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SOME RATIONAL F-CONTRACTIONS IN b-METRIC SPACES AND FIXED POINTS

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Abstract. In this paper, we introduce the notion of a new generalized type of rational F-contraction mapping. Further, the concept is used to obtain fixed points in a complete b-metric space. We also prove another unique fixed point theorem in the context of b-metric space. Our results are verified with example.

1. Introduction

Wardowski [27] introduced the concept of F-contraction and generalized the Banach fixed point theorem. For our discussion in this paper, we use the following notations: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{N}$ is the set of natural numbers.

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2. Preliminaries

Wardowski [27] defined the following:

Definition 2.1. ([27]) A self-mapping f in a metric space (Ω, d) is said to be an F-contraction if for all $\kappa, \delta \in \Omega$ and $d(f\kappa, f\delta) > 0$ implies

$$\tau + \mathcal{F}(d(f\kappa, f\delta)) \leq \mathcal{F}(d(\kappa, \delta)), \tag{2.1}$$

where $\tau > 0$ and $\mathcal{F} \in \mathfrak{F}$. Here $\mathfrak{F}$ is the family of all functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

($\mathcal{F}1$) $\mathcal{F}$ is increasing strictly;
($\mathcal{F}2$) $\lim_{n \to +\infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} \mathcal{F}(\alpha_n) = -\infty$ for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$;
($\mathcal{F}3$) for $0 < k < 1$, $\lim_{\alpha \to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$.

Many authors proved some interesting results and gave useful applications for the F-contraction mappings [1, 13, 16, 28]. Wardowski also pointed out that by considering different types of mappings in (2.1) variety of contractions can be obtained. He also remarked that from ($\mathcal{F}1$) and (2.1), it can be concluded that F-contraction mappings are contractive and hence continuous. Further, if $\mathcal{F}_1, \mathcal{F}_2$ be such that the properties ($\mathcal{F}1$)-($\mathcal{F}3$) in Definition 2.1 are satisfied. If $\mathcal{F}_1(\alpha) \leq \mathcal{F}_2(\alpha)$ for all $\alpha > 0$ and a mapping $G = \mathcal{F}_2 - \mathcal{F}_1$ is decreasing then every $\mathcal{F}_1$-contraction f is $\mathcal{F}_2$-contraction.

The following theorem was proved by Wardowski.

Theorem 2.2. ([27]) *If a self-mapping f is an F-contraction in a complete metric space (Ω, d), then for every $\kappa \in \Omega$, the sequence $\{f^n \kappa\}_{n \in \mathbb{N}}$ converges to $\kappa^* \in \Omega$ where κ^* is the unique fixed point of f.*

Secelean [23] replaced ($\mathcal{F}2$) of Definition 2.1 by either of the property given as under:

($\mathcal{F}2'$) $\inf \mathcal{F} = -\infty$ or
($\mathcal{F}2''$) a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive real numbers exist such that $\lim_{n \to \infty} \mathcal{F}(\alpha_n) = -\infty$.

Secelean [23] also proved the following:

Lemma 2.3. ([23]) *Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ be a increasing mapping and $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^+$. Then the following conditions hold true.*

(i) $\lim_{n \to \infty} \mathcal{F}(\alpha_n) = -\infty$ *implies* $\lim_{n \to \infty} \alpha_n = 0$;
(ii) $\inf \mathcal{F} = -\infty$ *and* $\lim_{n \to \infty} \alpha_n = 0$ *implies* $\lim_{n \to \infty} \mathcal{F}(\alpha_n) = -\infty$.

Wardowski also pointed out that Banach contractions are F-contractions and converse is not true.

F-contraction is introduced by Cosentino and Verto [7].

Definition 2.4. ([7]) Let (Ω, d) be a complete metric space. A self-mapping f is said to be a Hardy-Rogers type F-contraction if $\mathcal{F} \in \mathfrak{F}$ and $\tau > 0$ satisfies

$$\begin{aligned}\tau + \mathcal{F}(d(f\kappa, f\delta)) \leq \mathcal{F}(&\theta_1.d(\kappa, \delta) + \theta_2.d(\kappa, f\kappa) \\ &+ \theta_3.d(\delta, fu\delta) + \theta_4.d(\kappa, f\delta) + \theta_5.d(\delta, f\kappa))\end{aligned} \tag{2.2}$$

with $d(f\kappa, f\delta) > 0$ for all $\kappa, \delta \in \Omega$, where $\theta_1, \theta_2, \theta_3, \theta_4$ and θ_5 are non-negative numbers, $\theta_3 \neq 1$ and $\theta_1 + \theta_2 + \theta_3 + 2\theta_4 = 1$.

Theorem 2.5. ([7]) *Let (Ω, d) be a complete metric space. If a self-mapping f is a Hardy-Rogers-type contraction and $\theta_3 \neq 1$, then f has a fixed point. Further, f has a unique fixed point if $\theta_1 + \theta_4 + \theta_5 \leq 1$.*

In Definition 2.1, the condition $(\mathcal{F}3)$ was replaced by Piri and Kumam [14] as under:

$(\mathcal{F}3')$ $\mathcal{F}$ is continuous on $(0, +\infty)$.

