}_{\Sigma}^{q,\lambda}(1, 1, \alpha, \beta, x) =: \mathcal{K}_{\Sigma}^{\lambda}(\alpha, \beta, x)$, where $\mathcal{K}_{\Sigma}^{\lambda}(\alpha, \beta, x)$ represents the functions $f \in \Sigma$ that satisfy the

following conditions

$$1 + \frac{1}{\eta} \left(\frac{z \left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right)''}{\left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right)'} \right) \prec \mathcal{G}(x, z) \tag{1.23}$$

and

$$1 + \frac{1}{\eta} \left(\frac{w \left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right)''}{\left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right)'} \right) \prec \mathcal{G}(x, w). \tag{1.24}$$

(v) Putting $\eta = (1 - \delta) \cos \theta e^{-i\theta}$ ($|\theta| < \frac{\pi}{2}$; $0 \leq \delta < 1$), we obtain that

$$\mathcal{H}_\Sigma^{q,\lambda} \left((1 - \delta) \cos \theta e^{-i\theta}, \gamma, \alpha, \beta, x \right) =: \mathcal{R}_\Sigma^{q,\lambda} (\delta, \theta, \gamma, \alpha, \beta, x),$$

where $\mathcal{R}_\Sigma^{q,\lambda} (\delta, \theta, \gamma, \alpha, \beta, x)$ represents the functions $f \in \Sigma$ that satisfy the following conditions

$$e^{i\theta} \left(\frac{\gamma z \left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right)'' + \gamma \left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right)' + 1 - \gamma}{\left(\mathcal{M}_{\alpha,\beta}^\lambda f(z) \right)' } - 1 \right) \prec (\mathcal{G}(x, z) - 1) (1 - \delta) \cos \theta \tag{1.25}$$

and

$$e^{i\theta} \left(\frac{\gamma w \left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right)'' + \gamma \left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right)' + 1 - \gamma}{\left(\mathcal{M}_{\alpha,\beta}^\lambda g(w) \right)' } - 1 \right) \prec (\mathcal{G}(x, w) - 1) (1 - \delta) \cos \theta. \tag{1.26}$$

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{H}_\Sigma^{q,\lambda} (\eta, \gamma, \alpha, \beta, x)$

We recall the following lemma to prove our main results:

Lemma 2.1. ([27, p.172]) *If $w(z) = \sum_{k=1}^\infty p_k z^k$ is a Schwarz function for $z \in \mathbb{E}$, then*

$$|p_1| \leq 1, \quad |p_k| \leq 1 - |p_1|^2, \quad k \geq 1. \tag{2.1}$$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\gamma > 0$, $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $0 < \lambda \leq 1$; $0 < q < 1$; $\eta \in \mathbb{C}^*$ and $x \in \mathbb{R}$, the powers are understood as principle values.

Theorem 2.2. *Let f be given by (1.1) belongs to the class $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$. Then*

$$|a_2| \leq \frac{|\eta| |x| \sqrt{x}}{\sqrt{\left|(\gamma(2+q)-1)(1+q+q^2)\eta x_3^2 \Upsilon_3 - \left[\eta x^2 + \frac{(2\gamma-1)}{2}(3x^2-1)\right](2\gamma-1)(1+q)^2 \Upsilon_2^2\right|}}$$

and

$$|a_3| \leq \frac{|\eta| |x|}{(\gamma(2+q)-1)(1+q+q^2) \Upsilon_3} + \frac{|\eta|^2 x^2}{(1+q)^2 (2\gamma-1)^2 \Upsilon_2^2},$$

where $\Upsilon_k, k \in \{2, 3\}$, are given by (1.8).