They defined a family of functions $\mathfrak{F}$ satisfying $(\mathcal{F}1), (\mathcal{F}2')$ and $(\mathcal{F}3')$ and proved the following:

Theorem 2.6. ([14]) *Let f be a self- mapping in a complete metric space (Ω, d). Let $\mathcal{F} \in \mathfrak{F}$ satisfy that*

$$\forall \kappa, \delta \in \Omega, \ \ [d(f\kappa, f\delta) > 0 \quad \textit{implies} \quad \tau + \mathcal{F}(d(f\kappa, f\delta)) \leq \mathcal{F}(d(\kappa, \delta))],$$

where $\tau > 0$. Then f has a unique fixed point $\kappa^* \in \Omega$ and the sequence $\{f^n\kappa\}_{n\in\mathbb{N}}$ converges to κ^* for each $\kappa \in \Omega$.

Piri and Kumam [14] showed the independence of $(\mathcal{F}3)$ and $(\mathcal{F}3')$.

The next result was proved by Popescu and Gabrial [25] by generalizing the results in [7, 27].

Theorem 2.7. ([25]) *Let f be a self-mapping in a complete metric space (Ω, d). For $\tau > 0$, let $\kappa, \delta \in \Omega$, $d(f\kappa, f\delta) > 0$ implies*

$$\begin{aligned}\tau + \mathcal{F}(d(f\kappa, f\delta)) \leq \mathcal{F}(&\theta_1.d(\kappa, \delta) + \theta_2.d(\kappa, f\kappa) + \theta_3.d(\delta, f\delta) \\ &+ \theta_4.d(\kappa, f\delta) + \theta_5.d(\delta, f\kappa)),\end{aligned}$$

where the mapping $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ is increasing, $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ are non-negative numbers, $\theta_4 < 1/2, \theta_3 < 1, \theta_1 + \theta_2 + \theta_3 + 2\theta_4 = 1$, $0 < \theta_1 + \theta_4 + \theta_5 \leq 1$. Then f has a unique fixed point $\kappa^* \in \Omega$ and the sequence $\{f^n\kappa\}_{n\in\mathbb{N}}$ converges to κ^* for each $\kappa \in \Omega$.

Bakhtin [3] introduced b-metric space and later it was widely used by Czerwik [8].

Definition 2.8. ([3, 8]) Let $\Omega \neq \phi$ and $d : \Omega \times \Omega \to [0, +\infty)$ be a mapping satisfying:

(1) $d(\kappa, \delta) = 0$ if and only if $\kappa = \delta$ for all $\kappa, \delta \in \Omega$;
(2) $d(\kappa, \delta) = d(\delta, \kappa)$ for every $\kappa, \delta \in \Omega$;
(3) $d(\kappa, \delta) \leq s[d(\kappa, \mu) + d(\mu, \delta)]$ for every $\kappa, \delta, \mu \in \Omega$, where $s \geq 1$ is a real number.

Then d is called a b-metric on Ω and (Ω, d) a b-metric space.

Definition 2.9. ([3, 8]) A sequence $\{\kappa_n\}$ is in a b-metric space (Ω, d).

(1) $\{\kappa_n\}$ is called convergent in (Ω, d) if there exists a $\kappa \in \Omega$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $d(\kappa_n, \kappa) < \varepsilon$ for all $n > n_0$.
(2) $\{\kappa_n\}$ is a Cauchy sequence in (Ω, d) if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $d(\kappa_n, \kappa_m) < \varepsilon$ for all $n, m > n_0$.
(3) (Ω, d) is said to be complete, if every Cauchy sequence in Ω is convergent.

There are various results on contractive mapping. Rational type of contraction is also one such generalisation of contractive mappings. Some results of different types of contractive mappings can be seen in [2, 4, 5, 6, 9, 10, 11, 12, 15, 17, 18, 19, 20, 21, 22, 24, 26]. Here, we will prove some theorems on rational F-contractive mappings in b-metric spaces.

Next lemma is useful for b-metric space.

Lemma 2.10. ([2]) *Let (Ω, d) be a b-metric space. Let $\{\kappa_n\}$ and $\{\delta_n\}$ be b-convergent to $\kappa \in \Omega$ and $\delta \in \Omega$, respectively. Then we get*

$$\begin{aligned}\frac{1}{s^2} d(\kappa, \delta) &\leq \lim_{n\to\infty} \inf d(\kappa_n, \delta_n) \leq \lim_{n\to\infty} \sup d(\kappa_n, \delta_n) \\ &\leq s^2 d(\kappa, \delta).\end{aligned}$$

Particularly, if $\kappa = \delta$, then $\lim_{n\to\infty} d(\kappa_n, \delta_n) = 0$. Also, for each $\mu \in \Omega$, we get

$$\begin{aligned}\frac{1}{s} d(\kappa, \mu) &= \lim_{n\to\infty} \inf d(\kappa_n, \mu) \leq \lim_{n\to\infty} \sup d(\kappa_n, \mu) \\ &\leq s d(\kappa, \mu).\end{aligned}$$

3. Main results

We prove the following result.

Theorem 3.1. *Let (Ω, d) be a complete b-metric space and $s \geq 1$. Let $f : \Omega \to \Omega$ be a mapping and there exists $\tau > 0$ satisfying $d(f\kappa, f\delta) > 0$ implies*

$$\begin{aligned}\tau + \mathcal{F}(d(f\kappa, f\delta)) \quad \leq \quad & \mathcal{F}\Big(\theta_1.d(\kappa,\delta) + \theta_2\frac{d(\kappa, f\kappa)d(\delta, f\delta)}{1+d(\kappa,\delta)} \\ & +\theta_3\frac{d(\kappa, f\delta)d(\delta, f\kappa)}{1+d(\kappa,\delta)} + \theta_4\frac{d(f\kappa, f\delta)d(\kappa,\delta)}{1+d(\kappa,\delta)} \\ & +\theta_5\frac{d(\kappa, f\delta)d(\kappa,\delta)}{1+d(\kappa,\delta)} + \theta_6\frac{d(\delta, f\kappa)d(\kappa,\delta)}{1+d(\kappa,\delta)}\Big)\end{aligned}$$

for all $\kappa, \delta \in \Omega$, where the mapping $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ is increasing. $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ are non-negative numbers with $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2s\theta_5 + \theta_6 < 1$. Then, f has a unique fixed point $\kappa^* \in \Omega$ and the sequence $\{f^n\kappa\}_{n\in\mathbb{N}}$ converges to κ^* for every $\kappa \in \Omega$.

Proof. Consider an arbitrary point $\kappa_0 \in \Omega$, then sequence $\{\kappa_n\}_{n\in\mathbb{N}} \subset \Omega$ can be constructed as

$$\begin{aligned}&\kappa_1 = f\kappa_0, \\ &\kappa_2 = f\kappa_1 = f^2\kappa_0, \\ &\text{and so on} \\ &\kappa_n = f\kappa_{n-1} = f^n\kappa_0, \ \forall\, n \in \mathbb{N}. \qquad (3.1)\end{aligned}$$

Let $d(\kappa_n, f\kappa_n) = 0$, where $n \in \mathbb{N} \cup \{0\}$. Then we can conclude the proof. So let

$$0 < d(\kappa_n, f\kappa_n) = d(f\kappa_{n-1}, f\kappa_n), \ \forall\, n \in \mathbb{N}. \qquad (3.2)$$

Let us use the notation $d_n = d(\kappa_n, \kappa_{n+1})$. Due to the monotone property of $\mathcal{F}$ and assumption in the theorem for all $n \in \mathbb{N}$, we get

$$\begin{aligned}&\tau + \mathcal{F}(d_n) \\ &= \tau + \mathcal{F}(d(\kappa_n, \kappa_{n+1})) = \tau + \mathcal{F}(d(f\kappa_{n-1}, f\kappa_n)) \\ &\leq \mathcal{F}\Big(\theta_1 d(\kappa_{n-1}, \kappa_n) + \theta_2\frac{d(\kappa_{n-1}, f\kappa_{n-1})d(\kappa_n, f\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)} \\ &\quad + \theta_3\frac{d(\kappa_{n-1}, f\kappa_n)d(\kappa_n, f\kappa_{n-1})}{1+d(\kappa_{n-1},\kappa_n)} + \theta_4\frac{d(f\kappa_{n-1}, f\kappa_n)d(\kappa_{n-1},\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)} \\ &\quad + \theta_5\frac{d(\kappa_{n-1}, f\kappa_n)d(\kappa_{n-1},\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)} + \theta_6\frac{d(\kappa_n, f\kappa_{n-1})d(\kappa_{n-1},\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}\Big(\theta_1 d(\kappa_{n-1},\kappa_n) + \theta_2 \frac{d(\kappa_{n-1},\kappa_n)d(\kappa_n,\kappa_{n+1})}{1+d(\kappa_{n-1},\kappa_n)} \\
&\quad + \theta_3 \frac{d(\kappa_{n-1},\kappa_{n+1})d(\kappa_n,\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)} + \theta_4 \frac{d(\kappa_n,\kappa_{n+1})d(\kappa_{n-1},\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)} \\
&\quad + \theta_5 \frac{d(\kappa_{n-1},\kappa_{n+1})d(\kappa_{n-1},\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)} + \theta_6 \frac{d(\kappa_n,\kappa_n)d(\kappa_{n-1},\kappa_n)}{1+d(\kappa_{n-1},\kappa_n)}\Big) \\
&\le \mathcal{F}\Big(\theta_1 d(\kappa_{n-1},\kappa_n) + \theta_2 d(\kappa_n,\kappa_{n+1}) + 0 + \theta_4 d(\kappa_n,\kappa_{n+1}) \\
&\quad + \theta_5 d(\kappa_{n-1},\kappa_{n+1}) + 0\Big) \\
&\le \mathcal{F}\Big(\theta_1 d(\kappa_{n-1},\kappa_n) + \theta_2 d(\kappa_n,\kappa_{n+1}) + \theta_4 d(\kappa_n,\kappa_{n+1}) \\
&\quad + \theta_5 s[d(\kappa_{n-1},\kappa_n) + d(\kappa_n,\kappa_{n+1})]\Big) \\
&= \mathcal{F}((\theta_1 + s\theta_5)d(\kappa_{n-1},\kappa_n) + (\theta_2+\theta_4+s\theta_5)d(\kappa_n,\kappa_{n+1})).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{F}(d_n) &\le \mathcal{F}(\theta_1 + s\theta_5)d(\kappa_{n-1},\kappa_n) + (\theta_2+\theta_4+s\theta_5)d(\kappa_n,\kappa_{n+1})) - \tau \\
&< \mathcal{F}((\theta_1+s\theta_5)d_{n-1} + (\theta_2+\theta_4+s\theta_5)d_n)). \qquad (3.3)
\end{aligned}
$$