Proof. Since $f \in \mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$, there exist two analytic functions r and s in \mathbb{E} with $r(0) = s(0) = 0$, and $|r(z)| < 1, |s(w)| < 1$ for all $z, w \in \mathbb{E}$ given by

$$r(z) = \sum_{k=1}^{\infty} r_k z^k \quad \text{and} \quad s(w) = \sum_{k=1}^{\infty} s_k w^k.$$

From Lemma 2.1 we have

$$|r_k| \leq 1 \quad \text{and} \quad |s_k| \leq 1, \quad k \in \mathbb{N}. \tag{2.2}$$

In view of (1.15) and (1.16), we get

$$\frac{\gamma z D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right) \right) + \gamma D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right) + 1 - \gamma}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)} - 1 = \eta (\mathcal{G}(x, r(z)) - 1) \tag{2.3}$$

and

$$\frac{\gamma w D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right) \right) + \gamma D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right) + 1 - \gamma}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} g(w) \right)} - 1 = \eta (\mathcal{G}(x, s(w)) - 1). \tag{2.4}$$

Since

$$\begin{aligned} & \frac{\gamma z D_q \left(D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right) \right) + \gamma D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right) + 1 - \gamma}{D_q \left(\mathcal{M}_{\alpha,\beta}^{\lambda} f(z) \right)} - 1 \\ &= (1+q)(2\gamma-1) \Upsilon_2 a_2 z \\ &+ [(\gamma(2+q)-1)(1+q+q^2) \Upsilon_3 a_3 - (2\gamma-1)(1+q)^2 \Upsilon_2^2 a_2^2] z^2 + \dots, \end{aligned}$$

$$\begin{aligned} & \frac{\gamma w D_q \left(D_q \left(\mathcal{M}_{\alpha, \beta}^\lambda g(w) \right) \right) + \gamma D_q \left(\mathcal{M}_{\alpha, \beta}^\lambda g(w) \right) + 1 - \gamma}{D_q \left(\mathcal{M}_{\alpha, \beta}^\lambda g(w) \right)} - 1 \\ &= -(1+q)(2\gamma-1)\Upsilon_2 a_2 w \\ & \quad + [(\gamma(2+q)-1)(1+q+q^2)\Upsilon_3(2a_2^2) - a_3] \\ & \quad - (2\gamma-1)(1+q)^2 \Upsilon_2^2 a_2^2] w^2 + \dots \end{aligned}$$

and

$$\begin{aligned} \eta(\mathcal{G}(x, r(z)) - 1) &= \eta P_1(x) r_1 z + (P_1(x) r_2 + P_2(x) r_1^2) \eta z^2 + \dots, \\ \eta(\mathcal{G}(x, s(w)) - 1) &= \eta P_1(x) s_1 w + (P_1(x) s_2 + P_2(x) s_1^2) \eta w^2 + \dots \end{aligned}$$

Next, equating the corresponding coefficients of z and w in (2.3) and (2.4), we get

$$(1+q)(2\gamma-1)\Upsilon_2 a_2 = \eta P_1(x) r_1, \tag{2.5}$$

$$\begin{aligned} & (\gamma(2+q)-1)(1+q+q^2)\Upsilon_3 a_3 - (2\gamma-1)(1+q)^2 \Upsilon_2^2 a_2^2 \\ &= \eta P_1(x) r_2 + \eta P_2(x) r_1^2, \end{aligned} \tag{2.6}$$

$$-(1+q)(2\gamma-1)\Upsilon_2 a_2 = \eta P_1(x) s_1 \tag{2.7}$$

and

$$\begin{aligned} & (\gamma(2+q)-1)(1+q+q^2)\Upsilon_3(2a_2^2 - a_3) - (2\gamma-1)(1+q)^2 \Upsilon_2^2 a_2^2 \\ &= \eta P_1(x) s_2 + \eta P_2(x) s_1^2. \end{aligned} \tag{2.8}$$

From (2.6) and (2.7), we have

$$r_1 = -s_1. \tag{2.9}$$

By squaring (2.6) and (2.7), then adding the new relations we get

$$2(1+q)^2(2\gamma-1)^2 a_2^2 \Upsilon_2^2 = \eta^2 P_1^2(x) (r_1^2 + s_1^2). \tag{2.10}$$

If we add (2.5) and (2.8) we obtain

$$\begin{aligned} & 2 [(\gamma(2+q)-1)(1+q+q^2)\Upsilon_3 - (2\gamma-1)(1+q)^2 \Upsilon_2^2] a_2^2 \\ &= \eta P_1(x) (r_2 + s_2) + \eta P_2(x) (r_1^2 + s_1^2). \end{aligned}$$

We can rewrite (2.10) as

$$r_1^2 + s_1^2 = \frac{2(1+q)^2(2\gamma-1)^2}{\eta^2 P_1^2(x)} a_2^2 \Upsilon_2^2.$$

From above equation, we get

$$\begin{aligned} & 2 [(\gamma(2 + q) - 1) (1 + q + q^2) \eta P_1^2(x) \Upsilon_3 \\ & \quad - [\eta P_1^2(x) + (2\gamma - 1) P_2(x)] (2\gamma - 1) (1 + q)^2 \Upsilon_2^2] a_2^2 \\ & = \eta^2 P_1^3(x) (r_2 + s_2), \end{aligned}$$

it follows that

$$a_2^2 = \frac{\eta^2 P_1^3(x)(r_2+s_2)}{2[(\gamma(2+q)-1)(1+q+q^2)\eta P_1^2(x)\Upsilon_3 - (\eta P_1^2(x) + (2\gamma-1)P_2(x))(2\gamma-1)(1+q)^2\Upsilon_2^2]}. \tag{2.11}$$

Then taking the absolute value to the above equation and from (1.11) and (2.2), we obtain

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{[(\gamma(2+q)-1)(1+q+q^2)\eta x^2\Upsilon_3 - [\eta P_1^2(x) + \frac{(2\gamma-1)}{2}(3x^2-1)](2\gamma-1)(1+q)^2\Upsilon_2^2]}},$$

which gives the bound for $|a_2|$ as we asserted in our theorem.

To find the bound for $|a_3|$. Using (2.5) from (2.8), we have

$$\begin{aligned} & 2(\gamma(2 + q) - 1) (1 + q + q^2) \Upsilon_3 (a_3 - a_2^2) \\ & = \eta [P_1(x) (r_2 - s_2) + P_2(x) (r_1^2 - s_1^2)]. \end{aligned} \tag{2.12}$$

Form (2.9), (2.10) and (2.12), we obtain

$$a_3 = \frac{\eta P_1(x) (r_2 - s_2)}{2(\gamma(2 + q) - 1) (1 + q + q^2) \Upsilon_3} + \frac{\eta^2 P_1^2(x) (r_1^2 + s_1^2)}{2(1 + q)^2 (2\gamma - 1)^2 \Upsilon_2^2}. \tag{2.13}$$

Using (1.11) and (2.2), we get

$$|a_3| \leq \frac{|\eta| |x|}{(\gamma(2 + q) - 1) (1 + q + q^2) \Upsilon_3} + \frac{|\eta|^2 x^2}{(1 + q)^2 (2\gamma - 1)^2 \Upsilon_2^2}.$$

□

3. FEKETE-SZEGŐ PROBLEM FOR THE FUNCTION CLASS $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$

Theorem 3.1. *Let f be given by (1.1) and if $f \in \mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$, then*

$$|a_3 - \mu a_2^2| \leq |\eta||x| (|M + N| + |M - N|), \tag{3.1}$$

where

$$\begin{aligned} M &= \frac{(1-\mu)\eta x^2}{2[(\gamma(2+q)-1)(1+q+q^2)\eta x^2\Upsilon_3 - [\eta x^2 + \frac{(2\gamma-1)}{2}(3x^2-1)](2\gamma-1)(1+q)^2\Upsilon_2^2]}, \\ N &= \frac{1}{2(\gamma(2 + q) - 1) (1 + q + q^2) \Upsilon_3}, \end{aligned} \tag{3.2}$$

$\mu \in \mathbb{C}$, and $\Upsilon_k, k \in \{2, 3\}$, are given by (1.8).