From the property of $\mathcal{F}$,

$$d_n < (\theta_1+s\theta_5)d_{n-1} + (\theta_2+\theta_4+s\theta_5)d_n,$$

so

$$(1-\theta_2-\theta_4-s\theta_5)d_n \le (\theta_1+s\theta_5)d_{n-1}.$$

Since $\theta_1+\theta_2+\theta_3+\theta_4+2s\theta_5+\theta_6 < 1$, we have

$$
\begin{aligned}
d_n &\le \frac{\theta_1+s\theta_5}{1-\theta_2-\theta_4-s\theta_5} d_{n-1} \\
&\le d_{n-1}.
\end{aligned}
$$

Thus, $\{d_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence and hence $\lim_{n\to\infty} d_n = d$ exists.

Let $d > 0$. As $\mathcal{F}$ being increasing

$$\lim_{\kappa\to d_+} f(\kappa) = \mathcal{F}(d+0).$$

In inequality (3.3), taking the limit $n \to +\infty$,

$$\mathcal{F}(d+0) \le \mathcal{F}(d+0) - \tau,$$

which is a contradiction and hence

$$\lim_{n\to+\infty} d_n = 0. \qquad (3.4)$$

In order to prove that $\{\kappa_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if possible, let $\{k(n)\}_{n\in\mathbb{N}}$ and $\{l(n)\}_{n\in\mathbb{N}}$ as sequences where $k(n) > l(n) > n$ and $\varepsilon > 0$ with

$$d(\kappa_{k(n)}, \kappa_{l(n)}) > \varepsilon,\ d(\kappa_{k(n)-1}, \kappa_{l(n)}) \leq \varepsilon,\ \forall\, n \in \mathbb{N}. \tag{3.5}$$

By triangler inequality,

$$\varepsilon < d(\kappa_{k(n)}, \kappa_{l(n)}) \leq s[d(\kappa_{k(n)}, \kappa_{k(n)-1}) + d(\kappa_{k(n)-1}, \kappa_{l(n)})],$$

that is,

$$\frac{\varepsilon}{s} < \frac{1}{s} d(\kappa_{k(n)}, \kappa_{l(n)}) \leq d(\kappa_{k(n)}, \kappa_{k(n)-1}) + d(\kappa_{k(n)-1}, \kappa_{l(n)}).$$

Taking the limit $n \to +\infty$, we get

$$\begin{aligned}
\frac{\varepsilon}{s} &< \frac{1}{s} \lim_{n\to+\infty} d(\kappa_{k(n)}, \kappa_{l(n)}) \leq \frac{1}{s} \lim_{n\to+\infty} d(\kappa_{k(n)}, \kappa_{l(n)}) \leq \varepsilon, \\
\frac{\varepsilon}{s} &< \lim_{n\to+\infty} \inf \frac{1}{s} d(\kappa_{k(n)}, \kappa_{l(n)}) \leq \lim_{n\to+\infty} \sup \frac{1}{s} d(\kappa_{k(n)}, \kappa_{l(n)}) < \frac{\varepsilon s}{s},
\end{aligned}$$

which in turn implies

$$\lim_{n\to+\infty} d(\kappa_{k(n)}, \kappa_{l(n)}) = \varepsilon. \tag{3.6}$$

Since $d(\kappa_{k(n)}, \kappa_{l(n)}) > \varepsilon > 0$, by property of F we have