Proof. Let $f \in \mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$. As in the proof of Theorem 2.2, from (2.9) and (2.12), we have

$$a_3 - a_2^2 = \frac{\eta P_1(x)(r_2 - s_2)}{2(\gamma(2 + q) - 1)(1 + q + q^2)\Upsilon_3}, \tag{3.3}$$

and multiplying (2.11) by $(1 - \mu)$ we get

$$(1 - \mu)a_2^2 = \frac{(1 - \mu)\eta^2 P_1^3(x)(r_2 + s_2)}{2[(\gamma(2 + q) - 1)(1 + q + q^2)\eta P_1^2(x)\Upsilon_3 - [\eta P_1^2(x) + (2\gamma - 1)P_2(x)](2\gamma - 1)(1 + q)^2\Upsilon_2^2]}. \tag{3.4}$$

Adding (3.3) and (3.4) leads to

$$a_3 - \mu a_2^2 = \eta h_2 [(M + N)r_2 + (M - N)s_2], \tag{3.5}$$

where M and N are given by (3.2), and taking the absolute value of (3.5), from (2.2) we obtain the desired inequality (3.1). \square

Remark 3.2. A simple computation shows that the inequality $|M| \leq N$ is equivalent to

$$|\mu - 1| \leq \left| 1 - \frac{[\eta x^2 + \frac{(2\gamma - 1)}{2}(3x^2 - 1)](2\gamma - 1)(1 + q)^2\Upsilon_2^2}{\eta x^2(\gamma(2 + q) - 1)(1 + q + q^2)\Upsilon_3} \right|. \tag{3.6}$$

Thus, from Theorem 3.1 we get the next result:

If the function f given by (1.1) belongs to the class $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$, and $\eta \in \mathbb{C}^*$, then

$$|a_3 - \mu a_2^2| \leq \frac{\eta x}{(\gamma(2 + q) - 1)(1 + q + q^2)\Upsilon_3},$$

where $\mu \in \mathbb{C}$, with (3.6) and $\Upsilon_k, k \in \{2, 3\}$, are given by (1.8).

4. COROLLARIES AND ITS CONSEQUENCES

Allowing $q \rightarrow 1^-$, in view of Theorem 2.2 and Theorem 3.1, we obtain the following result:

Corollary 4.1. *Let the function f given by (1.1) belongs to the class $\mathcal{I}_{\Sigma}^{\lambda}(\eta, \gamma, \alpha, \beta, x)$. Then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{\left| 3(3\gamma - 1)\eta x_3^2\Upsilon_3 - 4\left[\eta x^2 + \frac{(2\gamma - 1)}{2}(3x^2 - 1)\right](2\gamma - 1)\Upsilon_2^2 \right|}}$$

and

$$|a_3| \leq \frac{|\eta||x|}{3(3\gamma - 1)\Upsilon_3} + \frac{|\eta|^2 x^2}{4(2\gamma - 1)^2\Upsilon_2^2}.$$

Also for $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq |\eta||x| (|M + N| + |M - N|), \quad (4.1)$$

where

$$M = \frac{(1-\mu)\eta x^2}{2[3(3\gamma-1)\eta x^2 \Upsilon_3 - 4[\eta x^2 + \frac{(2\gamma-1)}{2}(3x^2-1)](2\gamma-1)\Upsilon_2^2]},$$

$$N = \frac{1}{2(\gamma(2+q) - 1)(1+q+q^2)\Upsilon_3}$$

and Υ_k , $k \in \{2, 3\}$, are given by (1.8).

Fixing $\gamma = 1$, from Theorem 2.2 and Theorem 3.1 we get the following :

Corollary 4.2. *Let f given by (1.1) belongs to the class $\mathcal{K}_{\Sigma}^{q,\lambda}(\eta, \alpha, \beta, x)$. Then*

$$|a_2| \leq \frac{|\eta||x|\sqrt{x}}{\sqrt{|(1+q)(1+q+q^2)\eta x_3^2 \Upsilon_3 - [\eta x^2 + \frac{1}{2}(3x^2-1)](1+q)^2 \Upsilon_2^2|}}$$

and

$$|a_3| \leq \frac{|\eta||x|}{(q+1)(1+q+q^2)\Upsilon_3} + \frac{|\eta|^2 x^2}{(1+q)^2 \Upsilon_2^2}.$$

Also for $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq |\eta||x| (|M + N| + |M - N|), \quad (4.2)$$

where

$$M = \frac{(1-\mu)\eta x^2}{2[(q+1)(1+q+q^2)\eta x^2 \Upsilon_3 - [\eta x^2 + \frac{1}{2}(3x^2-1)](1+q)^2 \Upsilon_2^2]},$$

$$N = \frac{1}{2(q+1)(1+q+q^2)\Upsilon_3}$$

and Υ_k , $k \in \{2, 3\}$, are given by (1.8).