$$\begin{aligned}
\tau + \mathcal{F}(d(\kappa_{k(n)}, \kappa_{l(n)}) &= \tau + \mathcal{F}(d(f\kappa_{k(n)-1}, f\kappa_{l(n)-1})) \\
&\leq \mathcal{F}\Big(\theta_1 d(\kappa_{(n)-1}, \kappa_{l(n)-1}) \\
&\quad + \theta_2 \frac{d(\kappa_{k(n)-1}, f\kappa_{k(n)-1}) d(\kappa_{l(n)-1}, f\kappa_{l(n)-1})}{1 + d(\kappa_{k(n)-1}, \kappa_{l(n)-1})} \\
&\quad + \theta_3 \frac{d(\kappa_{k(n)-1}, f\kappa_{l(n)-1}) d(\kappa_{l(n)-1}, f\kappa_{k(n)-1})}{1 + d(\kappa_{k(n)-1}, \kappa_{l(n)-1})} \\
&\quad + \theta_4 \frac{d(f\kappa_{k(n)-1}, f\kappa_{l(n)-1}) d(\kappa_{k(n)-1}, \kappa_{l(n)-1})}{1 + d(\kappa_{k(n)-1}, \kappa_{l(n)-1})} \\
&\quad + \theta_5 \frac{d(\kappa_{k(n)-1}, f\kappa_{l(n)-1}) d(\kappa_{k(n)-1}, \kappa_{l(n)-1})}{1 + d(\kappa_{k(n)-1}, \kappa_{l(n)-1})} \\
&\quad + \theta_6 \frac{d(\kappa_{l(n)-1}, f\kappa_{k(n)-1}) d(\kappa_{k(n)-1}, \kappa_{l(n)-1})}{1 + d(\kappa_{k(n)-1}, \kappa_{l(n)-1})}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}\Big(\theta_1 d(\kappa_{P(n)-1},\kappa_{l(n)-1}) + \theta_2 \frac{d(\kappa_{k(n)-1},\kappa_{k(n)})d(\kappa_{l(n)-1},\kappa_{l(n)})}{1+d(\kappa_{k(n)-1},\kappa_{l(n)-1})} \\
&\quad +\theta_3 \frac{d(\kappa_{k(n)-1},\kappa_{l(n)})d(\kappa_{l(n)-1},\kappa_{k(n)})}{1+d(\kappa_{k(n)-1},\kappa_{l(n)-1})} \\
&\quad +\theta_4 \frac{d(\kappa_{k(n)},\kappa_{l(n)})d(\kappa_{k(n)-1},\kappa_{l(n)-1})}{1+d(\kappa_{k(n)-1},\kappa_{l(n)-1})} \\
&\quad +\theta_5 \frac{d(\kappa_{k(n)-1},\kappa_{l(n)})d(\kappa_{k(n)-1},\kappa_{l(n)-1})}{1+d(\kappa_{k(n)-1},\kappa_{l(n)-1})} \\
&\quad +\theta_6 \frac{d(\kappa_{l(n)-1},\kappa_{k(n)})d(\kappa_{k(n)-1},\kappa_{l(n)-1})}{1+d(\kappa_{k(n)-1},\kappa_{l(n)-1})}\Big) \\
&\leq \mathcal{F}\Big(\theta_1 d(\kappa_{k(n)-1},\kappa_{l(n)-1}) + \theta_2 d(\kappa_{k(n)-1},\kappa_{k(n)}) + \theta_3 d(\kappa_{l(n)-1},\kappa_{k(n)}) \\
&\quad +\theta_4 d(\kappa_{k(n)},\kappa_{l(n)}) + \theta_5 d(\kappa_{k(n)-1},\kappa_{l(n)}) + \theta_6 d(\kappa_{l(n)-1},\kappa_{k(n)})\Big) \\
&\leq \mathcal{F}\Big(\theta_1 s d(\kappa_{k(n)-1},\kappa_{l(n)}) + \theta_1 s d(\kappa_{l(n)},\kappa_{l(n)-1}) + \theta_2 d(\kappa_{k(n)-1},\kappa_{k(n)}) \\
&\quad +\theta_3 s d(\kappa_{l(n)-1},\kappa_{l(n)}) + \theta_3 s d(\kappa_{l(n)},\kappa_{k(n)}) \\
&\quad +\theta_4 d(\kappa_{k(n)},\kappa_{l(n)}) + \theta_5 s d(\kappa_{k(n)-1},\kappa_{k(n)}) \\
&\quad +\theta_5 s d(\kappa_{k(n)},\kappa_{l(n)}) + \theta_6 s d(\kappa_{l(n)-1},\kappa_{l(n)}) + \theta_6 s d(\kappa_{l(n)},\kappa_{k(n)})\Big) \\
&\leq \mathcal{F}\Big(\theta_1 s^2 d(\kappa_{k(n)-1},\kappa_{k(n)}) + \theta_1 s^2 d(\kappa_{k(n)},\kappa_{l(n)}) + \theta_1 s d(\kappa_{l(n)},\kappa_{l(n)-1}) \\
&\quad +\theta_2 d(\kappa_{k(n)-1},\kappa_{k(n)}) + \theta_3 s d(\kappa_{l(n)-1},\kappa_{l(n)}) \\
&\quad +\theta_3 s d(\kappa_{l(n)},\kappa_{k(n)}) + \theta_4 d(\kappa_{k(n)},\kappa_{l(n)}) \\
&\quad +\theta_5 s d(\kappa_{k(n)-1},\kappa_{k(n)}) + \theta_5 s d(\kappa_{k(n)},\kappa_{l(n)}) + \theta_6 s d(\kappa_{l(n)-1},\kappa_{l(n)}) \\
&\quad +\theta_6 s d(\kappa_{l(n)},\kappa_{k(n)})\Big) \\
&\leq \mathcal{F}\Big((\theta_1 s^2 + \theta_2 + \theta_5 s) d(\kappa_{k(n)-1},\kappa_{k(n)}) \\
&\quad +(\theta_1 s^2 + \theta_3 s + \theta_4 + \theta_5 s + \theta_6 s) d(\kappa_{k(n)},\kappa_{l(n)}) \\
&\quad +(\theta_1 s + \theta_6 s) d(\kappa_{l(n)-1},\kappa_{l(n)})\Big).
\end{aligned}
$$

Taking the limit $n \to +\infty$ we have

$$\tau + \mathcal{F}(\varepsilon + 0) \leq (\varepsilon + 0)$$

which is a contradiction and therefore, sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ is Cauchy. By completeness of Ω there is some $\kappa^* \in \Omega$ such that $\{\kappa_n\}_{n\in\mathbb{N}}$ is convergent to κ^*.