Taking $\eta = \gamma = 1$, from Theorem 2.2 and Theorem 3.1, we state the following:

Corollary 4.3. *Let f be given by (1.1) belongs to the class $\mathcal{K}_{\Sigma}^{q,\lambda}(\gamma, \alpha, \beta, x)$. Then*

$$|a_2| \leq \frac{|x|\sqrt{x}}{\sqrt{|(1+q)(1+q+q^2)x_3^2 \Upsilon_3 - [x^2 + \frac{1}{2}(3x^2-1)](1+q)^2 \Upsilon_2^2|}}$$

and

$$|a_3| \leq \frac{|x|}{(1+q)(1+q+q^2)\Upsilon_3} + \frac{x^2}{(1+q)^2 \Upsilon_2^2}.$$

Also, for $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq |x| (|M + N| + |M - N|),$$

where

$$M = \frac{(1-\mu)x^2}{2[(1+q)(1+q+q^2)x^2\Upsilon_3 - [x^2 + \frac{1}{2}(3x^2-1)](1+q)^2\Upsilon_2^2]},$$

$$N = \frac{1}{2(1+q)(1+q+q^2)\Upsilon_3}$$

and $\Upsilon_k, k \in \{2, 3\}$ are given by (1.8).

Taking $\eta = (1 - \delta) \cos \theta e^{-i\theta}$ ($|\theta| < \frac{\pi}{2}; 0 \leq \delta < 1$), from Theorem 2.2 and Theorem 3.1, we state the following:

Corollary 4.4. *Let f given by (1.1) belongs to the class $\mathcal{R}_{\Sigma}^{q,\lambda}(\delta, \theta, \gamma, \alpha, \beta, x)$. Then*

$$|a_2| \leq \frac{(1-\delta) \cos \theta |x| \sqrt{x}}{\sqrt{[(\gamma(2+q)-1)(1+q+q^2)(1-\delta) \cos \theta e^{-i\theta} x^2 \Upsilon_3 - [(1-\delta) \cos \theta e^{-i\theta} x^2 + \frac{(2\gamma-1)}{2}(3x^2-1)](2\gamma-1)(1+q)^2 \Upsilon_2^2]}}$$

and

$$|a_3| \leq \frac{(1-\delta) \cos \theta |x|}{(\gamma(2+q)-1)(1+q+q^2)\Upsilon_3} + \frac{(1-\delta)^2 \cos^2 \theta x^2}{(1+q)^2(2\gamma-1)^2 \Upsilon_2^2}.$$

Also, for $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq (1-\delta) \cos \theta |x| (|M + N| + |M - N|), \tag{4.3}$$

where

$$M = \frac{(1-\mu)(1-\delta) \cos \theta e^{-i\theta} x^2}{2[(\gamma(2+q)-1)(1+q+q^2)(1-\delta) \cos \theta e^{-i\theta} x^2 \Upsilon_3 - \{(1-\delta) \cos \theta e^{-i\theta} x^2 + \frac{(2\gamma-1)}{2}(3x^2-1)\}(2\gamma-1)(1+q)^2 \Upsilon_2^2]},$$

$$N = \frac{1}{2(\gamma(2+q)-1)(1+q+q^2)\Upsilon_3}$$

and $\Upsilon_k, k \in \{2, 3\}$ are given by (1.8).

Remark 4.5. We emphasize that general classes $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$ are completely new and not studied based on Mittag-Leffler-Type Borel Distribution involving the Legendre Polynomials. Suitably specializing the parameter γ, η and in Corollary 4.1, one can easily deduce the results for the new subclasses stated in Remark 1.4 based on Mittag-Leffler functions which are new and not studied so far.

Remark 4.6. We mention that all the above estimations for the coefficients $|a_2|$, $|a_3|$, and Fekete-Szegő problem for the function class $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta, \gamma, \alpha, \beta, x)$ are not sharp. To find the sharp upper bounds for the above functionals remains an interesting open problem, as well as those for $|a_n|$, $n \geq 4$.

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