If $\{k(n)\}_{n\in\mathbb{N}}$ be a sequence with $\kappa_{k(n)+1} = f\kappa_{k(n)} = f\kappa^*$, then $\lim_{n\to+\infty}\kappa_{k(n)+1} = \kappa^*$. Thus $f\kappa^* = \kappa^*$. Assuming $f\kappa^* \neq \kappa^*$ we have

$$\begin{aligned}
\tau + \mathcal{F}(d(f\kappa_n, f\kappa^*)) &\leq \mathcal{F}\Big(\theta_1 d(\kappa_n,\kappa^*) + \theta_2\frac{d(\kappa_n, f\kappa_n)d(\kappa^*, f\kappa^*)}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_3\frac{d(\kappa_n, f\kappa^*)d(\kappa^*, f\kappa_n)}{1+d(\kappa_n,\kappa^*)} + \theta_4\frac{d(f\kappa_n, f\kappa^*)d(\kappa_n,\kappa^*)}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_5\frac{d(\kappa_n, f\kappa^*)d(\kappa_n,\kappa^*)}{1+d(\kappa_n,\kappa^*)} + \theta_6\frac{d(\kappa^*, f\kappa_n)d(\kappa_n,\kappa^*)}{1+d(\kappa_n,\kappa^*)}\Big)\\
&= \mathcal{F}\Big(\theta_1 d(\kappa_n,\kappa^*) + \theta_2\frac{d(\kappa_n, \kappa_{n+1})d(\kappa^*, f\kappa^*)}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_3\frac{d(\kappa_n, f\kappa^*)d(\kappa^*, \kappa_{n+1})}{1+d(\kappa_n,\kappa^*)} + \theta_4\frac{d(\kappa_{n+1}, f\kappa^*)d(\kappa,\kappa^*)}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_5\frac{d(\kappa_n, f\kappa^*)d(\kappa_n,\kappa^*)}{1+d(\kappa_n,\kappa^*)} + \theta_6\frac{d(\kappa^*, \kappa_{n+1})d(\kappa_n,\kappa^*)}{1+d(\kappa_n,\kappa^*)}\Big)\\
&\leq \mathcal{F}\Big(\theta_1 d(\kappa_n,\kappa^*) + \theta_2\frac{d(\kappa_n, \kappa_{+1})d(\kappa^*, f\kappa^*)}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_3\frac{d(\kappa_n, f\kappa^*)d(\kappa^*, \kappa_{+1})}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_4 d(\kappa_{n+1}, f\kappa^*) + \theta_5 d(\kappa_n, f\kappa^*) + \theta_6 d(\kappa^*, \kappa_{n+1})\Big).
\end{aligned}$$

By increasing property of $\mathcal{F}$

$$\begin{aligned}
d(f\kappa_n, f\kappa^*) &< \theta_1 d(\kappa_n,\kappa^*) + \theta_2\frac{d(\kappa_n, \kappa_{n+1})d(\kappa^*, f\kappa^*)}{1+d(\kappa_n,\kappa^*)}\\
&\quad + \theta_3\frac{d(\kappa_n, f\kappa^*)d(\kappa^*, \kappa_{n+1})}{1+d(\kappa_n,\kappa^*)} + \theta_4 d(\kappa_{n+1}, f\kappa^*)\\
&\quad + \theta_5 d(\kappa_n, f\kappa^*) + \theta_6 d(\kappa^*, \kappa_{n+1}).
\end{aligned}$$

Letting n tends to $+\infty$, we get

$$\begin{aligned}
d(\kappa^*, f\kappa^*) &< \theta_4 d(\kappa^*, f\kappa^*) + \theta_5 d(\kappa^*, f\kappa^*)\\
&< d(\kappa^*, f\kappa^*),
\end{aligned}$$

which is a contradiction and therefore, $f\kappa^* = \kappa^*$. Let κ^* and δ be two distinct fixed points of f in Ω. Then, $d(f\kappa^*, f\delta) = d(\kappa^*, \delta) > 0$, we have

$$\begin{aligned}
\tau + \mathcal{F}(d(\kappa^*, \delta)) &= \tau + \mathcal{F}(d(f\kappa^*, f\delta)) \\
&\leq \mathcal{F}\Big(\theta_1 d(\kappa^*, \delta) + \theta_2 \frac{d(\kappa^*, f\kappa^*)d(\delta, f\delta)}{1 + d(\kappa^*, \delta)} \\
&\quad + \theta_3 \frac{d(\kappa^*, f\delta)d(\delta, f\kappa^*)}{1 + d(\kappa^*, \delta)} + \theta_4 \frac{d(f\kappa^*, f\delta)d(\kappa^*, \delta)}{1 + d(\kappa^*, \delta)} \\
&\quad + \theta_5 \frac{d(\kappa^*, f\delta)d(\kappa^*, \delta)}{1 + d(\kappa^*, \delta)} + \theta_6 \frac{d(\delta, f\kappa^*)d(\kappa^*, \delta)}{1 + d(\kappa^*, \delta)}\Big) \\
&\leq \mathcal{F}\Big(\theta_1 d(\kappa^*, \delta) + \theta_2 \frac{d(\kappa^*, f\kappa^*)d(\delta, f\delta)}{1 + d(\kappa^*, \delta)} \\
&\quad + \theta_3 \frac{d(\kappa^*, f\delta)d(\delta, f\kappa^*)}{1 + d(\kappa^*, \delta)} + \theta_4 d(f\kappa^*, f\delta) \\
&\quad + \theta_5 d(\kappa^*, f\delta) + \theta_6 d(\delta, f\kappa^*)\Big) \\
&= \mathcal{F}(\theta_1 d(\kappa^*, \delta) + \theta_4 d(\kappa^*, \delta) + \theta_5 d(\kappa^*, \delta) + \theta_6 d(\kappa^*, \delta)) \\
&= \mathcal{F}((\theta_1 + \theta_4 + \theta_5 + \theta_6) d(\kappa^*, \delta)) \\
&\leq \mathcal{F}(d(\kappa^*, \delta)),
\end{aligned}$$

which is a contradiction and hence fixed point is unique. □

Note: Taking $\theta_1 = 1$ and $\theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$ in Theorem 3.1, we obtain Theorem of Wardowski [27] in b-metric space.

Theorem 3.2. *Let f be a self-mapping in a complete b-metric space (Ω, d). Let $\mathcal{F} \in \mathfrak{F}$ satisfy*

$$\forall\ \kappa, \delta \in \Omega,\ [d(f\kappa, f\delta) > 0 \quad \text{implies} \quad \tau + \mathcal{F}(d(f\kappa, f\delta)) \leq \mathcal{F}(d(\kappa, \delta))], \tag{3.7}$$

where $\tau > 0$ and mapping $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfies $(\mathcal{F}2)$ and $(\mathcal{F}3'')$, where

$(\mathcal{F}3'')$ $\mathcal{F}$ is continuous on $(0, +\infty)$.

Then, f has a unique fixed point $\kappa^* \in \Omega$ and the sequence $\{f^n\kappa\}_{n\in\mathbb{N}}$ converges to κ^* for each $\kappa \in \Omega$.

Proof. Let $\kappa_0 \in \Omega$ be an arbitrary point and let us construct sequence $\{\kappa_n\}_{n\in\mathbb{N}} \subset \Omega$ as

$$\begin{aligned}
&\kappa_1 = f\kappa_0, \\
&\kappa_2 = f\kappa_1 = f^2\kappa_0, \\
&\text{and so on} \\
&\kappa_n = f\kappa_{n-1} = f^n\kappa_0,\ \forall\ n \in \mathbb{N}.
\end{aligned} \tag{3.8}$$

Let $d(\kappa_n, f\kappa_n) = 0$ where $n \in \mathbb{N} \cup \{0\}$. Then we can conclude the proof. So let

$$0 < d(\kappa_n, f\kappa_n) = d(f\kappa_{n-1}, f\kappa_n),\ \forall\ n \in \mathbb{N}. \tag{3.9}$$

We have

$$\tau + \mathcal{F}d(f\kappa_{n-1}, f\kappa_n)) = \mathcal{F}(d(f\kappa_{n-1}, \kappa_n)), \ \forall\, n \in \mathbb{N}, \tag{3.10}$$

that is,

$$\begin{aligned}\mathcal{F}(d(f\kappa_{n-1}, f\kappa_n)) &\leq \mathcal{F}(d(\kappa_{n-1}, \kappa_n) - \tau = \mathcal{F}(d(f\kappa_{n-2}, f\kappa_{n-1})) - \tau \\ &\leq \mathcal{F}(d(\kappa_{n-2}, \kappa_{n-1})) - 2\tau = \mathcal{F}(d(f\kappa_{n-3}, f\kappa_{n-2})) - 2\tau \\ &\leq \mathcal{F}(d(\kappa_{n-3}, \kappa_{n-2})) - 3\tau = \mathcal{F}(d(f\kappa_{n-4}, f\kappa_{n-3})) - 3\tau \\ &\ \vdots \\ &\leq \mathcal{F}(d(\kappa_0, \kappa_1)) - n\tau.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \mathcal{F}(d(\kappa_n, \kappa_{n+1})) = \lim_{n\to\infty} \mathcal{F}(d(f\kappa_{n-1}, f\kappa_n)) = -\infty.$$

By ($\mathcal{F}2$),

$$\lim_{n\to+\infty} d(\kappa_n, \kappa_{n+1}) = 0. \tag{3.11}$$

For proving Cauchyness of $\{\kappa_n\}_{n\in\mathbb{N}}$ if possible, let $\{k(n)\}_{n\in\mathbb{N}}$ and $\{l(n)\}_{n\in\mathbb{N}}$ be sequences with $k(n) > l(n) > n$ and $\varepsilon > 0$ be such that

$$k(n) > l(n) > n,\ d(\kappa_{k(n)}, \kappa_{l(n)}) \geq \varepsilon,\ d((\kappa_{k(n)-1}, \kappa_{l(n)-1}) < \varepsilon,\ \forall n \in \mathbb{N}. \tag{3.12}$$

Similar to Theorem 3.1, we have

$$\lim_{n\to+\infty} d(\kappa_{k(n)}, \kappa_{l(n)}) = \lim_{n\to+\infty} d(\kappa_{k(n)-1}, \kappa_{l(n)-1}) = \varepsilon. \tag{3.13}$$

So,

$$\tau + \mathcal{F}(d(f\kappa_{k(n)-1}, f\kappa_{l(n)-1})) \leq \mathcal{F}(d(\kappa_{k(n)-1}, \kappa_{l(n)-1})),\ \forall n \in \mathbb{N}.$$

Thus

$$\tau + \mathcal{F}(d(\kappa_{k(n)}, \kappa_{l(n)})) \leq \mathcal{F}(d(\kappa_{k(n)-1}, \kappa_{l(n)-1}),\ \forall n \in \mathbb{N}.$$

Taking the limit $n \to +\infty$, we have

$$\tau + \mathcal{F}(\varepsilon) \leq \mathcal{F}(\varepsilon)$$

being contradiction shows that $\{\kappa_n\}_{n\in\mathbb{N}}$ is Cauchy. By completeness of Ω, there is some $\kappa^* \in \Omega$ such that $\{\kappa_n\}_{n\in\mathbb{N}}$ is convergent.

Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence so that $\kappa_{k(n)+1} = f\kappa_{k(n)} = f\kappa^*$. Then $\lim_{n\to\infty} \kappa_{k(n)+1} = \kappa^*$. Thus $f\kappa^* = \kappa^*$. If $f\kappa^* \neq \kappa^*$, then we have

$$\tau + \mathcal{F}(d(\kappa_{n+1}, f\kappa^*)) \leq \mathcal{F}(d(\kappa_n, \kappa^*)),\ \forall n \geq N.$$

Taking the limit $n \to +\infty$, we get $\lim_{n\to+\infty} \mathcal{F}(d(\kappa_{n+1}, f\kappa^*)) = -\infty$. So, by ($\mathcal{F}2$), we get $\lim_{n\to+\infty} d(\kappa_{n+1}, f\kappa^*) = 0$. Thus $d(\kappa^*, f\kappa^*) = 0$ which is a contradiction. This shows that f has a fixed point κ^*. Uniqueness part is same as in Theorem 3.1. □

Example 3.3. Let $\Omega = \{P, Q, R, U, V\}$ and define $d : \Omega \times \Omega \to [0, +\infty)$ by

$$d(t,t) = 0, \ \forall t \in \Omega, \tag{3.14}$$

$$d(t,u) = d(u,t), \ \forall t,u \in \Omega, \tag{3.15}$$

$$d(P,Q) = d(P,R) = d(P,U) = d(Q,R) = d(Q,U) = 2, \tag{3.16}$$

$$d(P,V) = d(Q,V) = d(R,U) = 3, \ d(R,V) = d(U,V) = \frac{3}{2}. \tag{3.17}$$

For $s = 2$, (Ω, d) is a b-metric and complete.

Let us define a mapping $f : \Omega \to \Omega$ as $fP = R,\ fQ = U,\ fR = fU = fV = V$. Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ and satisfies relation (3.7). As $\mathcal{F}(d(fP, fQ)) = \mathcal{F}(d(R,U)) = \mathcal{F}(3)$, $\mathcal{F}(d(P,Q)) = \mathcal{F}(2)$, $\mathcal{F}$ cannot be increasing and hence $(\mathcal{F}1)$ does not hold.

Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{F}(p) = \begin{cases} -\frac{1}{t}, & t \in (0, \frac{3}{2}, \\ t - \frac{13}{6}, & t \in (\frac{3}{2}, \frac{5}{2}), \\ \frac{-4t+11}{3}, & t \in (\frac{5}{2}, 5], \\ t - 8, & t \in (5, \infty). \end{cases}$$

Then, $\mathcal{F}$ satisfy $(\mathcal{F}3'')$ and $(\mathcal{F}2)$.

For $t = P,\ u = R$ or $t = P,\ u = U$ or $t = Q,\ u = R$ or $t = Q,\ u = U$ we have $\mathcal{F}(d(ft, fu)) = \mathcal{F}(\frac{3}{2}) = -\frac{2}{3}$ and $\mathcal{F}(d(t,u)) = \mathcal{F}(2) = -\frac{1}{6}$ so we have $\tau - \frac{2}{3} \leq -\frac{1}{6}$ or $\tau \leq \frac{1}{2}$.

For $t = P,\ u = V$ or $t = Q,\ u = V$ we get $\mathcal{F}(d(ft, fu)) = \mathcal{F}(\frac{3}{2}) = -\frac{2}{3}$ and $\mathcal{F}(d(t,u)) = \mathcal{F}(3) = -\frac{1}{3}$ so we have $\tau - \frac{2}{3} \leq -\frac{1}{3}$ or $\tau \leq \frac{1}{3}$. Choosing $\tau = \frac{1}{6}$, $\mathcal{F}$ satisfies conditions of Theorem 3.2.

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